Ambiguities of low-x evolution equations at NLO

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Introduction

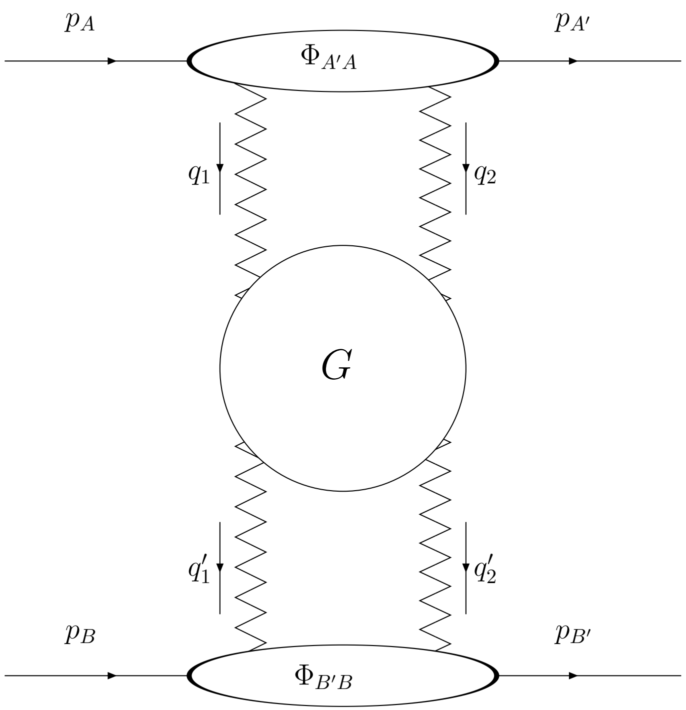
Talking about low-x evolution equations I mean the BFKL equation and the evolution equation of color dipoles. For brevity I'll call it BK equations.
From the Summary of my talk at the preceding Workshop:

- In the case of scattering of colourless objects the BFKL kernel can be written in the dipole form (Möbius representation).
- The dipole form is greatly simplified in comparison with the BFKL kernel in the momentum representation.
- The quark contribution to the dipole form agrees with corresponding contribution to the BK kernel.
- It would be extremely interesting to compare corresponding gluon contributions. Unfortunately, this contribution to the BK kernel is not yet known.

Introduction

In the BFKL approach scattering amplitudes $\mathcal{A}_{AB}^{A'B'}$ are presented in the form :

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}.$$



The impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describing transitions $A \rightarrow A'$ and $B \rightarrow B'$ depend on properties of scattering particles.

All energy dependence is contained in the Green's function G for two interacting Reggeized gluons.

Originally the approach was formulated in the momentum space. The impact factors and the kernel kernel of the BFKL equation for the Green's function are defined in the transverse momentum space.

The kernel is known now in the NLO for $t \neq 0$ and all possible t-channel colour states.

Introduction

For scattering of colourless objects the BFKL equation can be written in the Möbius invariant form
L.N. Lipatov, 1986.
The Möbius invariance can be made evident by transformation from the transverse momentum to the transverse coordinate representation.
Moreover, in the Möbius invariant form the LO BFKL kernel in the coordinate representation coincides with the kernel of the colour dipole approach
N.N. Nikolaev and B.G. Zakharov, 1994,
A. H. Mueller, 1994.
Actually this kernel can be written as early as 1985, when Lipatov considered the BFKL equation in the coordinate space and demonstrated that for scattering of colourless objects its Green function in this space can be taken in conformal invariant form. Only by chance he did not write explicitly the kernel in this space.

