Four-dimensional regularization (FDR)

Roberto Pittau

CERN and University of Granada

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Outline

1. Intro and motivations
2. **UV** divergences
3. **IR** infinities

Dealing with infinities is **costly:**

- Regularization
- Renormalization
- Counterterms
- Subtractions . . .

A great deal of intermediate steps!

The aim of **FDR** is to provide a shortcut to the physical answer, avoiding the avoidable steps of the calculation

- R.P., **arXiv:1208.5457** (first paper)
- A.M. Donati and R.P., **arXiv:1302.5668**
(1-loop **EW** in $H \to \gamma\gamma$)
- R.P., **arXiv:1307.0705**
(1-loop massless **QCD** in $H \to gg$)
- A.M. Donati and R.P., **arXiv:1311.5500**
(2-loop **QCD** in $H \to \gamma\gamma$)
- R.P., **arXiv:1408.5345**
(integration-by-parts identities)

UV

Imagine to **define** a **four-dimensional** multi-loop integration

$$\int [d^4 q_1] \ldots [d^4 q_\ell] \, J(q_1, \ldots, q_\ell)$$

- **coinciding** with normal integration for UV finite integrals
- always **UV finite**
- **independent of any UV cutoff**, but dependent on the (arbitrary) renormalization scale μ_R

IF such a definition fulfills

i) **Invariance under shift of any integration variable:**

$$\int [d^4q_1]\ldots[d^4q_\ell]\,J(q_1,\ldots,q_\ell) = \int [d^4q_1]\ldots[d^4q_\ell]\,J(q_1+p_1,\ldots,q_\ell+p_\ell)$$

ii) **Simplifications among numerators and denominators:**

$$\int [d^4q_1]\ldots[d^4q_\ell]\,\frac{\bar{q}_i^2 - m_i^2}{(\bar{q}_i^2 - m_i^2)^m \ldots} = \int [d^4q_1]\ldots[d^4q_\ell]\,\frac{1}{(\bar{q}_i^2 - m_i^2)^{m-1} \ldots}$$

$$(\bar{q}^2 = q^2 + i0 \equiv q^2 - \mu^2)$$

then Gauge Invariance (and other symmetries of the **QFT**) are guaranteed **because**

- usual manipulations hold at the integrand level
- any graphical proof of Ward-Identities holds

IF such a definition exists **UNLIKE*** other four-dimensional approaches (*Pauli-Villars, Implicit Regularization, Dimensional Reduction, FDH...*)

Regularization $\Leftrightarrow$ Renormalization

- **QFT** renormalized by re-interpreting loop integrals
- μ_R drops after fixing free parameters in $\mathcal{L}$ in terms of observables (**finite renormalization**)

* Due to the absence of UV cutoff

Such a definition **exists**

$$\int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} \equiv \lim_{\mu \to 0} \int_{\mathrm{R}} d^4q \left(\frac{1}{(\bar{q}^2 - M^2)^2} - \left[\frac{1}{\bar{q}^4} \right] \right) \Bigg|_{\mu = \mu_R}$$

$$= \lim_{\mu \to 0} \int d^4q \left(\frac{M^2}{\bar{q}^4(\bar{q}^2 - M^2)} + \frac{M^2}{\bar{q}^2(\bar{q}^2 - M^2)^2} \right) \Bigg|_{\mu = \mu_R}$$

- Induced IR divergence **provisionally** regulated by

$$\bar{q}^2 = q^2 + i0 \equiv q^2 - \mu^2$$

- Dependence on UV regulator R **canceled** by **Partial Fractioning** (**FDR defining expansion**)

$$\frac{1}{\bar{q}^2 - M^2} = \frac{1}{\bar{q}^2} + \frac{M^2}{\bar{q}^2(\bar{q}^2 - M^2)}$$

- $\mu \to \mu_R$ possible at the **integrand** level (e.g. for numerical evaluations)

$$\int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2}$$
$$\equiv \int_{\mathrm{R}} d^4q \left(\frac{1}{(q^2 - M^2)^2} - \left[\frac{1}{(q^2 - \mu_R^2)^2} \right] \right)$$

UV **finite** four-dimensional integral

Renormalization scale

- **In general:** take the **integrand** of an ℓ-loop function

$$J(q_1,\ldots,q_\ell) = [J_{\rm INF}(q_1,\ldots,q_\ell)] + J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$$

- The divergent loop **integrands** in $[J_{\rm INF}(q_1,\ldots,q_\ell)]$ allowed to depend on μ, **but not on physical scales**

$$\Rightarrow \textbf{physics in } J_{{\rm F},\ell}(q_1,\ldots,q_\ell)$$

- The FDR integral over $J(q_1,\ldots,q_\ell)$ is **defined** as

$$\int [d^4q_1]\ldots[d^4q_\ell]\, J(q_1,\ldots,q_\ell) \equiv \lim_{\mu\to 0} \int d^4q_1 \ldots d^4q_\ell\, J_{{\rm F},\ell}(q_1,\ldots q_\ell)$$

A two-loop FDR defining expansion with $\bar{D}_i = \bar{q}_i^2 - m_i^2$

$$\frac{1}{\bar{D}_1\bar{D}_2\bar{D}_{12}} = \left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_{12}^2}\right] \qquad (q_{12} = q_1 + q_2)$$

$$+ \quad m_1^2\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_{12}^2}\right] + \frac{m_1^4}{(\bar{D}_1\bar{q}_1^4)}\left[\frac{1}{\bar{q}_2^4}\right] - m_1^4\frac{q_1^2 + 2(q_1\cdot q_2)}{(\bar{D}_1\bar{q}_1^4)\bar{q}_2^4\bar{q}_{12}^2}$$

$$+ \quad m_2^2\left[\frac{1}{\bar{q}_1^2\bar{q}_2^4\bar{q}_{12}^2}\right] + \frac{m_2^4}{(\bar{D}_2\bar{q}_2^4)}\left[\frac{1}{\bar{q}_1^4}\right] - m_2^4\frac{q_2^2 + 2(q_1\cdot q_2)}{\bar{q}_1^4(\bar{D}_2\bar{q}_2^4)\bar{q}_{12}^2}$$

$$+ \quad m_{12}^2\left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_{12}^4}\right] + \frac{m_{12}^4}{(\bar{D}_{12}\bar{q}_{12}^4)}\left[\frac{1}{\bar{q}_1^4}\right] - m_{12}^4\frac{q_{12}^2 - 2(q_1\cdot q_{12})}{\bar{q}_1^4\bar{q}_2^2(\bar{D}_{12}\bar{q}_{12}^4)}$$

$$+ \quad \frac{m_1^2m_2^2}{(\bar{D}_1\bar{q}_1^2)(\bar{D}_2\bar{q}_2^2)\bar{q}_{12}^2} + \frac{m_1^2m_{12}^2}{(\bar{D}_1\bar{q}_1^2)\bar{q}_2^2(\bar{D}_{12}\bar{q}_{12}^2)} + \frac{m_2^2m_{12}^2}{\bar{q}_1^2(\bar{D}_2\bar{q}_2^2)(\bar{D}_{12}\bar{q}_{12}^2)}$$

$$+ \quad \frac{m_1^2m_2^2m_{12}^2}{(\bar{D}_1\bar{q}_1^2)(\bar{D}_2\bar{q}_2^2)(\bar{D}_{12}\bar{q}_{12}^2)}$$

