Two-loop calculations with massive quarks

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- **Soft and collinear corrections**
- **Resummation and NNLO expansions**
- **Top pair production at the Tevatron and the LHC**
- **Two-loop eikonal calculations**

Soft and collinear gluon corrections

Incomplete cancellations of infrared divergences between virtual diagrams and real diagrams with soft (low-energy) gluons

For the process $p_1 + p_2 \to p_3 + p_4$

define $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_2 - p_3)^2$ and $s_4 = s + t + u - m_3^2 - m_4^2$, $z = Q^2/s$

At threshold $s_4 \to 0$ (1PI) or $z \to 1$ (PIM)

Soft corrections $\left[\frac{\ln^k(s_4/M^2)}{s_4}\right]_+$ or $\left[\frac{\ln^k(1-z)}{1-z}\right]_+$

$k \le 2n - 1$ for the $\mathcal{O}(\alpha_s^n)$ corrections

Resum (exponentiate) these soft corrections

At NLL accuracy requires one-loop calculations in the eikonal approximation

Also purely collinear corrections $\ln^k(s_4/M^2)$ or $\ln^k(1-z)$

Soft and collinear gluon corrections

The n-th order corrections in the partonic cross section

$$\hat{\sigma}^{(n)}(z) = V^{(n)}\,\delta(1-z) + \sum_{k=0}^{2n-1} S_k^{(n)} \left[\frac{\ln^k(1-z)}{1-z}\right]_+ + \sum_{k=0}^{2n-1} C_k^{(n)} \ln^k(1-z)$$

Near threshold soft corrections are dominant and provide excellent approximations to the full cross section

Examples: top pair and single top production

jet, direct photon, or W production at high p_T

In other cases purely collinear corrections also have to be included to get a good approximation (e.g. Higgs production).

The hadronic cross section

$$\sigma = \sum_f \int dx_1 dx_2\, \phi_{f_1/p}(x_1,\mu_F)\, \phi_{f_2/\bar{p}}(x_2,\mu_F)\, \hat{\sigma}(s,t,u,\mu_F,\mu_R,\alpha_s)$$

Resummed cross section

Resummation follows from factorization properties of the cross section

- performed in moment space

$$\hat{\sigma}^{res}(N) = \exp\left[\sum_i E^{f_i}(N_i)\right] \exp\left[\sum_j E'^{f_j}(N_j)\right] \exp\left[\sum_i 2\int_{\mu_F}^{\sqrt{s}} \frac{d\mu}{\mu}\, \gamma_{i/i}(N_i, \alpha_s(\mu))\right]$$

$$\times \exp\left[\sum_i 2 d_{\alpha_s} \int_{\mu_R}^{\sqrt{s}} \frac{d\mu}{\mu}\, \beta(\alpha_s(\mu))\right] H^{f_i f_j}(\alpha_s(\mu_R))$$

$$\times \exp\left[\int_{\sqrt{s}}^{\sqrt{s}/\tilde{N}_j} \frac{d\mu}{\mu}\, \Gamma_S^{\dagger\, f_i f_j}(\alpha_s(\mu))\right] \tilde{S}^{f_i f_j}\left(\alpha_s\left(\frac{\sqrt{s}}{\tilde{N}_j}\right)\right) \exp\left[\int_{\sqrt{s}}^{\sqrt{s}/\tilde{N}_j} \frac{d\mu}{\mu}\, \Gamma_S^{f_i f_j}(\alpha_s(\mu))\right]$$

where

$$\sum_i E^{f_i}(N_i) = -\sum_i C_i \int_0^1 dz \frac{z^{N_i-1}-1}{1-z} \left\{\int_{(1-z)^2}^1 \frac{d\lambda}{\lambda} \frac{\alpha_s(\lambda s)}{\pi} + \frac{\alpha_s((1-z)^2 s)}{\pi}\right\} + \mathcal{O}(\alpha_s^2)$$

$$\sum_j E'^{f_j}(N_j) = \sum_j \int_0^1 dz \frac{z^{N_j-1}-1}{1-z} \left\{C_j \int_{(1-z)^2}^{1-z} \frac{d\lambda}{\lambda} \frac{\alpha_s(\lambda s)}{\pi} - B_j^{(1)} \frac{\alpha_s((1-z)s)}{\pi} - C_j \frac{\alpha_s((1-z)^2 s)}{\pi}\right\} + \mathcal{O}(\alpha_s^2)$$

$C_i = C_F = (N_c^2-1)/(2N_c)$, $B_q^{(1)} = 3C_F/4$ **for quarks;** $C_i = C_A = N_c$, $B_g^{(1)} = \beta_0/4$ **for gluons**