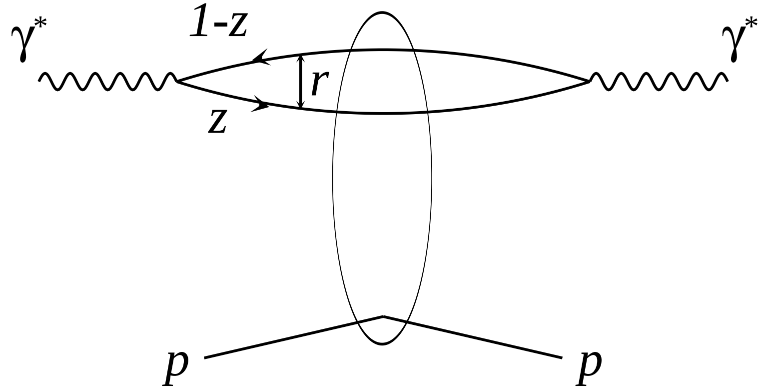
Introduction

In the color dipole approach

N.N. Nikolaev and B.G. Zakharov, 1994,

A. H. Mueller, 1994

γ^* scattering is considered as γ^* splitting into a $q\bar{q}$ colour dipole with subsequent $q\bar{q}$ scattering. The important point is conservation of transverse coordinates of the dipole components.



$$\sigma_{\gamma^*}(x,Q^2) = \int d^2r \int_0^1 dz |\Psi_{\gamma^*}(r,z,Q^2)|^2 \sigma_{dp}(r,x),$$

$x = Q^2/s$, $\Psi_{\gamma^*}(r,z,Q^2)$ is the photon wave function, z is the longitudinal momentum fraction carried by the quark, $\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{r}_1$ and $\vec{r}_2$ are the quark and antiquark transverse coordinates, $\sigma_{dp}(r,x)$ is the dipole cross section,

Introduction

$$\sigma_{dp}(r,x) = 2\int d^2b\, \mathcal{N}(\vec{r}_1,\vec{r}_2;Y)\,;$$

$\vec{b} = (\vec{r}_1 + \vec{r}_2)/2$ is the impact parameter, $Y = \log(1/x)$, $\mathcal{N}(\vec{r}_1,\vec{r}_2;Y)$ is the the imaginary part of the dipole scattering amplitude obeying the equation

$$\frac{\partial \mathcal{N}}{\partial Y} = \widehat{\mathcal{K}}_{dip}\mathcal{N},$$

$$\langle \vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_{dip}|\vec{r}_1{}'\vec{r}_2{}'\rangle = \frac{\alpha_s N_c}{2\pi^2}\int d^2\rho \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\left(\delta(\vec{r}_{11'})\delta(\vec{r}_{2\rho}) + \delta(\vec{r}_{22'})\delta(\vec{r}_{1\rho}) - \delta(\vec{r}_{11'})\delta(\vec{r}_{22'})\right),$$

$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j,\;\; \vec{r}_{i\rho} = \vec{r}_i - \vec{\rho},\;\; \vec{r}_{ij'} = \vec{r}_i - \vec{r}_j{}',$

with the non-linear extension (BK equation) for $S = 1 - \mathcal{N}$:

$$\frac{\partial S(\vec{r}_1,\vec{r}_2;Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\rho}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\left[S(\vec{r}_1,\vec{\rho},Y)S(\vec{\rho},\vec{r}_2,Y) - S(\vec{r}_1,\vec{r}_2,Y)\right]$$

Ia. Balitsky, 1996, Yu. Kovchegov, 1999.

Möbius representation of the BFKL kernel

Generally speaking, the BFKL kernel is more general than the the dipole one. This is clear, because the BFKL kernel can be applied not only in the case of scattering of colourless objects.

However, when it is applied to the latter case, one can use the "dipole" and "gauge invariance" properties of targets and projectiles
L. N. Lipatov, 1989,
and omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the "dipole" property, i.e. the property which provides vanishing of cross-sections for scattering of zero-size dipoles.

The coordinate representation of the kernel obtained in such a way is what we call dipole or Möbius form of the BFKL kernel.