"Sub-vacua" must be canceled by one-loop counterterms!

Proof of shift-invariance and num/den simplifications

i)

FDR integrals as finite differences of **shift invariant** UV divergent integrals

$$\int [d^4q_1]\dots[d^4q_\ell]\, J(\{q_i\})$$
$$= \lim_{\mu\to 0} \mu_R^{-\ell\epsilon} \int d^nq_1 \dots d^nq_\ell \left(J(\{q_i\}) - [J_{\rm INF}(\{q_i\})] \right)$$

r.h.s. regulated in DR *(but any regulator* R *would give same result)* ii)

Provided any q_i^2 appearing in the numerator from Feynman rules is also shifted $q_i^2 \to \bar{q}_i^2$ *(Global Prescription)*. For instance

$$\int [d^4q] \frac{\bar{q}^2 - M^2}{(\bar{q}^2 - M^2)^3} = \int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} \qquad (1)$$

Extra integrals containing μ^2 appear, and Eq. (1) holds **only if** the **same** subtraction is performed in front of μ^2 **as if it was** $q^\alpha q^\beta$

$$\int [d^4q] \frac{\mu^2}{(\bar{q}^2 - M^2)^3} \equiv \lim_{\mu \to 0} \int_R d^4q \left(\frac{\mu^2}{(\bar{q}^2 - M^2)^3} - \left[\frac{\mu^2}{\bar{q}^6} \right] \right) = \frac{i\pi^2}{2}$$

Manipulating FDR integrals

Need to reduce the problem to simpler **Master Integrals** (MIs)

- **Tensor reduction** ($\bar{D}_0 = \bar{q}^2 - m_0^2$) compatible with **consistency conditions**

$$\int [d^4q] \frac{q^\alpha q^\beta}{\bar{D}_0^3} = \frac{g^{\alpha\beta}}{4} \int [d^4q] \frac{q^2}{\bar{D}_0^3} = \frac{g^{\alpha\beta}}{4} \int [d^4q] \frac{1}{\bar{D}_0^2}$$

- **Integration-by-parts identities** ($\bar{D}_1 = (q+p)^2 - m_1^2 - \mu^2$)

$$0 = \int [d^4q] \frac{\partial}{\partial q^\alpha} \frac{q^\alpha}{\bar{D}_0 \bar{D}_1} = \int [d^4q] \left\{ \frac{4}{\bar{D}_0 \bar{D}_1} - 2\frac{q^2}{\bar{D}_0^2 \bar{D}_1} - 2\frac{q^2 + (q \cdot p)}{\bar{D}_0 \bar{D}_1^2} \right\}$$

In both cases $q^2 = \bar{D}_0 + m_0^2 + \mu^2$ needed to simplify with $\bar{D}_0$ and

$$\int [d^4q] \frac{\mu^2}{\bar{D}_0^3} = \int [d^4q] \frac{\mu^2}{\bar{D}_0^2 \bar{D}_1} = \int [d^4q] \frac{\mu^2}{\bar{D}_0 \bar{D}_1^2} = \frac{i\pi^2}{2}$$

Pause & Recap

1. The **FDR** approach to the **UV** problem **defines** a **four-dimensional and finite loop-integration** in a way compatible with shift and gauge invariance

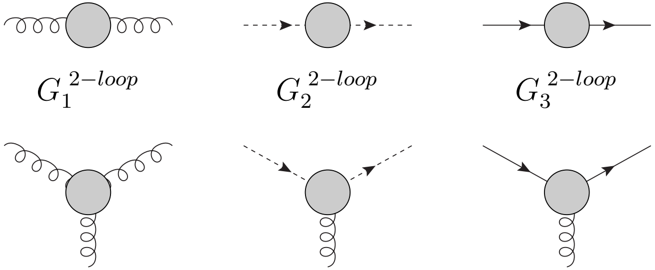
2. This is achieved by

   encoding the UV subtraction
   in the definition of the loop integrals

3. **Algebraic integrand manipulations** are allowed to reduce FDR integrals to MIs **without using their explicit definition**: **No** reference to $[J_{\rm INF}]$ $\Rightarrow$ subtracted integrands **"irrelevant"**

Two-loop off-shell QCD amplitudes in FDR

WORK IN PROGRESS (Page, R.P.):



G_1^{2-loop} G_2^{2-loop} G_3^{2-loop}

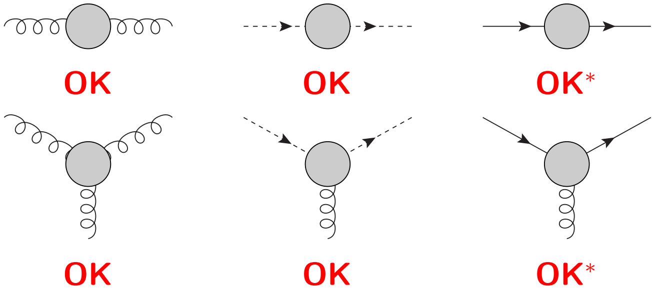
G_4^{2-loop} G_5^{2-loop} G_6^{2-loop}

- The splitting $J(q_1,q_2) = [J_{\rm INF}(q_1,q_2)] + J_{\rm F}(q_1,q_2)$ can be thought **both** as a way of extracting UV infinities in **DR** **and** as the starting point to define the **FDR** integration

- **FDR** **must** be equivalent to a particular **DR** renormalization scheme formulated in terms of **CTs** $(Z_G, Z_{Gh}, Z_\Psi, Z_{\alpha_S}, Z_m)$