Γ_S **is the soft anomalous dimension - a matrix in color space**

NNLO expansions of resummed cross section

Invert back to momentum space and expand to arbitrary order

NLO soft and collinear gluon corrections

$$\hat{\sigma}^{(1)} = F^B \frac{\alpha_s(\mu_R^2)}{\pi} \left\{ c_3 \left[\frac{\ln(1-z)}{1-z}\right]_+ + c_2 \left[\frac{1}{1-z}\right]_+ + c_1^{\mu}\, \delta(1-z) + c_3^c \ln(1-z) + c_2^c \right\}$$

NNLO soft and collinear gluon corrections

$$\hat{\sigma}^{(2)} = F^B \frac{\alpha_s^2(\mu_R^2)}{\pi^2} \left\{ \frac{1}{2} c_3^2 \left[\frac{\ln^3(1-z)}{1-z}\right]_+ + \cdots + \frac{1}{2} c_3 c_3^c \ln^3(1-z) + \cdots \right\}$$

Top quark production

Dominant process is pair production $q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$

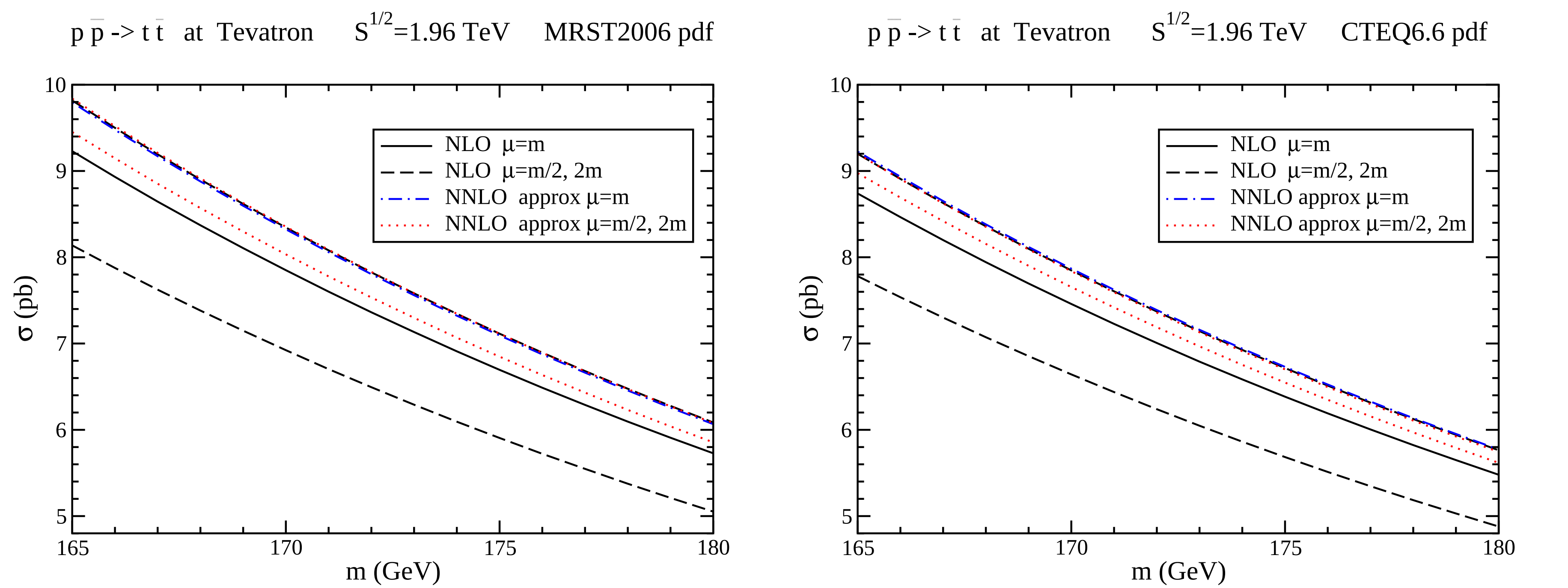
Very good agreement of theory (with soft-gluon corrections) with Tevatron data

