Geometric scaling from DGLAP evolution

Fabrizio Caola

Università degli Studi di Milano

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Outline

Geometric scaling from DGLAP evolution: theory

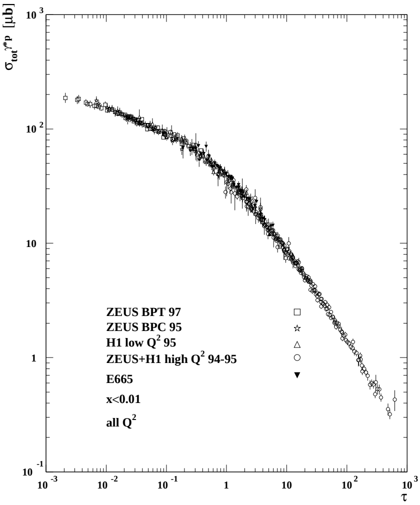
- Geometric scaling, saturation and DGLAP evolution
- Can geometric scaling be produced by DGLAP evolution?
- A simple fixed coupling analysis
- Introducing running coupling
- G.S. can in fact be produced by DGLAP evolution

Phenomenology: is the HERA scaling a DGLAP–based scaling?

- The geometric scaling kinematic window
- Theoretical vs. phenomenological scaling

[Based on Stefano Forte & F.C., 0802.1878 (hep-ph), accepted by Phys. Rev. Lett.]

Geometric scaling



Stasto, Golec-Biernat, Kwiecinski, hep-ph/0007192

Geometric scaling

$$\sigma\left(x, Q^2\right) = F_2/Q^2 = \sigma(\tau),$$

with $\tau = Q^2 x^\lambda$ or

$$\tau = Q^2 \exp\left[-\lambda\sqrt{\log(1/x)}\right]$$

How can we explain geometric scaling?

Three possible scenarios:

1. Geometric scaling is a saturation–based phenomenon. What we are seeing at HERA are saturation effects. If so, big problems with our PDFs!
2. Geometric scaling is generated by saturation physics at some low scale and then it is preserved by DGLAP evolution [see e.g. Kwiecinski, Stasto, PRD 66:014013,2002]
3. Geometric scaling is generated by DGLAP evolution. There exists a region where geometric scaling can be explained by pure DGLAP evolution, without need of saturation

A toy model without saturation: the LO DGLAP evolution at small x

- At small x the evolution is dominated by the large eigenvalue of the a.d. matrix in the singlet sector
- Consider only the singlet parton density

$$G(x,t) = x\left[g(x,Q^2) + k_q \otimes q(x,Q^2)\right]$$

with as usual $t \equiv \log Q^2/Q_0^2$

The LO DGLAP equation for G in Mellin space

$$\frac{d}{dt}G(N,t) = \alpha_s \gamma_0(N) G(N,t)$$

GS from DGLAP evolution: the fixed coupling case

The DGLAP solution

$$G(\xi, t) = \int \frac{dN}{2\pi i} G_0(N) \exp\left[\alpha_s \gamma_0(N)\ t + N \log(1/x)\right]$$

In the saddle point approximation

$G \approx e^{\alpha_s \gamma_0(N_s)\ t + N_s \log(1/x)}$, leading to the double log result

$$\sigma = \exp\left[2\sqrt{\overline{\alpha}_s t \log(1/x)} - (1 + \overline{\alpha}_s) t\right],$$

with $\overline{\alpha}_s \equiv N_c/\pi\ \alpha_s$ and $t \equiv \log Q^2/Q_0^2$

Apparently no geometric scaling!