- This happens **IF** 2-loop "sub-vacua" are canceled by 1-loop **CTs**

- We find



* require the addition of extra-integrals – readable form the original 2-loop integrand – to maintain **consistency conditions**

IR (virtual)

μ^2 in propagators also regulates IR infinities

- **CL/UV** singularities compensate each other

$$B^{\rm FDR}(p^2=0,0,0) = \int [d^4q] \frac{1}{\bar{q}^2((q+p)^2-\mu^2)} = \mathbf{0}$$

- Due to a **full** cancellation between **CL** and **UV** regulators

$$B^{\rm FDR}(p^2,0,0) = -i\pi^2 \lim_{\mu\to 0}\int_0^1 dx\,\left[\ln(\mu^2 - p^2x(1-x)) - \ln(\mu^2)\right]$$

- **FDR scale-less UV divergent integrals vanish** (as in DR)

- Overlapping **Soft/CL** infinities at one loop

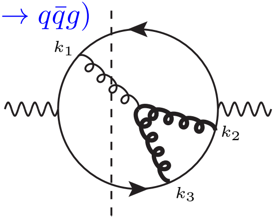
If $\bar{D}_i = (q+p_i)^2 - \mu^2$ with $p_i^2 = 0$ and $s = -2(p_1 \cdot p_2)$

$$C(s) = \int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} = \lim_{\mu \to 0} \int d^4q \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2}$$
$$= \frac{i\pi^2}{s} \left[\frac{\ln^2(\mu^2/s) - \pi^2}{2} + i\,\pi \ln(\mu^2/s) \right]$$

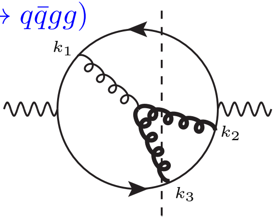
IR (real)

How $\mu^2 \to \mu^2 - i\epsilon$ regulator in virtuals matches the reals?

$V(\gamma^* \to q\bar{q}g)$



$R(\gamma^* \to q\bar{q}gg)$

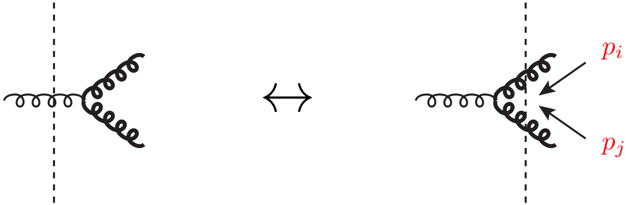


$$V = -\int 2\pi\delta_+(k_1^2)\,\frac{-i}{\bar{\mathbf{k}}_2^2 - i\epsilon}\,\frac{-i}{\bar{\mathbf{k}}_3^2 - i\epsilon}\,f(k_j)\delta^4(k_1+k_2+k_3)\,dk_1^0dk_2^0dk_3^0$$

$$R = \int \frac{i}{k_1^2 + i\epsilon}\,2\pi\delta_-(\bar{\mathbf{k}}_2^2)\,2\pi\delta_-(\bar{\mathbf{k}}_3^2)\,f(k_j)\delta^4(k_1+k_2+k_3)\,dk_1^0dk_2^0dk_3^0$$

One proves that **IR logs** match and cancel in $(V+R)$ via

$$\frac{1}{k_1^2 + i\epsilon} = -2\pi i\delta_+(k_1^2) + \left(\frac{1}{k_1^2 - i\epsilon k_1^0}\right) \quad \text{and} \quad 2\pi i\delta_-(\bar{\mathbf{k}}_{2,3}^2) = \frac{1}{\bar{\mathbf{k}}_{2,3}^2 - i\epsilon} - \left(\frac{1}{\bar{\mathbf{k}}_{2,3}^2 - i\epsilon k_{2,3}^0}\right)$$



$$\frac{1}{q^2 - \mu^2} \leftrightarrow \delta(q^2 - \mu^2)$$

- $p_{i,j}^2 = \mu^2 \to 0$ in Phase-Space boundaries (μ-massive PS)
- The massless (gauge invariant) $|M|^2_{\rm R}(\{\hat{p}_i\})$ has to be integrated over a μ-massive PS. E.g. with **fudge factors**

$$\frac{1}{2(\hat{p}_i \cdot \hat{p}_j)} \to \frac{1}{2(\hat{p}_i \cdot \hat{p}_j)} \left[\frac{2(\hat{p}_i \cdot \hat{p}_j)}{(p_i + p_j)^2}\right] \qquad (\hat{p}_{i,j}^2 = 0)$$

- **Unobserved (Virtual or Real) particles are given a mass μ**

Two general treatments of IR infinities at NLO

1. $\ln \mu^2$ terms extracted **analytically** from virtuals and matched with logs obtained **numerically** from reals

   (as Phase-Space slicing but **no analytic subtraction of $1/\epsilon$ IR poles needed nor approximations in Soft/CL regions**)

WORK IN PROGRESS (Donati, Moretti, Piccinini, R.P.)**:**

2. **Disintegrating** the virtuals by rewriting $\ln \mu^2$s appearing there as **local** counterterms of the reals

   (counterterms automatically generated during the computation of the loop part. Once added to reals $\mu \to 0$ **possible in μ-massive PS**)

- **Soft/CL** counterterms **directly read** from virtuals

$$\int d\Phi_2 \mathcal{R}e \left(\int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} \right) = \int_\mu d\Phi_3 \frac{1}{s_{13} s_{32}}$$

which means to rewrite the virtual $\ln^2 \mu^2$ as

$$\ln^2 \frac{\mu^2}{s_{ijk}} - \pi^2 = \frac{8s}{\pi^2} \int_\mu d\Phi_3 \frac{1}{s_{ij} s_{jk}}$$

In practice one sets $\int [d^4q] 1/(\bar{q}^2 \bar{D}_i \bar{D}_k) = 0$ and compensates by subtracting $1/(s_{ij} s_{jk})$ in the real part

- **CL** s_{ij}^n / s_{ik} counterterms determined by universal collinear behavior encoded in the Altarelli-Parisi splitting functions

The knowledge of the coefficients of the IR divergent 1-loop 3-point functions is enough to construct **all** the local counterterms

- Numerical stability of this approach **under study**

- Extension to initial state radiation **under construction**

The aim is to extend this treatment to NNLO and construct a general fully four-dimensional and local subtraction scheme for IR divergences

Summary and outlook

1. **QFT** renormalized by **defining** a new mathematical object (**FDR** integral) compatible with shift and gauge invariance
2. Results of renormalizable **QFT** reproduced, only **finite** and **global** renormalization needed, $\mathcal{L}$ **untouched**, no order-by-order **UV** counterterms
3. **IR** divergences naturally fit the **FDR** scheme
4. Investigation under way on the numerical stability and speed of the four-dimensional local **FDR** subtraction of the **IR** infinities at one loop
5. Work in progress toward **numerical** applications at two loops fully exploiting the four-dimensionality of **FDR**

Thank you!