Recent evidence for single top production - cross section consistent with theory

Opportunities for study of electroweak properties of the top

Top quark mass value lowered to $\sim$ 172 GeV

Top quark cross section at the Tevatron

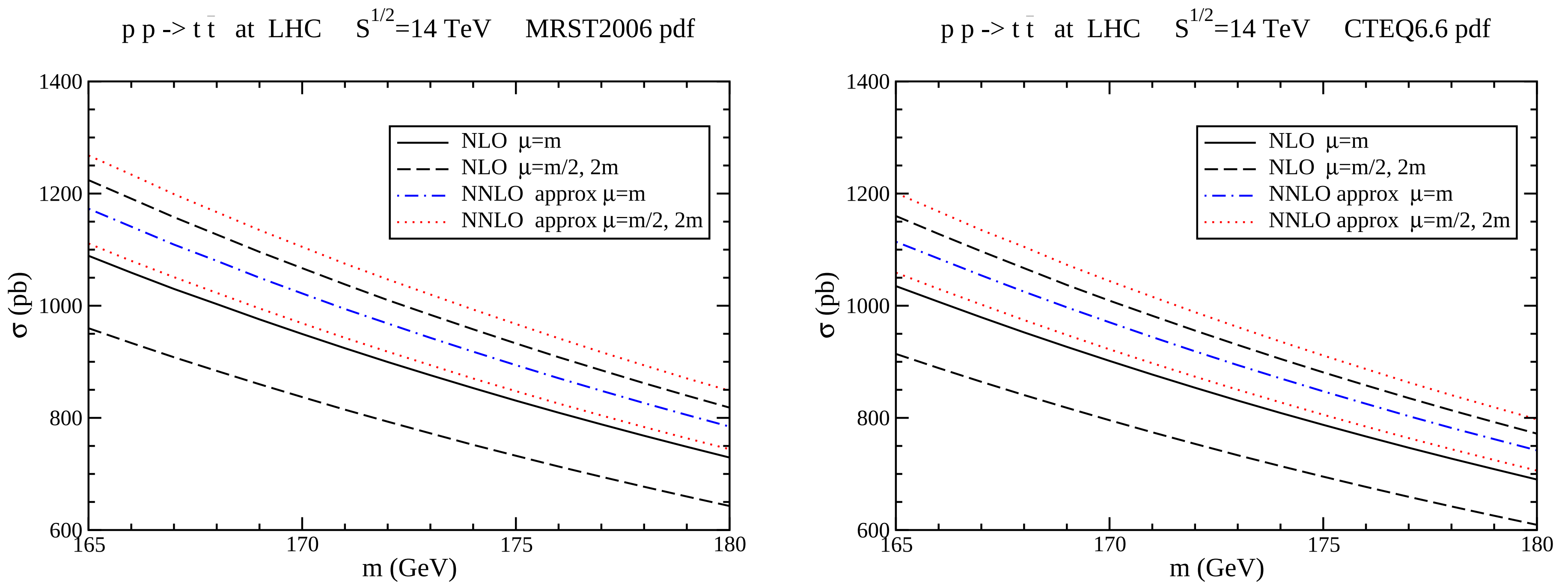


$$\sigma^{\mathrm{NNLOapprox}}_{p\bar{p}\to t\bar{t}}(1.96\,\mathrm{TeV}, m=172\,\mathrm{GeV}, \mathrm{MRST}) = 7.80 \pm 0.31\,{}^{+0.03}_{-0.27}\,{}^{+0.23}_{-0.19}\,\mathrm{pb} = 7.80\,{}^{+0.39}_{-0.45}\,\mathrm{pb}$$

$$\sigma^{\mathrm{NNLOapprox}}_{p\bar{p}\to t\bar{t}}(1.96\,\mathrm{TeV}, m=172\,\mathrm{GeV}, \mathrm{CTEQ}) = 7.39 \pm 0.30\,{}^{-0.03}_{-0.20}\,{}^{+0.48}_{-0.37}\,\mathrm{pb} = 7.39\,{}^{+0.57}_{-0.52}\,\mathrm{pb}$$

Kinematics uncertainty, scale variation, pdf errors **(NK, R. Vogt)**

Top quark cross section at the LHC

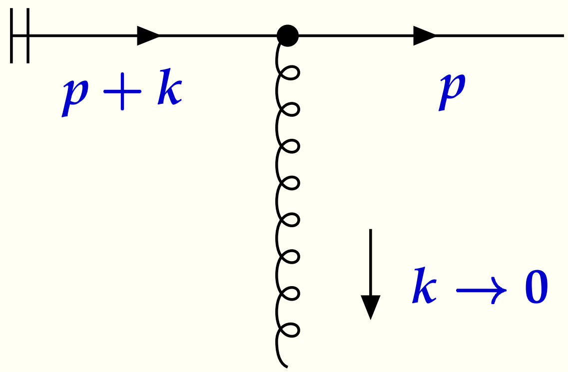


$$\sigma^{\rm NNLOapprox}_{pp\to t\bar{t}}(14\,{\rm TeV}, m=172\,{\rm GeV}, {\rm MRST}) = 968 \pm 4\,{}^{+79}_{-50}\,{}^{+12}_{-13}\,{\rm pb} = 968\,{}^{+80}_{-52}\,{\rm pb}$$

$$\sigma^{\rm NNLOapprox}_{pp\to t\bar{t}}(14\,{\rm TeV}, m=172\,{\rm GeV}, {\rm CTEQ}) = 919 \pm 4\,{}^{+70}_{-45}\,{}^{+29}_{-31}\,{\rm pb} = 919\,{}^{+76}_{-55}\,{\rm pb}$$