Möbius representation of the BFKL kernel

Indeed, for colourless objects the impact factors in the representation

$$\delta(\vec{q}_A - \vec{q}_B) disc_s \mathcal{A}_{AB}^{A'B'} = \frac{i}{4(2\pi)^{D-2}} \langle A'\bar{A}| e^{Y\widehat{\mathcal{K}}} \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle$$

are "gauge invariant":

$$\langle A'\bar{A}|\vec{q}, 0\rangle = \langle A'\bar{A}|0, \vec{q}\rangle = 0 \; .$$

Therefore $\langle A'\bar{A}|\Psi\rangle = 0$ if $\langle \vec{r}_1, \vec{r}_2|\Psi\rangle$ does not depend either on $\vec{r}_1$ or on $\vec{r}_2$.
$\langle A'\bar{A}|\widehat{\mathcal{K}}$ is "gauge invariant" as well, because $\langle \vec{q}_1, \vec{q}_2|\widehat{\mathcal{K}}_r|\vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime}\rangle$ vanishes at $\vec{q}_1^{\,\prime} = 0$ or $\vec{q}_2^{\,\prime} = 0$.
It means that we can change $|In\rangle \equiv \left(\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}\right)^{-1} |\bar{B}'B\rangle$ for $|In_d\rangle$, where $|In_d\rangle$ has the "dipole " property $\langle \vec{r}, \vec{r}|In_d\rangle = 0$.
After this one can omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel becomes conserving the "dipole " property.

Möbius representation of the BFKL kernel

In the NLO the Möbius form can be written as

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_d^{NLO} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{\alpha_s^2(\mu) N_c^2}{4\pi^3} \Bigg[\delta(\vec{r}_{11'}) \delta(\vec{r}_{22'}) \int d\vec{\rho}\, g^0(\vec{r}_1, \vec{r}_2; \vec{\rho})$$

$$+ \delta(\vec{r}_{11'}) g(\vec{r}_1, \vec{r}_2; \vec{r}_2{}') + \delta(\vec{r}_{22'}) g(\vec{r}_2, \vec{r}_1; \vec{r}_1{}') + \frac{1}{\pi} g(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') \Bigg]$$

with the functions g turning into zero when their first two arguments coincide.

The first three terms contain ultraviolet singularities which cancel in their sum, as well as in the LO, with account of the “dipole” property of the “target” impact factors. The coefficient of $\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})$ is written in the integral form in order to make the cancellation evident.

The term $g(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}')$ is absent in the LO because the LO kernel in the momentum space does not contain terms depending on all three independent momenta simultaneously.

Möbius representation of the BFKL kernel

For the gluon contribution

V.S. F, R. Fiore, A.V. Grabovsky, A. Papa, 2007

$$g_G(\vec{r}_1, \vec{r}_2; \vec{r}_2\,') \;=\; \frac{11}{6}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{r_G^2}\right) + \frac{11}{6}\left(\frac{1}{\vec{r}_{22'}^{\,2}} - \frac{1}{\vec{r}_{12'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)$$

$$+\frac{1}{2\vec{r}_{22'}^{\,2}}\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{2\,\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right),$$

$$\ln r_G^2 = 2\psi(1) - \ln\frac{\mu^2}{4} - \frac{3}{11}\left(\frac{67}{9} - 2\zeta(2)\right).$$

$$g_G^0(\vec{r}_1, \vec{r}_2; \rho) = g_G(\vec{r}_1, \vec{r}_2; \rho) + \frac{1}{2}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\ln\left(\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{12}^{\,2}}\right)\ln\left(\frac{\vec{r}_{2\rho}^{\,2}}{\vec{r}_{12}^{\,2}}\right)$$

Both $g_G^0(\vec{r}_1, \vec{r}_2; \vec{\rho})$ and $g_G(\vec{r}_1, \vec{r}_2; \vec{\rho})$ vanish at $\vec{r}_1 = \vec{r}_2$. Then, these functions turn into zero for $\vec{\rho}^{\,2} \to \infty$ faster than $(\vec{\rho}^{\,2})^{-1}$ to provide the infrared safety. The ultraviolet singularities of these functions at $\vec{\rho} = \vec{r}_2$ and $\vec{\rho} = \vec{r}_1$ cancel on account of the “dipole” property of the “target” impact factors.