A closer look at the saddle point approximation

The saddle condition reads

$$\alpha_s \left. \frac{d}{dN}\gamma_0(N) \right|_{N=N_s} = -\frac{\xi}{t} \longrightarrow N_s(t,\xi) = N_s(\xi/t),$$

where $\xi \equiv \log(1/x)$

Hence

$$\sigma \sim \exp\left[\alpha_s \gamma_0(N_s)\ t + N_s \xi - t\right] = \exp\left[f(t/\xi)\xi\right],$$

with

$$f(z) = (\alpha_s \gamma_0(N_s) - 1)\ z + N_s.$$

Geometric scaling from the saddle point approximation

Now expand $f(z)$ around $t/\xi = z_0 = \lambda$ such that $f(z_0) = 0$:

$$\sigma \sim \exp\left[f'(\lambda)(z - z_0)\xi + O\left((z - z_0)^2\right)\right]$$

As long as we can neglect higher terms in this expansion

$$\sigma \sim \exp\left[f'(\lambda)\left(\frac{t}{\xi} - \lambda\right)\xi\right] = \exp\left[f'(\lambda)(t - \lambda\xi)\right]$$

Geometric scaling!

$$\sigma(t, \xi) = \sigma(t - \lambda\xi) = \sigma\left(Q^2 x^\lambda\right)$$

A few comments

- Analitically, this is the same argument proposed by Iancu et al. in a BFKL context, [NPA 708:327-352,2002]
- However: Iancu et al. impose the condition $\sigma(t = \lambda\xi) = const$ as a consequence of parton saturation. At the DGLAP level, this condition is automatically fulfilled with the LO anomalous dimension γ_0 (and more in general with any reasonable anomalous dimension)
- Note that G_0 does not enter in our equations. We have implicitly assumed that the boundary condition is washed out by the perturbative evolution

Running coupling

DGLAP–BFKL duality in the leading twist sector

Write the DGLAP solution in the "dual" form

$$G(\xi, t) \approx \int \frac{dM}{2\pi i} \exp\left(Mt + \sqrt{\xi \frac{-2 \int_{M_0}^{M} \chi(\alpha_s, M') dM'}{\beta_0 \alpha_s}} \right)$$

where χ is a suitable kernel "dual" to γ.

We can repeat the previous saddle point argument, with the only replacement

$$\xi \to \sqrt{\xi}$$

A new scaling variable!

$$\log \tau = t - \lambda \sqrt{\xi} \to \tau = Q^2 \exp\left[-\lambda \sqrt{1/x} \right]$$

Summarizing our results so far...

G.S. is an approximation to the full DGLAP solution!

- Fixed coupling G.S. variable: $\log\tau = t - \lambda\log(1/x)$
- Running coupling G.S. variable: $\log\tau = t - \lambda\sqrt{\log(1/x)}$

The third scenario is possible!

Geometric scaling can be generated by perturbative DGLAP evolution

How good our approximations are?

The arguments so far involved several approximations:

- Saddle point evaluation of the integral
- Truncated Taylor expansion
- Fixed coupling analysis

To assess their accuracy:

1. Introduce the variable $\zeta = t + \lambda\xi$
2. Search for $\lambda = \lambda(t, \xi)$ such that

$$\frac{d\sigma}{d\zeta} = 0$$

3. If $\lambda(t, \xi) = const$, then we have exact geometric scaling

An analytical argument: running coupling scaling

The derivative argument

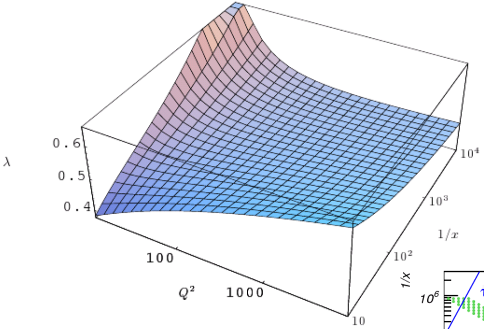
Determine λ from the condition $\frac{d}{d\zeta}\sigma = 0$. The leading term:

$$\lambda = \frac{2\gamma t \log(t/t_0)}{(t+\gamma^2)\sqrt{\log(t/t_0)} - \gamma\sqrt{\xi}}$$