Backup slides

- Notice that $J(q_1, \ldots, q_\ell)$ can also be a **tensor**. E. g.

$$\int [d^4q] \frac{q^\alpha q^\beta}{(\bar{q}^2 - M^2)^3}$$

$$\equiv \lim_{\mu \to 0} \int_{\mathrm{R}} d^4q \left(\frac{q^\alpha q^\beta}{(\bar{q}^2 - M^2)^3} - \left[\frac{q^\alpha q^\beta}{\bar{q}^6} \right] \right)$$

$$= \lim_{\mu \to 0} \int d^4q \, q^\alpha q^\beta \left(\frac{M^2}{\bar{q}^6 (\bar{q}^2 - M^2)} + \frac{M^2}{\bar{q}^4 (\bar{q}^2 - M^2)^2} + \frac{M^2}{\bar{q}^2 (\bar{q}^2 - M^2)^3} \right)$$

Dependence on μ of FDR integrals

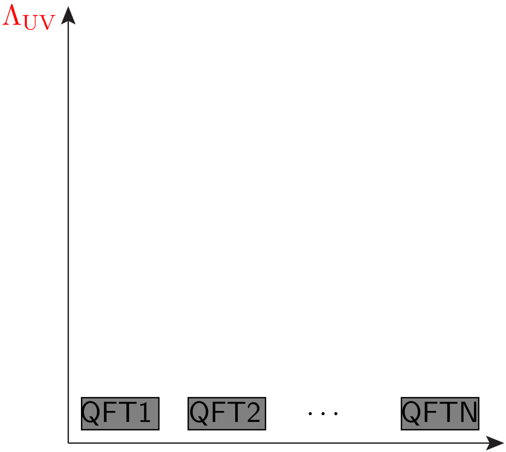
$$\int [d^4q_1]\dots[d^4q_\ell]\,J(\{q_i\}) = \lim_{\mu\to 0}\mu_R^{-\ell\epsilon}\int d^nq_1\dots d^nq_\ell\left(J(\{q_i\}) - [J_{\rm INF}(\{q_i\})]\right)$$

1. First term in r.h.s. independent of μ ($\mu \to 0$ in *integrand*)
2. Polynomially divergent integrals in $[J_{\rm INF}]$ **do not contribute** (*proportional to positive powers of* μ)
3. $\ln(\mu/\mu_R)$s **generated by log divergent integrals in** $[J_{\rm INF}]$:
   - FDR integrals depend on μ *logarithmically*
   - By sidestepping the subtraction of the $\ln(\mu/\mu_R)$s, $\lim_{\mu\to 0}$ formally taken by trading $\ln(\mu)$ for $\ln(\mu_R)$

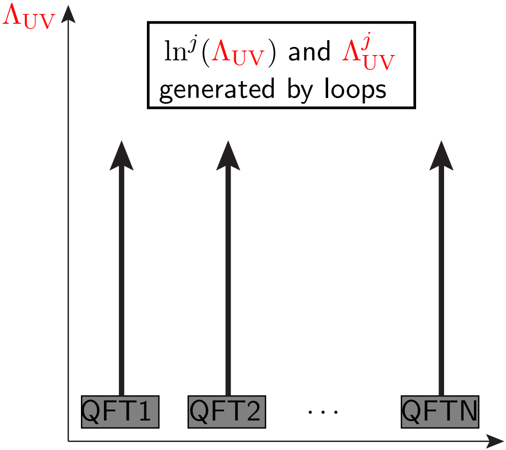
FDR integrals do not depend on any cutoff but only on the renormalization scale μ_R

Renormalization in FDR

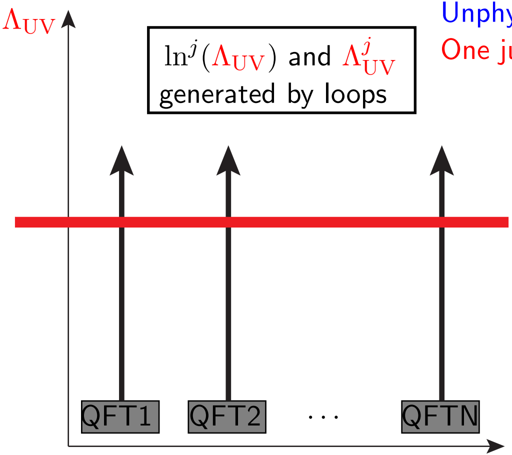
QFTs vs UV cutoff (I)



QFTs vs UV cutoff (II)



QFTs vs UV cutoff (III)



Unphysical if $\Lambda_{\rm UV} \to \infty$

One just **ignores** them

The right question is not "Where do the infinities go?" but

What is the cost of **ignoring** infinities?

- No cost for polynomially divergent infinities (decoupling)
- Only logarithmic infinities influence the physical spectrum ($\ln \mu_R$ pops up in $J_{\mathrm{F},\ell}(q_1, \ldots, q_\ell)$ when separating them)
- Physics at Λ_{UV} scale manifests itself only logarithmically at lower energies

Polynomial divergences are unobservable

Global Finite Renormalization

Consider the Lagrangian of a renormalizable QFT dependent on m parameters p_i $(i = 1 : m)$

$$\mathcal{L}(p_1, \ldots, p_m)$$

Before an observable $\mathcal{O}^{\rm TH}_{m+1}$ can be calculated, p_i must be fixed by means of m measurements

$$\mathcal{O}^{\rm TH}_i(p_1, \ldots, p_m) = \mathcal{O}^{\rm EXP}_i$$

which determine p_i in terms of observables $\mathcal{O}^{\rm EXP}_i$ and corrections computed at the loop level ℓ one is working:

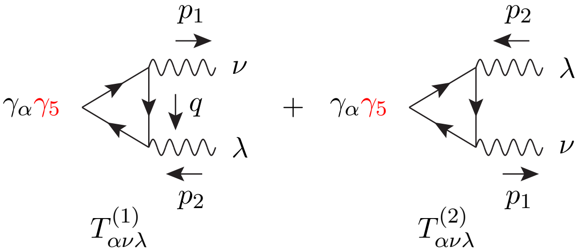
$$p_i = p_i^{\ell-loop}(\mathcal{O}^{\rm EXP}_1, \ldots, \mathcal{O}^{\rm EXP}_m) \equiv \bar{p}_i$$