Kinematics uncertainty, scale variation, pdf errors **(NK, R. Vogt)**

Eikonal approximation

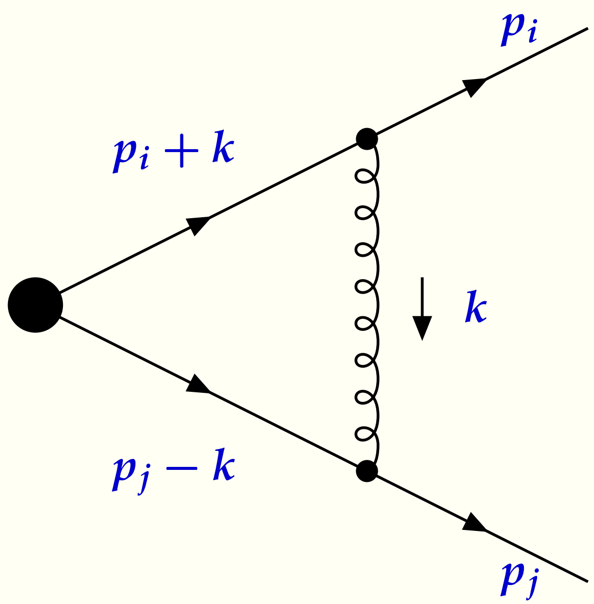


$$\bar{u}(p)\,(-ig_s T_F^c)\,\gamma^\mu \frac{i(\not p + \not k + m)}{(p+k)^2 - m^2 + i\epsilon} \to \bar{u}(p)\, g_s T_F^c\, \gamma^\mu \frac{\not p + m}{2p\cdot k + i\epsilon} = \bar{u}(p)\, g_s T_F^c \frac{v^\mu}{v\cdot k + i\epsilon}$$

with $p \propto v$, T_F^c generators of SU(3)

Perform calculation for massive quarks in Feynman gauge **(NK, P. Stephens)**

One-loop diagram



$$I_{1l} = g_s^2 \int \frac{d^n k}{(2\pi)^n} \frac{(-i) g^{\mu\nu}}{k^2} \frac{v_i^\mu}{v_i \cdot k} \frac{(-v_j^\nu)}{(-v_j \cdot k)}$$

Using Feynman parametrization, this can be rewritten as

$$I_{1l} = -2 i g_s^2 \frac{v_i \cdot v_j}{(2\pi)^n} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^n k}{\left[x k^2 + y v_i \cdot k + (1-x-y) v_j \cdot k\right]^3}$$

which, after the integration over k, gives

$$I_{1l} = g_s^2\, v_i \cdot v_j\, 2^{6-2n}\, \pi^{-n/2}\, \Gamma\left(3-\frac{n}{2}\right) \int_0^1 dx\, x^{3-n} \times \int_0^{1-x} dy \left[-y^2 v_i^2 - (1-x-y)^2 v_j^2 - 2y\, v_i \cdot v_j (1-x-y)\right]^{n/2-3}$$

After several manipulations, and with $n = 4 - \epsilon$ and $\beta = \sqrt{1-4m^2/s}$,

$$I_{1l} = \frac{\alpha_s}{\pi} (-1)^{-1-\epsilon/2}\, 2^{5\epsilon/2}\, \pi^{\epsilon/2}\, \Gamma\left(1+\frac{\epsilon}{2}\right) (1+\beta^2) \int_0^1 dx\, x^{-1+\epsilon} (1-x)^{-1-\epsilon} \times \left\{ \int_0^1 dz \left[4z\beta^2(1-z)+1-\beta^2\right]^{-1} - \frac{\epsilon}{2} \int_0^1 dz \frac{\ln\left[4z\beta^2(1-z)+1-\beta^2\right]}{4z\beta^2(1-z)+1-\beta^2} + \mathcal{O}\left(\epsilon^2\right) \right\}$$