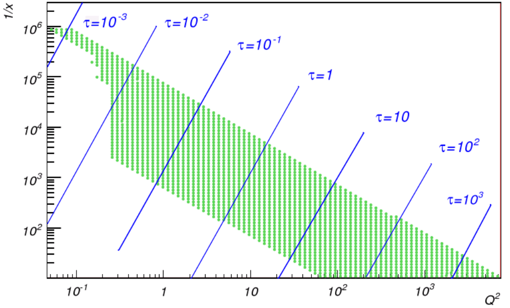
- If $(t+\gamma^2)^2 \log(t/t_0) \gg \gamma^2\xi$, then λ does not depend on x
- As t increases λ becomes more and more a constant

This geometric scaling is a large Q^2 – "large" x phenomenon!

A numerical argument, fixed coupling scaling



The HERA small x region →



How to extract λ: the quality factor method

[Gelis et al., PLB 647:376-379,2007]

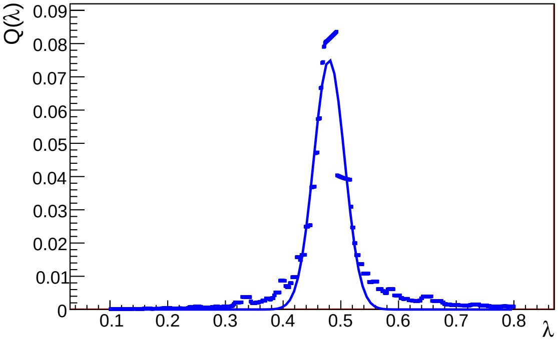
How can we extract the best value for λ?

Define $Q(\lambda)^{-1} \equiv \sum_i \left[\left([\sigma_{tot}^{\gamma^* p}]_{i+1} - [\sigma_{tot}^{\gamma^* p}]_i \right)^2 \Big/ \left((\tau_{i+1} - \tau_i)^2 + \epsilon \right) \right]$

From a gaussian fit:

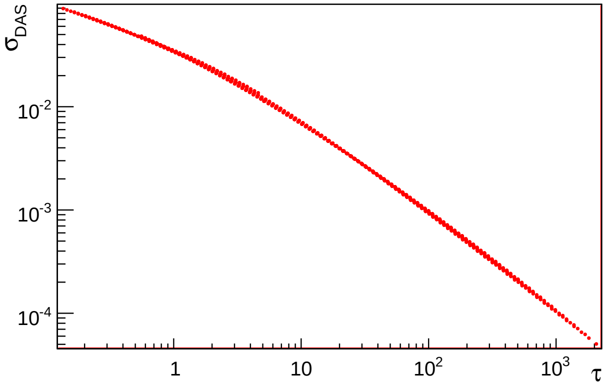
$$\lambda_{fix} = 0.48 \pm 0.02$$

$$\lambda_{run} = 2.18 \pm 0.22$$



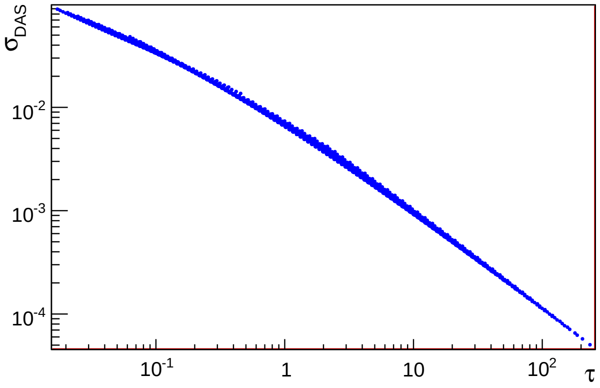
Scaling plot – fixed coupling scaling

The LO DGLAP form for σ in the HERA region,
$x < 0.1$, $Q^2 > 10\ \text{GeV}^2$ and $\log\tau = t - \lambda\xi$, $\lambda = 0.48$



Scaling plot – running coupling scaling

Same as before, but with $\log\tau = t - \lambda\sqrt{\xi}$, $\lambda = 2.18$



The DGLAP solution exhibits geometric scaling!