Then

$$\mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m) \quad \text{with} \quad \frac{\partial\, \mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m)}{\partial \mu_R} = 0$$

is a **prediction** of the QFT

A 1-loop warm-up: The ABJ anomaly

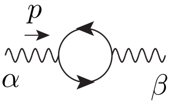


$$p^\alpha T_{\alpha\nu\lambda} = -i\frac{e^2}{4\pi^4}\mathrm{Tr}[\gamma_5 \not{p}_2\gamma_\lambda\gamma_\nu\not{p}_1]\int [d^4q]\,\mu^2\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}$$

$$p^\alpha T_{\alpha\nu\lambda} = \frac{e^2}{8\pi^2}\mathrm{Tr}[\gamma_5 \not{p}_2\gamma_\lambda\gamma_\nu\not{p}_1]$$

A 2-loop warm-up: LL γ self-energy in QED

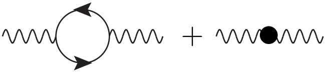
It is obtained by squaring the diagram



$$= i\, T_{\alpha\beta}\, \Pi(p^2) \qquad T_{\alpha\beta} = g_{\alpha\beta} p^2 - p_\alpha p_\beta$$

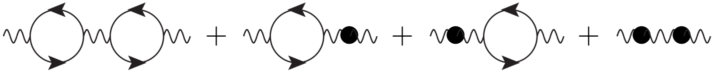
$$\Pi(p^2) = \frac{1}{\epsilon}\, \Pi_{-1} + \Pi_0 + \epsilon\, \Pi_1$$

In DR, one-loop UV counterterms are needed to avoid $\Pi_{-1}\Pi_1$



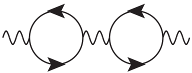
$$= i\, T_{\alpha\beta}\, \Pi_0 + \mathcal{O}(\epsilon)$$

Therefore, up to terms $\mathcal{O}(\epsilon)$



$$= iT_{\alpha\beta}\,\Pi_0^2$$

In FDR, the product of two one-loop diagrams **is the product of the two finite parts**, so that one obtains **without counterterms**



$$= i\,T_{\alpha\beta}\,\Pi^2_{\mathrm{FDR}}(p^2)$$

with $\Pi_{\mathrm{FDR}}(p^2) = \Pi_0 = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\,\ln\frac{m^2-p^2x(1-x)}{\mu_R^2}$

$\Longrightarrow$ No order-by-order renormalization in FDR

- The previous example also shows that **ℓ-loop integrals are directly re-usable in (ℓ+1)-loop calculations**

- For instance, the two-loop factorizable FDR integral

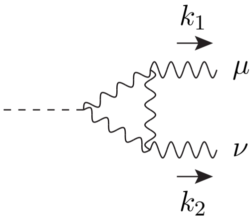
$$\int \frac{[d^4q_1]}{(\bar{q}_1^2 - m_1^2)^\alpha} \times \int \frac{[d^4q_2]}{(\bar{q}_2^2 - m_2^2)^\beta}$$

  is simply the product of two one-loop FDR integrals

- That **is not** the case in DR, where further expanding in ϵ is in general required

Example 1: $H \to \gamma(k_1^\mu)\,\gamma(k_2^\nu)$ (generic R_ξ gauge)

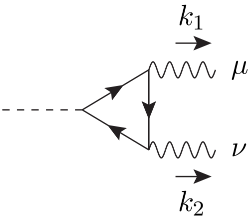
Alice M. Donati and R.P., arXiv:1302.5668



$\widetilde{\mathcal{M}}_W(\beta)$

26 diagrams

$$\beta = \frac{4\,M_W^2}{M_H^2}$$



$\widetilde{\mathcal{M}}_f(\eta)$

2 diagrams

$$\eta = \frac{4\,m_f^2}{M_H^2}$$

$$
\begin{aligned}
\mathcal{M}^{\mu\nu}(\beta,\eta) &= \left(\widetilde{\mathcal{M}}_W(\beta) + \sum_f N_c Q_f^2\, \widetilde{\mathcal{M}}_f(\eta)\right) T^{\mu\nu} \\
T^{\mu\nu} &= k_1^\nu k_2^\mu - (k_1 \cdot k_2)\, g^{\mu\nu} \\
\widetilde{\mathcal{M}}_W(\beta) &= \frac{i\, e^3}{(4\pi)^2 s_W M_W} \left[2 + 3\beta + 3\beta(2-\beta) f(\beta) \right] \\
\widetilde{\mathcal{M}}_f(\eta) &= \frac{-i\, e^3}{(4\pi)^2 s_W M_W}\, 2\eta \left[1 + (1-\eta) f(\eta) \right]
\end{aligned}
$$

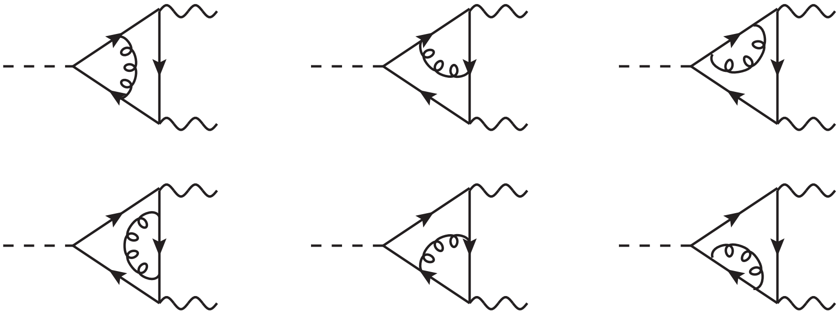
$$
f(x) = -\frac{1}{4} \ln^2 \left(\frac{1+\sqrt{1-x+i\varepsilon}}{-1+\sqrt{1-x+i\varepsilon}} \right)
$$

NOTE:

$$
\int [d^4q] \frac{\bar{q}^2 g_{\mu\nu} - 4 q_\mu q_\nu}{(\bar{q}^2 - M^2)^3} = \int [d^4q] \frac{-\mu^2}{(\bar{q}^2 - M^2)^3} g_{\mu\nu} = -\frac{i\pi^2}{2} g_{\mu\nu}
$$