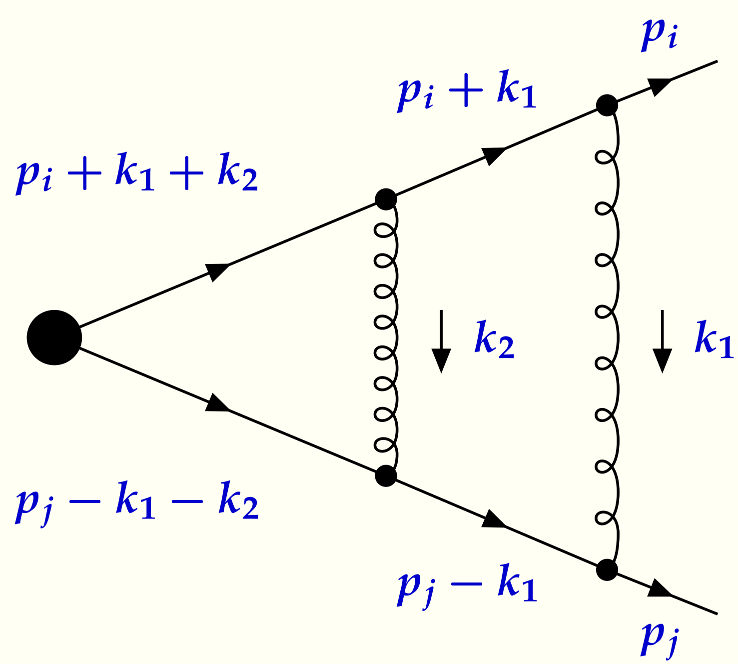
integral over x contains both UV and IR singularities - isolate UV singularities

$$\int_0^1 dx\, x^{-1+\epsilon} (1-x)^{-1-\epsilon} = \frac{1}{\epsilon} + \mathrm{IR}$$

Then

$$I_{1l}^{UV} = \frac{\alpha_s}{\pi} \frac{(1+\beta^2)}{2\beta} \left\{ \frac{1}{\epsilon} \ln\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{2} (4\ln 2 + \ln \pi - \gamma_E - i\pi) \ln\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{4}\ln^2(1+\beta) - \frac{1}{4}\ln^2(1-\beta) - \frac{1}{2}\mathrm{Li}_2\left(\frac{1+\beta}{2}\right) + \frac{1}{2}\mathrm{Li}_2\left(\frac{1-\beta}{2}\right) \right\}$$

Two-loop diagrams



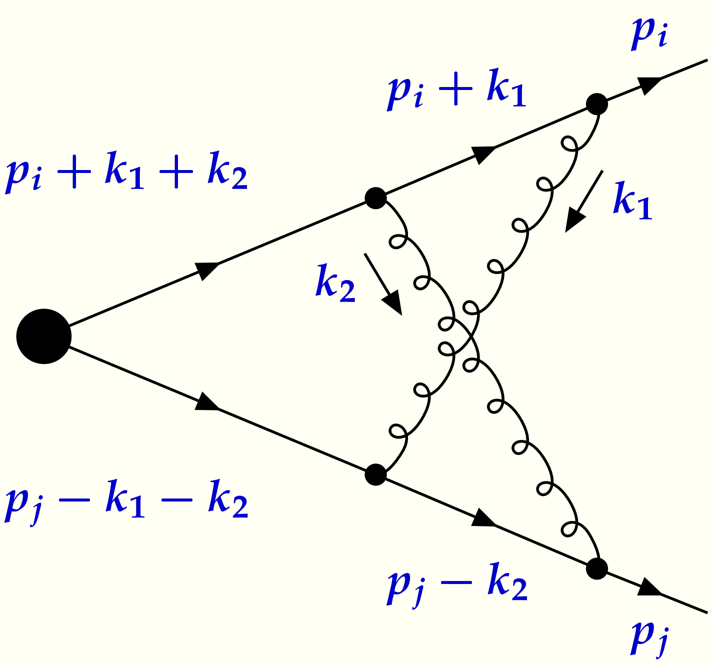
$$I_1 = g_s^4 \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(-i) g^{\mu\nu}}{k_1^2} \frac{(-i) g^{\rho\sigma}}{k_2^2} \frac{v_i^\mu}{v_i \cdot k_1} \frac{v_i^\rho}{v_i \cdot (k_1 + k_2)} \frac{(-v_j^\nu)}{-v_j \cdot k_1} \frac{(-v_j^\sigma)}{-v_j \cdot (k_1 + k_2)}$$

I_1 is symmetric under $k_1 \leftrightarrow k_2$ as is crossed diagram I_2 in the next section. Then

$$I_1 = \frac{1}{2} (I_{1l})^2 - I_2$$

(color factors must also be included in final result)

Two-loop diagrams



$$I_2 = g_s^4 \int \frac{d^n k_1}{(2\pi)^n} \frac{d^n k_2}{(2\pi)^n} \frac{(-i) g^{\mu\nu}}{k_1^2} \frac{(-i) g^{\rho\sigma}}{k_2^2} \frac{v_i^\mu}{v_i \cdot k_1} \frac{v_i^\rho}{v_i \cdot (k_1 + k_2)} \frac{(-v_j^\nu)}{-v_j \cdot (k_1 + k_2)} \frac{(-v_j^\sigma)}{-v_j \cdot k_2}$$