What we have seen so far

The LO DGLAP solution exhibits geometric scaling

- Spectacular scaling behaviour both in the fixed and in the running coupling variables
- This scaling is generated by the DGLAP evolution
- The scaling behaviour persists in a wide kinematic window
- In particular GS persists at large Q^2 and "large" $x \longrightarrow$

Different from saturation–based scaling!

What about the real world?

Can we use our theoretical results to explain the phenomenological geometric scaling observation?

Yes, as long as the DGLAP evolution is a good approximation to the full QCD evolution. This is true if

- x should be small, but not so small ✓
- Q^2 should be large enough to justify a f.o. calculation ✓
- Boundary condition effects should be small enough ✓
- The "small" eigenvector of the a.d. matrix should be really suppressed X

✓: OK in the small x HERA region for $Q^2 > 10$ GeV^2

DGLAP evolution at the quark–gluon coupled level

Only the largest eigenvector:

$$F_2 = \frac{\gamma}{\rho} G$$

Only a trivial overall constant K must be fitted to the data

Both the contributions:

$$F_2 = \frac{\gamma}{\rho} G + \bar{G}$$

with

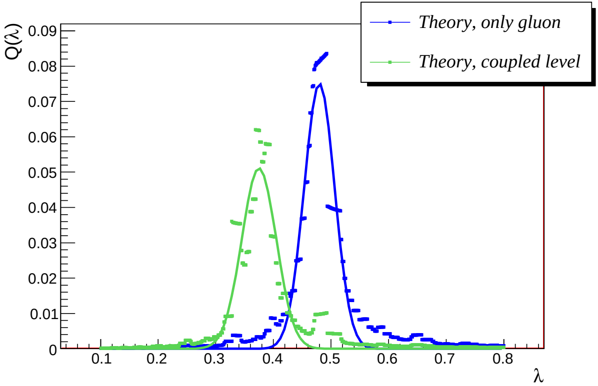
$$\bar{G} = k \exp\left[-16 \frac{n_f}{27\beta_0} \log(t/t_0)\right]$$

k must be fitted to the data. From a global fit we obtain $k = 0.18$ (only large Q^2 – "large" x data fitted)

The small eigenvector and geometric scaling

The new term $\bar{G}$ violates G.S., hence we expect that the scaling behaviour of the full solution deteriorates slightly.

Indeed, this is just the case:



$Q^2 > 25$ GeV^2
(so $n_f = 5$)

The effects of the small eigenvector

$\bar{G}$ deteriorates slightly geometric scaling, but we are forced to consider it if we want to explain data!

Considering all data with $Q^2 > 10$ GeV^2

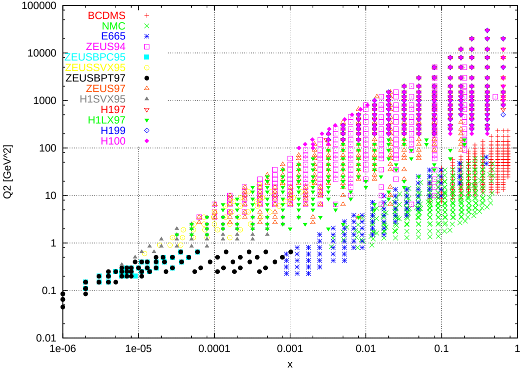
- $\lambda_{fix} = 0.34 \pm 0.02$
- $\lambda_{run} = 1.68 \pm 0.26$