Möbius representation of the BFKL kernel

$$g_G(\vec r_1,\vec r_2;\vec r_1^{\,\prime},\vec r_2^{\,\prime}) = \left[\frac{(\vec r_{22'}\ \vec r_{12})}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}} - \frac{2\,(\vec r_{22'}\ \vec r_{11'})}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}} + \frac{2\,(\vec r_{22'}\ \vec r_{12'})\,(\vec r_{11'}\ \vec r_{12'})}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}\vec r^{\,2}_{12'}}\right]\ln\left(\frac{\vec r^{\,2}_{12'}}{\vec r^{\,2}_{1'2'}}\right)$$

$$+\frac{1}{2\vec r^{\,2}_{1'2'}}\left[\frac{(\vec r_{11'}\,\vec r_{22'})}{\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}} + \frac{(\vec r_{21'}\,\vec r_{12'})}{\vec r^{\,2}_{21'}\,\vec r^{\,2}_{12'}} - \frac{2(\vec r_{22'}\ \vec r_{21'})}{\vec r^{\,2}_{22'}\vec r^{\,2}_{21'}}\right]\ln\left(\frac{\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}}{\vec r^{\,2}_{1'2'}\vec r^{\,2}_{12}}\right) + \frac{(\vec r_{11'}\ \vec r_{22'})}{2\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}}\ln\left(\frac{\vec r^{\,2}_{21'}\,\vec r^{\,2}_{12'}}{\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}}\right)$$

$$+\frac{1}{d\,\vec r^{\,2}_{1'2'}}\left[\frac{(\vec r_{1'2'}\,\vec r_{12'})\vec r^{\,2}_{12}}{\vec r^{\,2}_{11'}} + \frac{2(\vec r_{22'}\,\vec r_{21'})(\vec r_{12}\,\vec r_{21'})}{\vec r^{\,2}_{21'}} + \frac{(\vec r_{22'}\,\vec r_{12'})(\vec r_{11'}\,\vec r_{21'})}{\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}}\vec r^{\,2}_{1'2'} - \vec r^{\,2}_{1'2'}\right]\ln\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}}\right)$$

$$+\frac{1}{2\vec r^{\,4}_{1'2'}}\left(\frac{\vec r^{\,2}_{11'}\,\vec r^{\,2}_{22'}}{d}\ln\left(\frac{\vec r^{\,2}_{12'}\,\vec r^{\,2}_{21'}}{\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}}\right) - 1\right) + \frac{1}{\vec r^{\,2}_{11'}}\left(\frac{(\vec r_{12}\,\vec r_{21'})}{\vec r^{\,2}_{1'2'}\vec r^{\,2}_{21'}} - \frac{(\vec r_{11'}\,\vec r_{12})}{\vec r^{\,2}_{1'2'}\vec r^{\,2}_{22'}} - \frac{(\vec r_{11'}\,\vec r_{21'})}{\vec r^{\,2}_{22'}\vec r^{\,2}_{21'}}\right)\ln\left(\frac{\vec r^{\,2}_{12'}}{\vec r^{\,2}_{11'}}\right)$$

$$-\frac{(\vec r_{12}\ \vec r_{22'})}{\vec r^{\,2}_{1'2'}\vec r^{\,2}_{22'}\vec r^{\,2}_{12'}}\ln\left(\frac{\vec r^{\,2}_{11'}}{\vec r^{\,2}_{1'2'}}\right) + (1\leftrightarrow 2),$$

$$d = \vec r^{\,2}_{12'}\vec r^{\,2}_{21'} - \vec r^{\,2}_{11'}\vec r^{\,2}_{22'}.$$

This term also vanishes at $\vec r_1 = \vec r_2$, so that it possesses the “dipole” property. It has ultraviolet singularity only at $\vec r_{1'2'} = 0$ and tends to zero at large $\vec r_1^{\,\prime\,2}$ and $\vec r_2^{\,\prime\,2}$ sufficiently quickly in order to provide the infrared safety.