Example 2: gluonic corrections to $\Gamma(\mathbf{H} \to \gamma\gamma)$

Alice M. Donati and R.P., arXiv:1311.5500



12 diagrams

Important facts

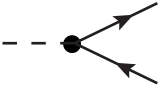
- $$\mathcal{M}^{(2-loop)} = \underbrace{\mathcal{M}^{(1-loop)}}_{\frac{i\alpha}{3\pi v}} \left(1 - \frac{\alpha_S}{\pi}\right) \quad (\text{when } m_{\text{top}} \to \infty)$$

- **No** integral by integral correspondence between FDR and DR and results coincide only at the very end

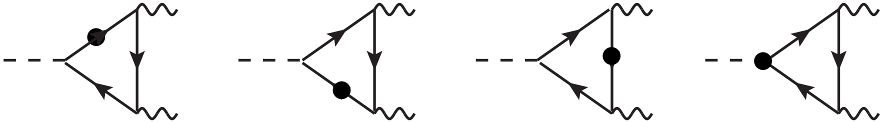
- If $m_{\text{top}} \to \infty$ **no** finite renormalization needed in FDR

- In DR no renormalization (of sub-divergences) with UV counterterms gives a **wrong** result

$= -i\,\delta m$



$= -i\,\dfrac{\delta m}{v}$



$$= \begin{cases} 0 \times \delta m & \text{in FDR} \quad \text{with } \delta m \propto \ln \mu_R \\ \mathcal{O}(\epsilon) \times \delta m & \text{in DR} \quad \text{with } \delta m \propto 1/\epsilon \end{cases}$$

Example 3: $\Gamma(\mathbf{H} \to \mathbf{gg})$

R. P., arXiv:1307.0705

- FDR is used to compute the NLO QCD corrections to $\mathbf{H} \to \mathbf{gg}$ in the large top mass limit (effective ggH theory)
- The well known fully inclusive result

$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2)) \left[1 + \frac{95}{4} \frac{\alpha_S}{\pi}\right]$$

is re-derived **both analytically and numerically**, where

$$\Gamma^{(0)}(\alpha_S(M_H^2)) = \frac{G_F \alpha_S^2(M_H^2)}{36\sqrt{2}\pi^3} M_H^3$$

- **UV**, **Soft** and **CL** divergences, besides α_S **renormalization**

- One obtains

$$\Gamma_R(\mathbf{H} \to \mathbf{ggg}) = \frac{3}{2}\frac{\alpha_S}{\pi}\,\Gamma^{(0)}(\alpha_S) \times \left[\ln^2\frac{M_H^2}{\mu^2} - \pi^2 + \frac{73}{6} - \frac{11}{3}\ln\frac{M_H^2}{\mu^2}\right]$$

and (accounting for the finite renormalization term $\left(1+\frac{11}{4}\frac{\alpha_S}{\pi}\right)$ in the ggH effective coupling)

$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma_V(\mathbf{H} \to \mathbf{gg}) + \Gamma_R(\mathbf{H} \to \mathbf{ggg})$$

$$= \Gamma^{(0)}(\alpha_S)\left[1 + \frac{\alpha_S}{\pi}\left(\frac{95}{4} - \frac{11}{2}\ln\frac{M_H^2}{\mu^2}\right)\right]$$

α_S renormalization

- The residual μ^2 is a universal dependence on the renormalization scale $(\mu = \mu_R)$
- $\ln(\mu_R^2)$ can be reabsorbed in the gluonic running of the strong coupling constant **(Finite Global Renormalization)**

$$\begin{aligned}\Gamma^{(0)}(\alpha_S) &\to \Gamma^{(0)}(\alpha_S(\mu_R^2)) \\ \alpha_S(M_H^2) &= \frac{\alpha_S(\mu_R^2)}{1 + \frac{\alpha_S}{2\pi}\frac{11}{2}\ln\frac{M_H^2}{\mu_R^2}}\end{aligned}$$

$$\Gamma(\mathbf{H} \to \mathbf{gg}) = \Gamma^{(0)}(\alpha_S(M_H^2))\left[1 + \frac{95}{4}\frac{\alpha_S}{\pi}\right]$$

quod erat demostrandum

Looking inside the garbage bin

Independently of the number of external legs!

1. $\left[\frac{1}{\bar{q}^4}\right]$ is the only possible **subtracted** 1-loop log divergent scalar

Vacuum Integrand $\Leftrightarrow$ Vacuum Bubble

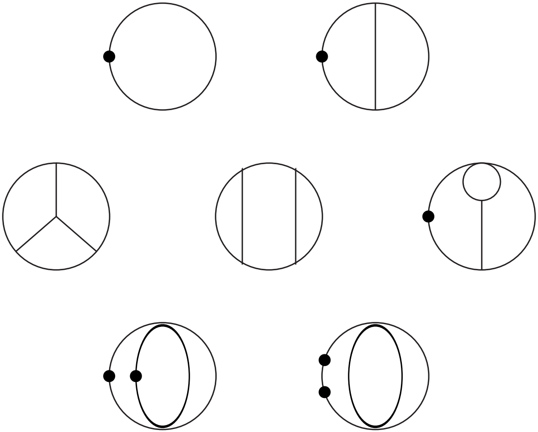
2. At 2 loops $\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_{12}^2}\right]$ is log divergent

3. Five additional log divergent vacuum integrands at 3 loops

$$\left[\frac{1}{\bar{q}_1^2\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{13}^2((q_2-q_3)^2-\mu^2)}\right] \quad \left[\frac{1}{\bar{q}_1^2\bar{q}_3^2\bar{q}_2^4\bar{q}_{12}^2\bar{q}_{23}^2}\right]$$

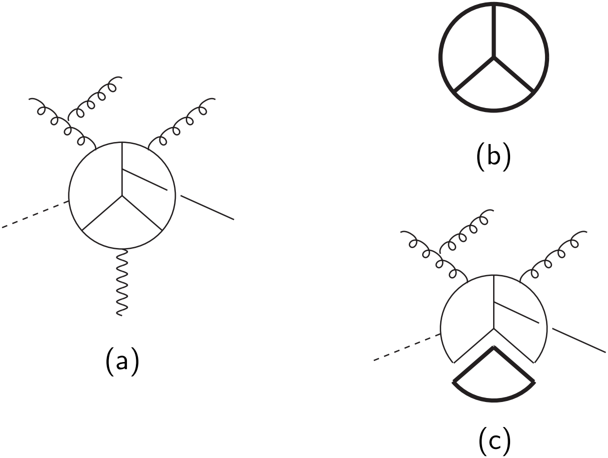
$$\left[\frac{1}{\bar{q}_1^4\bar{q}_2^2\bar{q}_3^2\bar{q}_{12}^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^4\bar{q}_2^4\bar{q}_3^2\bar{q}_{123}^2}\right] \quad \left[\frac{1}{\bar{q}_1^6\bar{q}_2^2\bar{q}_3^2\bar{q}_{123}^2}\right]$$

Corresponding 1-, 2- and 3-loop log topologies



By tensor reduction divergent tensors are reducible to combinations of those scalar topologies plus finite constants

Vacuum inside loops (pictorially)



(b) *and* (c) *are* **Vacuum Bubbles** *generated by the generic diagram* (a).
They do not contribute to the interaction and are discarded (irrelevant!)