These are our final predictions for λ

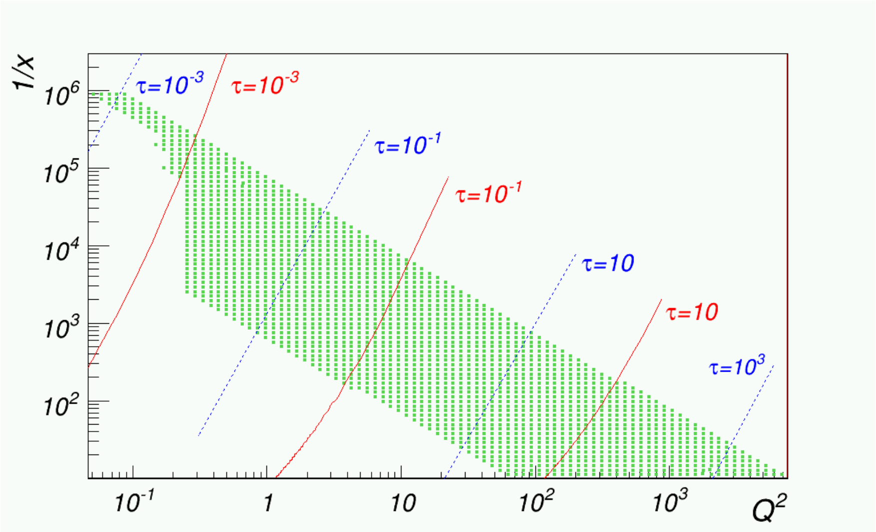
Phenomenology I: The neural network approach

The neural neural network parametrization of F_2
[NNPDF Collaboration, JHEP 0503(2005) 080]

- More flexible analysis
- Reliable results as long as we stay in the "populated" region

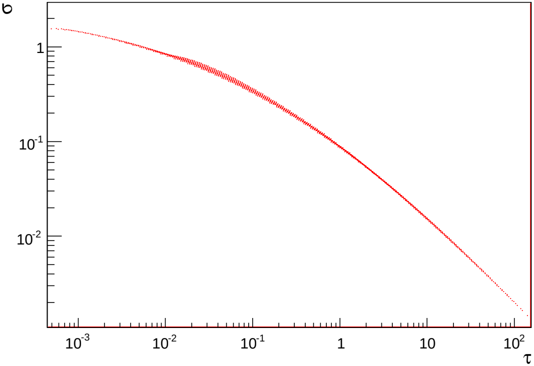


Phenomenology II: Our sample



Geometric scaling in the original kinematic window

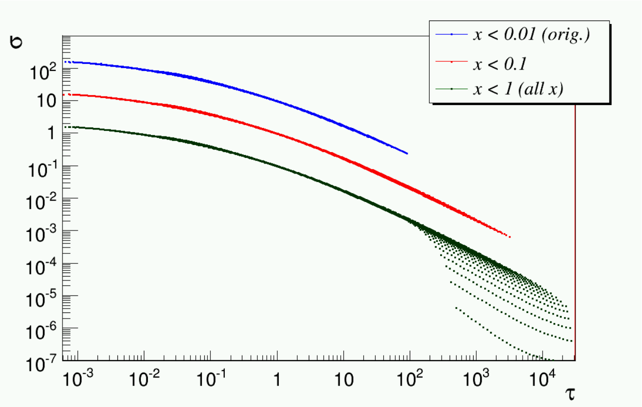
- $x < 0.01,\ Q^2 < 450\ \text{GeV}^2$
- $\lambda = \lambda_{fix} = 0.34 \qquad \lambda_{exp} = 0.32 \pm 0.06$



Geometric scaling in an extended window

Is this scaling a DGLAP like scaling?

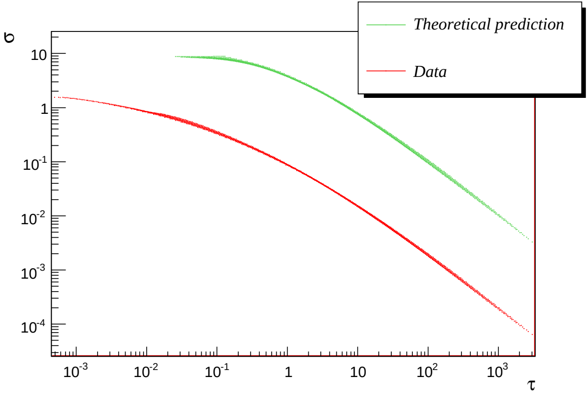
If so, it should be valid in a wider kinematic region, say $x < 0.1$