The kernel of the linearized BK equation

$$g(\vec{r}_1, \vec{r}_2; \vec{r}_2{}') = \frac{11}{6} \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2} \vec{r}_{12'}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{r_{\mu_B}^2}\right) + \frac{11}{6}\left(\frac{1}{\vec{r}_{22'}^{\,2}} - \frac{1}{\vec{r}_{12'}^{\,2}}\right) \ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)$$

$$-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2} \vec{r}_{12'}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right) \ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right),$$

$$\ln r_{\mu_B}^2 = -\ln \mu^2 - \frac{3}{11}\left(\frac{67}{9} - 2\zeta(2)\right)$$

$$g^0(r_1, r_2, \rho) = -g_B(r_1, r_2, \rho)$$

$$g_B(\vec{r}_1, \vec{r}_2; \vec{r}_1{}', \vec{r}_2{}') = \ln\left(\frac{\vec{r}_{12'}^{\,2} \vec{r}_{21'}^{\,2}}{\vec{r}_{11'}^{\,2} \vec{r}_{22'}^{\,2}}\right) \left[\frac{\vec{r}_{11'}^{\,2} \vec{r}_{22'}^{\,2} - 2\vec{r}_{12}^{\,2} \vec{r}_{1'2'}^{\,2}}{d\, \vec{r}_{1'2'}^{\,4}} + \frac{1}{2\vec{r}_{1'2'}^{\,4}}\right.$$

$$\left. + \frac{1}{4\vec{r}_{11'}^{\,2} \vec{r}_{22'}^{\,2}} \left(\frac{\vec{r}_{12}^{\,4}}{d} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right) + \frac{1}{4\vec{r}_{12'}^{\,2} \vec{r}_{21'}^{\,2}} \left(\frac{\vec{r}_{12}^{\,4}}{d} + \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)\right] - \frac{1}{\vec{r}_{1'2'}^{\,4}}.$$

Ambiguities of the NLO kernel

The discontinuities

$$\delta(\vec{q}_A - \vec{q}_B) disc_s \mathcal{A}_{AB}^{A'B'} = \frac{i}{4(2\pi)^{D-2}} \langle A'\bar{A}| e^{Y\widehat{\mathcal{K}}} \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle$$

are invariant under the operator transformation

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{O}}^{-1} \widehat{\mathcal{K}} \widehat{\mathcal{O}}, \ \langle A'\bar{A}| \to \langle A'\bar{A}|\widehat{\mathcal{O}}, \ \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle \to \widehat{\mathcal{O}}^{-1} \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle \ .$$

In the LO the kernel is fixed by the requirement of the Möbius invariance of its dipole form. But even after this transformations with $\hat{\mathcal{O}} = 1 + \hat{O}$, where $\hat{O} \sim g^2$, are still possible. At that

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}} - [\widehat{\mathcal{K}}^B \hat{O}].$$

These transformations rearrange NLO corrections to the kernel and impact factors. They can be used for simplification of the dipole form.

Ambiguities of the NLO kernel

$$\left(\frac{s}{s_0}\right)^{\widehat{\mathcal{K}}} \simeq \left(1+\hat{O}_L\right)\left(\frac{s}{\hat{f}_L \hat{f}_R}\right)^{\widehat{\mathcal{K}}}\left(1+\hat{O}_R\right)$$

$$\hat{O}_L = \ln\left(\frac{\hat{f}_L}{\sqrt{s_0}}\right)\widehat{\mathcal{K}}^B$$

$$\hat{O}_R = \widehat{\mathcal{K}}^B \ln\left(\frac{\hat{f}_R}{\sqrt{s_0}}\right)$$

$$\left(\frac{s}{s_0}\right)^{\widehat{\mathcal{K}}} \simeq \left(\frac{s}{\hat{f}_L \hat{f}_R}\right)^{\widehat{\mathcal{K}}'}\left(1-\hat{O}_L\right)\left(1+\hat{O}_R\right)$$

$$\widehat{\mathcal{K}}' = \widehat{\mathcal{K}} - [\widehat{\mathcal{K}}^B, \hat{O}_L]$$

Summary

- There is an evident discrepancy between the Möbius form of the BFKL kernel and the BK kernel.
- The discrepancy can be removed (at least partly) by the ambiguities of the NLO kernel.
- The ambiguities are concerned with definition of the impact factors and the energy scale.
- Till now we do not managed to remove the discrepancy using these ambiguities.