- Infinities are put back into the vacuum, rather than absorbed in the parameter of the Lagrangian $\mathcal{L}$

Order-by-order **vacuum redefinition*** *instead of order-by-order redefinition of* $\mathcal{L}$

- This is possible because no cutoff is left in FDR integrals to be compensated by counterterms in $\mathcal{L}$

***dubbed Topological Renormalization**

FDR versus BPHZ

1. The FDR subtraction is obtained by a formal expansion of the original loop integrands **around poles in** $\bar{q}_i^2$, and not via a Taylor expansion in the external momenta
2. In FDR poles in $\bar{q}_i^2$ giving rise to UV divergences are subtracted **without any attempt of re-introducing them into the Lagrangian**
3. **Gauge invariance is automatically respected** in FDR, while it must be enforced by hand in BPHZ

Shift invariance of one-loop FDR integrals

Given

$$\bar{D} = q^2 - M^2 - \mu^2$$
$$\bar{D}_p = (q+p)^2 - M^2 - \mu^2$$

and

$$I^{(0)} = \int [d^4q] \frac{1}{\bar{D}^2}\,, \qquad I_p^{(0)} = \int [d^4q] \frac{1}{\bar{D}_p^2}$$
$$I^{(2)} = \int [d^4q] \frac{1}{\bar{D}}\,, \qquad I_p^{(2)} = \int [d^4q] \frac{1}{\bar{D}_p}$$

I prove that

$$I^{(0)} = I_p^{(0)} \quad \text{and} \quad I^{(2)} = I_p^{(2)}$$

$$I^{(0)} = I_p^{(0)}$$

From the FDR defining expansions one obtains

$$\frac{1}{\bar{D}^2} = \left[\frac{1}{\bar{q}^4}\right] + J_F^{(0)}$$
$$\frac{1}{\bar{D}_p^2} = \left[\frac{1}{\bar{q}^4}\right] + J_{F,p}^{(0)}$$

Then

$$I^{(0)} = \lim_{\mu \to 0} \int d^n q \left(\frac{1}{\bar{D}^2} - \frac{1}{\bar{q}^4}\right) = \lim_{\mu \to 0} \int d^n q \left(\frac{1}{\bar{D}_p^2} - \frac{1}{\bar{q}^4}\right) = I_p^{(0)}$$

$$I^{(2)} = I_p^{(2)}$$

From the FDR defining expansions one obtains

$$\frac{1}{\bar{D}} = \left[\frac{1}{\bar{q}^2}\right] + M^2\left[\frac{1}{\bar{q}^4}\right] + J_F^{(2)}$$

$$\frac{1}{\bar{D}_p} = \left[\frac{1}{\bar{q}^2}\right] + (M^2 - p^2)\left[\frac{1}{\bar{q}^4}\right] - 2p^\alpha\left[\frac{q_\alpha}{\bar{q}^4}\right] + 4p^\alpha p^\beta\left[\frac{q_\alpha q_\beta}{\bar{q}^6}\right] + J_{F,p}^{(2)}$$

Then

$$I^{(2)} = \lim_{\mu\to 0}\int d^n q\left(\frac{1}{\bar{D}} - \frac{1}{\bar{q}^2} - \frac{M^2}{\bar{q}^4}\right)$$

and

$$I_p^{(2)} = I^{(2)} + \underbrace{\int d^n q\left(\frac{p^2}{\bar{q}^4} + 2\frac{(q\cdot p)}{\bar{q}^4} - 4\frac{(q\cdot p)^2}{\bar{q}^6}\right)}_{=0}$$

This is because

$$\int d^nq \frac{1}{q^2-\mu^2} = \int d^nq \frac{1}{(q+p)^2-\mu^2} =$$

$$\int d^nq \frac{1}{q^2-\mu^2}\left[1-\underbrace{\left(\frac{p^2+2(q\cdot p)}{\bar{q}^2}-4\frac{(q\cdot p)^2}{\bar{q}^4}\right)}_{\propto\, p^2 \text{ when integrated}}+\mathcal{O}(p^3)\right]$$

Then

$$\int d^nq \left(\frac{p^2}{\bar{q}^4}+2\frac{(q\cdot p)}{\bar{q}^4}-4\frac{(q\cdot p)^2}{\bar{q}^6}\right)=0$$

which can also be tested by a direct computation

Equivalence of FDR and DR (in $\overline{\mathrm{MS}}$) at one loop

When computed in DR in $n = 4+\epsilon$ dimensions, the subtracted one-loop tensors obey *gauge preserving consistency relations*

$$\int d^n q \left[\frac{q^\mu q^\nu}{\bar{q}^6}\right] = \frac{g^{\mu\nu}}{4} \int d^n q \left[\frac{1}{\bar{q}^4}\right]$$

$$\int d^n q \left[\frac{q^\mu q^\nu q^\rho q^\sigma}{\bar{q}^8}\right] = \frac{(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})}{24} \int d^n q \left[\frac{1}{\bar{q}^4}\right]$$

For **both** scalars and tensors $J_{\mathrm{INF}}(q)$ is proportional to

$$\mu_R^{-\epsilon} \int d^n q \left[\frac{1}{\bar{q}^4}\right] = i\pi^2 \left(-\frac{2}{\epsilon} - \gamma_E - \ln \pi - \ln \frac{\mu^2}{\mu_R^2}\right)$$

In FDR all terms but $\ln \frac{\mu^2}{\mu_R^2}$ are subtracted, as in $\overline{\mathrm{MS}}$