Our final results:

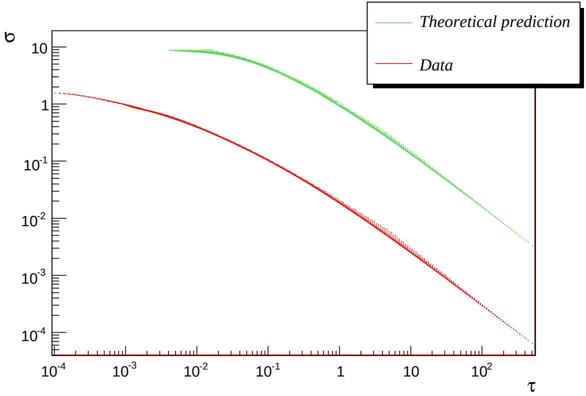
Fixed–coupling scaling

$\lambda = \lambda_{fix} = 0.32,\ x < 0.1,\ Q^2 > 1\ \text{GeV}^2$ for the theoretical curve



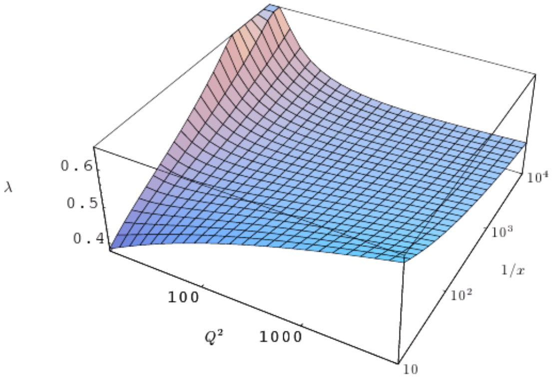
The same with running–coupling scaling

$\lambda = \lambda_{run} = 1.66 \qquad \lambda_{exp} = 1.62 \pm 0.25$



The DGLAP evolution can explain GS in a wide kinematic window!

What about the small x region?



At small x

Perturbative resummations!

By far more involved

At HERA: small $x \rightarrow$ small Q^2, hence higher order and higher twist effects.

What can we learn from perturbative resummation

- $\alpha_s \log 1/x$ resummation effects are smaller than one could naively expect
- Seizable effects only for $x < 10^{-4}$
- A quadratic expansion of the BFKL kernel near its minimus is a vary good approximation
- The BFKL equation must be considered at the running coupling level

see e.g. Altarelli, Ball, Forte, NPB 742 (2006); Ciafaloni, Colferai, Salam Stasto, JHEP 0708 (2007) and Chris White's talk

Resummations and geometric scaling

Resummation of a quadratic BFKL kernel at running coupling

- First approximation: the a.d. has a simple pole located at $N_0 \sim 0.1 - 0.3$ leading to a fixed coupling GS with $\lambda = N_0$
- If we consider the leading Q^2 dependence of the pole: approximate running coupling GS with $\lambda \sim 1.2 - 1.7$ (Airy resummation)
- Running coupling II scaling with $\lambda_{RC_{II}} = \lambda_{RC}^2$

Still compatible with the phenomenological observation!

This way a DGLAP–based GS could extend down to $Q^2 \approx 5$ GeV^2

Conclusions and outlook

So...

- In a wide kinematic region, say $Q^2 > 10$ GeV^2 the geometric scaling seen at HERA seems indeed a DGLAP–based scaling
- 5 GeV$^2 \lesssim Q^2 \lesssim 10$ GeV^2: perturbative resummations may provide an explanation for GS (Handle with care!)
- For yet lower Q^2 G.S. may provide genuine evidence for parton saturation

How can we improve these results?