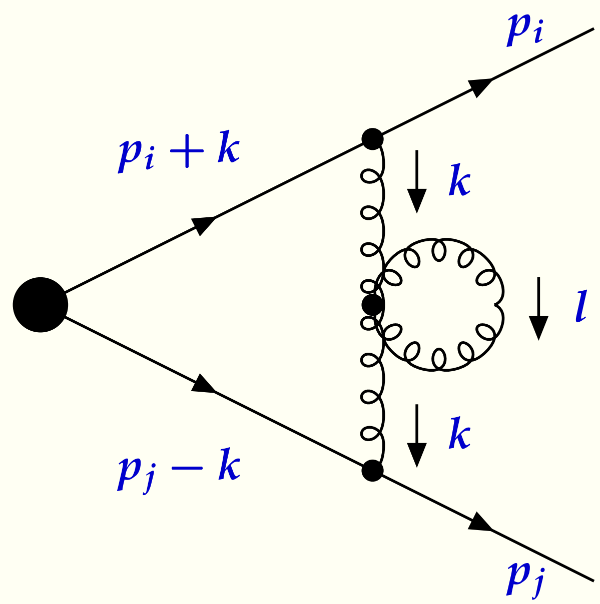
We can set $\epsilon = 0$ in the k_2 integral since it is UV finite. Then

$$I_2 = -i\frac{\alpha_s^2}{\pi^2}2^{-4}\pi^{-2}(1+\beta^2)^2\int_0^1 dz\int_0^1 \frac{dy}{2\beta^2(1-y)^2z^2-2\beta^2(1-y)z-\frac{(1-\beta^2)}{2}}$$
$$\times\int\frac{d^nk_1}{k_1^2\, v_i\cdot k_1\,[(v_i-v_j)z+v_j]\cdot k_1}$$

Now we proceed with the k_1 integral. Isolate UV and IR poles.

After many steps

$$I_2^{UV} = -\frac{\alpha_s^2}{\pi^2}\frac{(1+\beta^2)^2}{4\beta^2}\frac{1}{\epsilon}\left\{\ln\left(\frac{1-\beta}{1+\beta}\right)\left[2\,\mathrm{Li}_2\left(\frac{2\beta}{1+\beta}\right)+4\,\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)+2\,\mathrm{Li}_2\left(\frac{-(1-\beta)}{1+\beta}\right)\right.\right.$$
$$\left.-\ln(1+\beta)\,\ln(1-\beta)-\zeta_2\right]$$
$$\left.-2\ln^2\left(\frac{1-\beta}{1+\beta}\right)\ln\left(\frac{1+\beta}{2\beta}\right)+\frac{1}{3}\ln^3(1-\beta)-\frac{1}{3}\ln^3(1+\beta)-\mathrm{Li}_3\left(\frac{(1-\beta)^2}{(1+\beta)^2}\right)+\zeta_3\right\}$$

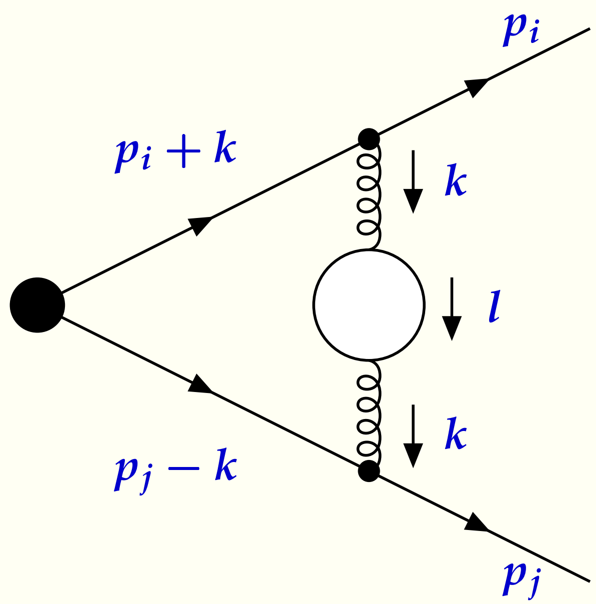


$$I = -i g_s^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \frac{(-i) g^{\mu\mu'}}{k^2} \frac{(-i) g^{\nu\nu'}}{k^2} \frac{(-i) g^{\rho\rho'}}{l^2} \frac{v_i^\mu}{v_i \cdot k} \frac{(-v_j^\nu)}{(-v_j \cdot k)}$$
$$\times \left[g^{\mu'\rho'} g^{\rho\nu'} + g^{\mu'\rho} g^{\rho'\nu'} - 2 g^{\mu'\nu'} g^{\rho\rho'} \right]$$