UV divergences versus $\ln(\mu_R)$ in FDR integrals

The absence of UV infinities in $[J_{\rm INF}]$ is a sufficient **but not necessary** condition for the absence of $\ln(\mu_R)$ in $J_{{\rm F},\ell}$. For instance

$$\int [d^4q_1][d^4q_2]\left(\frac{2}{\bar D_1^2\bar D_2\bar D_{12}}-\frac{1}{\bar D_1^2\bar D_2^2}+\frac{4m^2}{\bar D_1^3\bar D_2^2}\right)=2\pi^4 f$$

with $\bar D_i=\bar q_i^2-m^2$ and $f=\frac{i}{\sqrt{3}}\left({\rm Li}_2(e^{i\frac{\pi}{3}})-{\rm Li}_2(e^{-i\frac{\pi}{3}})\right)$. While

$$\mu_R^{-2\epsilon}\int d^nq_1d^nq_2\left[\frac{2}{\bar D_1^2\bar D_2\bar D_{12}}-\frac{1}{\bar D_1^2\bar D_2^2}+\frac{4m^2}{\bar D_1^3\bar D_2^2}\right]_{\rm INF}$$
$$=\pi^4\left\{-2\left(\frac{1}{\epsilon}+\ln\pi+\gamma_E+\ln\frac{m^2}{\mu_R^2}\right)-3+2f\right\}$$

IBP identity at two loops in FDR

- With $D_i = q_i^2 - m_i^2$ and $q_{12} = q_1 + q_2$

$$
\begin{aligned}
0 &= \int [d^4q_1][d^4q_2] \frac{\partial}{\partial q_1^\alpha} \frac{q_1^\alpha q_1^\beta q_1^\gamma}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} \\
&= \int [d^4q_1][d^4q_2]\, q_1^\beta q_1^\gamma \left\{ \frac{6}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}} - \frac{6 q_1^2}{\bar{D}_1^4 \bar{D}_2 \bar{D}_{12}} - 2 \frac{(q_1 \cdot q_{12})}{\bar{D}_1^3 \bar{D}_2 \bar{D}_{12}^2} \right\}
\end{aligned}
$$

Naive treatment of scaleless integrals in DR

$$B^{\mathrm{DR}}(p^2,0,0) = \int d^n q \frac{1}{q^2 (q+p)^2} \qquad (p^2=0)$$

$$\frac{1}{(q+p)^2} = \frac{1}{q^2-M^2} - \left(\frac{1}{q^2-M^2} - \frac{1}{(q+p)^2}\right)$$
$$= \frac{1}{q^2-M^2} - \frac{M^2+2(q\cdot p)}{(q^2-M^2)(q+p)^2}$$

$$B^{\mathrm{DR}}(p^2,0,0) = \underbrace{\int d^n q \frac{1}{q^2(q^2-M^2)}}_{\text{defined if } \epsilon<0} - \underbrace{\int d^n q \frac{M^2+2(q\cdot p)}{q^2(q^2-M^2)(q+p)^2}}_{\text{defined if } \epsilon>0}$$

They cancel but **do they define $B^{\mathrm{DR}}(p^2,0,0)$?**

(**NO** ϵ can be found for which they simultaneously exist)

Non-renormalizable QFTs (an alternative view)

Extending the FDR framework to a non-renormalizable QFT described by a Lagrangian $\mathcal{L}_{NR}(p_1, \ldots, p_m)$

R.P., arXiv:1305.0419

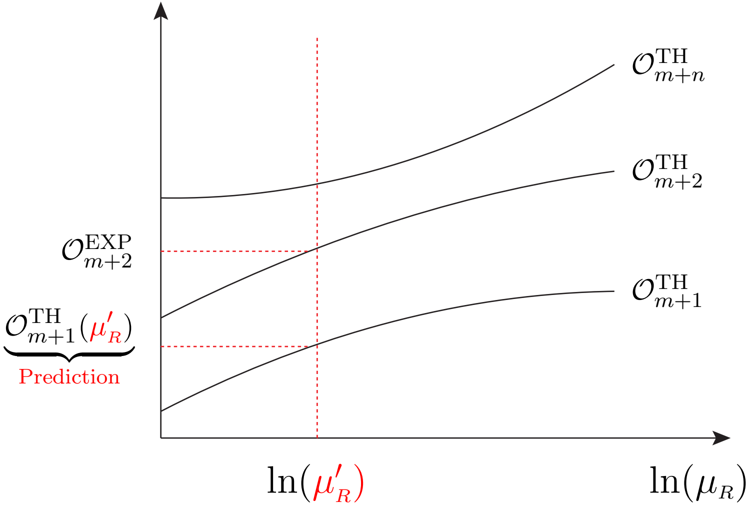
1. Now $\ln(\mu_R)$ *might* appear when computing observables

$$\mathcal{O}^{\rm TH}_{m+1}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu_R))$$

2. However, combinations of observables in which μ_R disappears can be unambiguously predicted by $\mathcal{L}_{NR}$. E. g. (at one loop)

$$\begin{aligned}
\mathcal{O}^{\rm TH}_{m+1} &= \alpha \ln(\mu_R) + k_1 \\
\mathcal{O}^{\rm TH}_{m+2} &= \beta \ln(\mu_R) + k_2 \\
\mathcal{O}_{\rm Predictable} &= \frac{\mathcal{O}_{m+1}}{\alpha} - \frac{\mathcal{O}_{m+2}}{\beta} = \frac{k_1}{\alpha} - \frac{k_2}{\beta}
\end{aligned}$$

3. This is equivalent to extracting $\ln(\mu_R)$ from $\mathcal{O}^{\rm EXP}_{m+2} = \mathcal{O}^{\rm TH}_{m+2}$ and inserting it in $\mathcal{O}^{\rm TH}_{m+1}$



4. **At any loop order just one** additional measurement needed to fix μ_R by solving

$$\mathcal{O}^{\rm EXP}_{m+2} = \mathcal{O}^{\rm TH}_{m+2}(\bar{p}_1, \ldots, \bar{p}_m, \ln(\mu'_R))$$

and setting $\mu_R = \mu'_R$ in $\mathcal{O}^{\rm TH}_{m+1}, \cdots, \mathcal{O}^{\rm TH}_{m+n}$

5. **Predictivity** restored in the infinite loop limit (unlike in Weinberg's approach to QFTs)

6. Higher dimensional non-renormalizable interactions **solely generated by loops** (bare $\mathcal{L}_{NR}$ *frozen*)

7. **The physical meaning of the extra measurement is**

disentangling from the physical spectrum the effects of the unknown UV completion of $\mathcal{L}_{NR}$