- Focus on the small Q^2 region
- Subasymptotic corrections in order to disentangle DGLAP and saturation–based scaling

Backup slides

A note on the running coupling derivation

Consider again the DGLAP solution in the dual form

$$G(\xi, t) \approx \int \frac{dM}{2\pi i} \exp\left(Mt + \sqrt{\xi \frac{-2\int_{M_0}^{M} \chi(\alpha_s, M')dM'}{\beta_0 \alpha_s}} \right)$$

- The running coupling solution in the dual form is valid only if the kernel χ is linear in α_s
- OK in the collinear approximation
- OK if χ is a generic LO BFKL kernel
- Not OK with a generic LO DGLAP kernel! Less general than the fixed coupling case

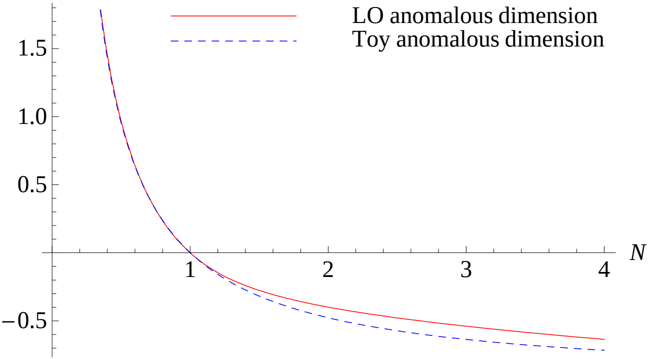
The toy model

Consider a LO DGLAP evolution with anomalous dimension γ given by

$$\gamma(\alpha_s, N) = \alpha_s \frac{N_c}{\pi} \left(\frac{1}{N} - 1 \right)$$

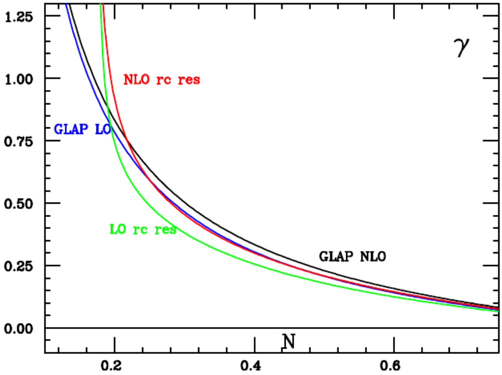
- Simple pole at $N = 0 \rightarrow$ OK for not so small x (see e.g. Guido Altarelli's talk)
- $\gamma(\alpha_s, 1) = 0 \rightarrow$ OK with momentum conservation
- No saturation at all
- Can be solved analytically

Not so bad for a toy model!



Quite accurate in a wide kinematic region
(say $x \lesssim 0.1$, $Q^2 \gtrsim 10\ \text{GeV}^2$)

The toy model and resummations



[ALTARELLI, BALL, FORTE, NPB 742:1-40,2006.]

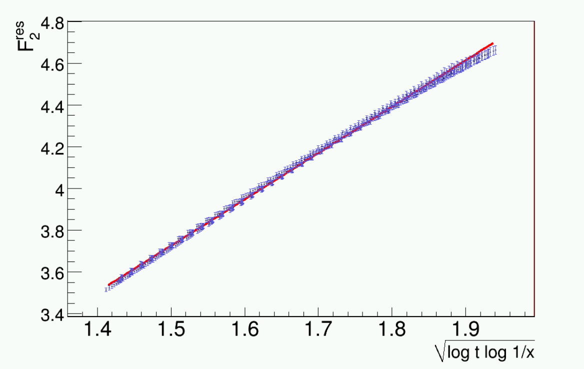
OK down to $x \sim 10^{-4}$

LO DGLAP evolution: a comparison with data

Only one eigenvector

QCD prediction: $F_2 \approx f\,(t, \log(1/x))\exp\left[2\gamma\sqrt{\log t \log(1/x)}\right]$