This can be rewritten as

$$I = \frac{2 g_s^4 (n-1)}{(2\pi)^{2n}} v_i \cdot v_j \int \frac{d^n k}{k^4 \, v_i \cdot k \, v_j \cdot k} \int \frac{d^n l}{l^2}$$

But $\int d^n l / l^2 = 0$ and thus $I_4 = 0$.



$$I = (-1) n_f g_s^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \frac{v_i^\mu}{v_i \cdot k} \frac{(-v_j^\rho)}{(-v_j \cdot k)} \frac{(-i) g^{\mu\nu}}{k^2} \frac{(-i) g^{\rho\sigma}}{k^2} \mathrm{Tr}\left[-i\gamma^\nu \frac{i\not{l}}{l^2} (-i)\gamma^\sigma i \frac{(\not{l} - \not{k})}{(l-k)^2}\right]$$

After a few manipulations involving the trace we can write this integral as

$$I = -n_f \frac{n g_s^4}{(2\pi)^{2n}} \left[I^a + I^b + I^c + I^d + I^e\right]$$

where

$$I^a = v_i \cdot v_j \int \frac{d^n k}{v_i \cdot k \; v_j \cdot k \; k^4} \int \frac{d^n l}{(l-k)^2} \qquad I^b = -v_i \cdot v_j \int \frac{d^n k}{v_i \cdot k \; v_j \cdot k \; k^4} \int d^n l \frac{l \cdot k}{l^2 (l-k)^2}$$

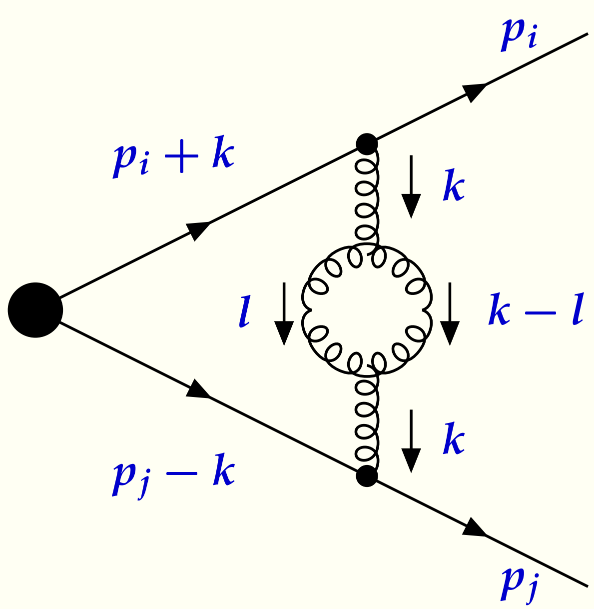
$$I^c = -2 \int \frac{d^n k}{v_i \cdot k \; v_j \cdot k \; k^4} \int d^n l \frac{v_i \cdot l \; v_j \cdot l}{l^2 (l-k)^2} \qquad I^d = \int \frac{d^n k}{v_i \cdot k \; k^4} \int d^n l \frac{v_i \cdot l}{l^2 (l-k)^2}$$

$$I^e = \int \frac{d^n k}{v_j \cdot k \; k^4} \int d^n l \frac{v_j \cdot l}{l^2 (l-k)^2}$$

Now $I^a = 0$, $I^d = 0$, $I^e = 0$.

After many steps extract UV poles

$$\begin{aligned} I^{UV} \quad = \quad & -n_f \frac{\alpha_s^2}{\pi^2} \frac{(1+\beta^2)}{6\beta} \left\{ \frac{1}{\epsilon^2} \ln\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{\epsilon} \left[\mathrm{Li}_2\left(\frac{2}{1+\beta}\right) - \mathrm{Li}_2\left(\frac{2}{1-\beta}\right) + \ln^2(1+\beta) \right.\right. \\ & \left.\left. - \ln^2(1-\beta) + \left(\frac{7}{12} + 5\ln 2 + \ln\pi - \gamma_E\right) \ln\left(\frac{1-\beta}{1+\beta}\right) \right] \right\} \end{aligned}$$