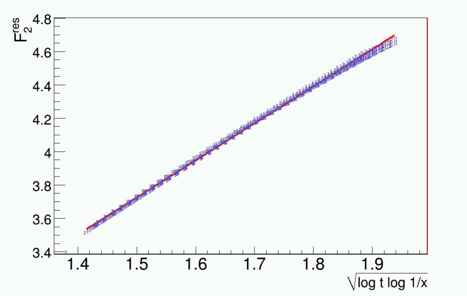
Define $F_2^{res} \equiv \log(F_2/f)$ and plot the experimental F_2^{res}



$\gamma_{fit} = 2.22 \pm 0.004$
$\gamma_{th} = 2.4 \; (n_f = 4)$

Both eigenvectors

This time $F_2^{res} \equiv \log\left[(F_2 - \bar{G})/f\right]$.

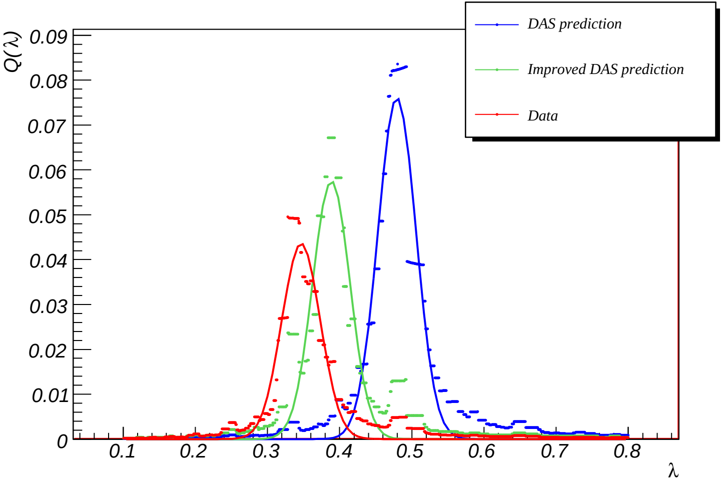


$\gamma_{fit} = 2.42 \pm 0.004$
$\gamma_{th} = 2.4 \ (n_f = 4)$

Good agreement theory/phenomenology

Up to our level of accuracy, the (improved) toy model is in good agreement with data

The quality factor: Comparison with data



GS and resummations: the Airy case

Consider a quadratic BFKL kernel

$$\chi(\alpha_s, M) = \alpha_s \left[c + k/2\,(M - M_0)^2\right]$$

then the r.c. resummed anomalous dimension reads

$$\gamma_A = \frac{3\beta_0 N_0^2 \alpha_s(t)}{4\pi\beta_0 + 8\pi c \alpha_s(t)} \frac{1}{N - N_0} + O\left[(N - N_0)^0\right]$$

Leading behaviour of the solution

$$\mathcal{M}^{-1}\left[\exp(A/(N - N_0))\right] \approx \exp\left[N_0\xi + 2\sqrt{A\xi}\right]$$

Approximate GS (modulo logarithmic deviations)

$$\sigma \approx \exp(-t + N_0\xi)$$

Taking into accout the (leading) Q^2 dependence of N_0

$$N_0 : \quad \left(\frac{2\beta_0 N_0}{4\pi k}\right)^{1/3} \frac{4\pi}{\beta_0}\left[\frac{1}{\alpha_s(t)} - \frac{c}{N_0}\right] = z_0,$$

with $z_0 = -2.338$ the first zero of the Airy Function. At large t:

$$N_0(t) = c\alpha_s(t)\left[1 + z_0\left(\frac{\beta_0^2}{32\pi^2}\frac{k}{c}\right)^{1/3}\alpha_s(t)^{2/3} + ...\right]$$

Search for the "geometric line" $N_0(t_s)\xi - t_s = 0$:

$$t_s(\xi) = \sqrt{4\pi c/\beta_0}\sqrt{\xi} + O\left(\xi^{1/6}\right)$$

R.c. geometric scaling with $\lambda = \sqrt{4\pi c/\beta_0}$

DGLAP evolution: subleading contributions

Low x – moderate Q^2 data for F_2 rescaled by the theoretical DGLAP prediction