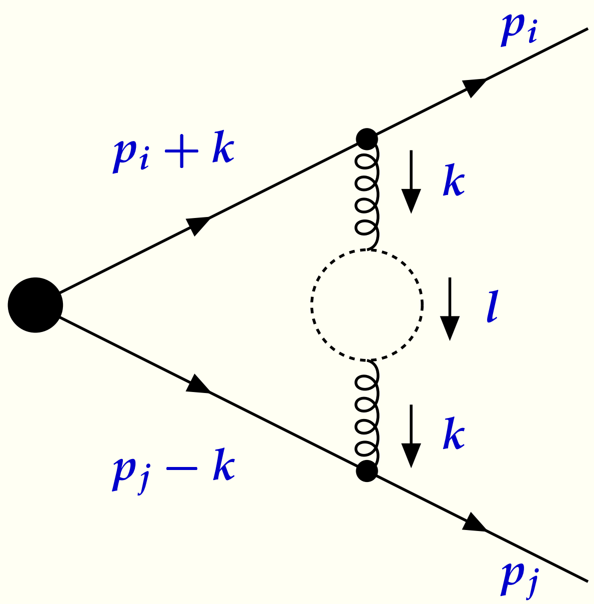
$$
\begin{aligned}
I \quad = \quad & \frac{1}{2} g_s^4 \int \frac{d^n k}{(2\pi)^n} \frac{d^n l}{(2\pi)^n} \frac{v_i^\mu}{v_i \cdot k} \frac{(-v_j^\nu)}{(-v_j \cdot k)} \frac{(-i) g^{\mu\mu'}}{k^2} \frac{(-i) g^{\rho\rho'}}{l^2} \frac{(-i) g^{\sigma\sigma'}}{(k-l)^2} \frac{(-i) g^{\nu\nu'}}{k^2} \\
& \times \left[g^{\mu'\rho} (k+l)^\sigma + g^{\rho\sigma} (k-2l)^{\mu'} + g^{\sigma\mu'} (-2k+l)^\rho \right] \\
& \times \left[g^{\rho'\nu'} (l+k)^{\sigma'} + g^{\nu'\sigma'} (-2k+l)^{\rho'} + g^{\sigma'\rho'} (k-2l)^{\nu'} \right]
\end{aligned}
$$

After several manipulations we can write this integral as

$$I = \frac{1}{2} g_s^4 \int \frac{d^n k}{(2\pi)^{2n}} \left\{ \left[\frac{4\, v_i \cdot v_j}{v_i \cdot k\, v_j \cdot k\, k^2} + \frac{(n-6)}{k^4} \right] \int \frac{d^n l}{l^2 (k-l)^2} \right.$$
$$\left. - \frac{(2n-3)}{k^4} \left(\frac{v_i^\mu}{v_i \cdot k} + \frac{v_j^\mu}{v_j \cdot k} \right) \int d^n l \frac{l_\mu}{l^2 (k-l)^2} + \frac{(4n-6)}{v_i \cdot k\, v_j \cdot k} \frac{v_i^\mu v_j^\nu}{k^4} \int d^n l \frac{l_\mu l_\nu}{l^2 (k-l)^2} \right\}$$

Calculate UV poles

$$I^{UV} = -\frac{19}{96} \frac{\alpha_s^2}{\pi^2} \frac{(1+\beta^2)}{\beta} \left\{ \frac{1}{\epsilon^2} \ln\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{\epsilon} \left[\mathrm{Li}_2\left(\frac{2}{1+\beta}\right) - \mathrm{Li}_2\left(\frac{2}{1-\beta}\right) + \ln^2(1+\beta) \right. \right.$$
$$\left. \left. - \ln^2(1-\beta) + \left(\frac{58}{57} + 5\ln 2 + \ln\pi - \gamma_E \right) \ln\left(\frac{1-\beta}{1+\beta}\right) \right] \right\}$$



$$I=(-1)g_s^4\int\frac{d^nk}{(2\pi)^n}\frac{d^nl}{(2\pi)^n}\frac{v_i^\mu}{v_i\cdot k}\frac{(-v_j^\rho)}{(-v_j\cdot k)}\frac{i}{l^2}l^\nu\frac{i}{(l-k)^2}(l-k)^\sigma\frac{(-i)g^{\mu\nu}}{k^2}\frac{(-i)g^{\rho\sigma}}{k^2}$$

Then

$$I^{UV} = -\frac{\alpha_s^2}{\pi^2}\frac{(1+\beta^2)}{96\beta}\left\{\frac{1}{\epsilon^2}\ln\left(\frac{1-\beta}{1+\beta}\right)+\frac{1}{\epsilon}\left[\mathrm{Li}_2\left(\frac{2}{1+\beta}\right)-\mathrm{Li}_2\left(\frac{2}{1-\beta}\right)+\ln^2(1+\beta)\right.\right.$$
$$\left.\left.-\ln^2(1-\beta)+\left(\frac{4}{3}+5\ln 2+\ln\pi-\gamma_E\right)\ln\left(\frac{1-\beta}{1+\beta}\right)\right]\right\}$$

Summary

- **Soft and collinear corrections important in cross sections**
- **Resummation and NNLO expansions**
- **Top pair cross section at the Tevatron and the LHC**
- **Two-loop calculations in eikonal approximation**
- **Massive quarks involve further complications**
- **UV poles calculated for several diagrams**
- **Include self-energies, counterterms, color factors**
- **Work is ongoing ...**