Heavy flavor corrections to deep-inelastic scattering at three loops

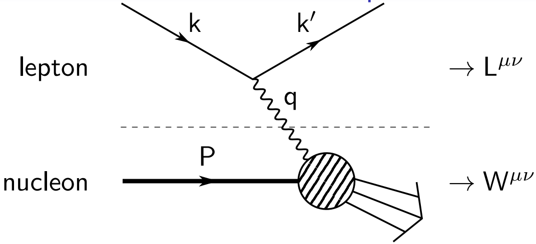
A. Behring

DESY

April 17th, 2015

based on
[J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn,
A. von Manteuffel, M. Round, C. Schneider, F. Wißbrock '14 [Nucl. Phys. B 886 (2014) 733]]
and
[J. Ablinger, A. Behring, J. Blümlein, A. De Freitas,
A. von Manteuffel, C. Schneider '14 [Nucl. Phys. B 890 (2014) 48]]

Heavy flavor contributions to deep-inelastic scattering



Cross section: $$\frac{d\sigma}{dx dQ^2} \propto L_{\mu\nu} W^{\mu\nu}$$

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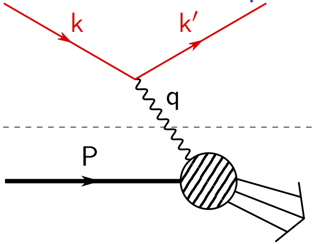

lepton $\rightarrow L^{\mu\nu}$

nucleon $\rightarrow W^{\mu\nu}$

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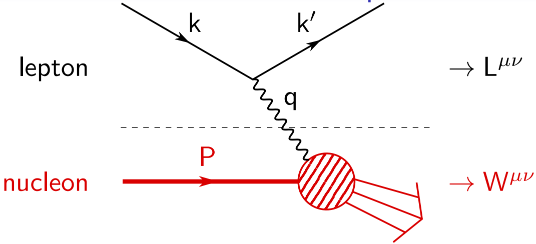
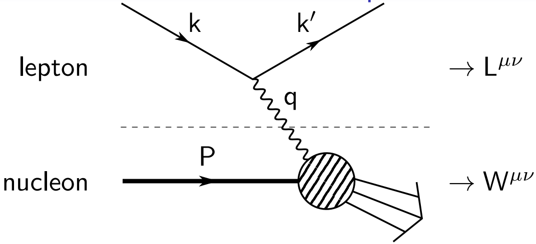
Heavy flavor contributions to deep-inelastic scattering



Cross section: $\frac{d\sigma}{dx\,dQ^2} \propto L_{\mu\nu}\boxed{W^{\mu\nu}}$

Hadronic tensor: $\boxed{W_{\mu\nu} = (\dots)_{\mu\nu}F_L(x, Q^2) + (\dots)_{\mu\nu}F_2(x, Q^2)}$

Heavy flavor contributions to deep-inelastic scattering



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Hadronic tensor: $$W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} F_2(x, Q^2)$$

Structure functions contain light and heavy quark contributions.

Motivation for NNLO heavy quark corrections to DIS:

- Precision determinations of α_s and m_c require heavy flavor corrections at $\mathcal{O}(\alpha_s^3)$ due to the precision of the world data.

Heavy flavor contributions to deep-inelastic scattering

Hadronic tensor: $W_{\mu\nu} = (\dots)_{\mu\nu}\, F_L(x, Q^2) + (\dots)_{\mu\nu}\, F_2(x, Q^2)$

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Structure functions: $\boxed{F_2(x) \;=\; x \sum_j \mathbb{C}_{2,j}(x) \otimes f_j(x)}$

Wilson coefficients
perturbative

PDFs
non-perturbative

Heavy flavor contributions to deep-inelastic scattering

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x- and N-space are connected by a Mellin transformation:

$$M[f(x)](N) = \int_0^1 \mathrm{d}x \, x^{N-1} f(x)$$

Representation simplifies in Mellin space.

Heavy flavor contributions to deep-inelastic scattering

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Structure functions: $\boxed{F_2(N-1) = \sum_j \underset{\substack{\uparrow \\ \text{Wilson coefficients} \\ \text{perturbative}}}{\mathbb{C}_{2,j}(N)} \cdot \underset{\substack{\uparrow \\ \text{PDFs} \\ \text{non-perturbative}}}{f_j(N)}}$

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Heavy flavor contributions to deep-inelastic scattering

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Wilson coefficients: $\boxed{\mathbb{C}_{2,j}(N) = C_{2,j}(N) + H_{2,j}(N)}$

massless
Wilson coefficients
NNLO: [Moch, Vermaseren, Vogt '05]

heavy-flavor
Wilson coefficients

Heavy flavor contributions to deep-inelastic scattering

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For $Q^2/m^2 \geq 10$ the heavy flavor Wilson coefficients factorize:
[Buza, Matiounine, Smith, Migneron, van Neerven '96]

Heavy flavor
Wilson coefficients:

$$\boxed{H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)}$$

massive operator matrix elements (OMEs)

massless Wilson coefficients

LO: [Witten '76; Babcock, Sievers '78; Shifman, Vainshtein, Zakharov '78; Leveille, Weiler '79; Glück, Reya '79; Glück, Hoffmann, Reya '82]
NLO: [Laenen, van Neerven, Riemersma, Smith '93; Buza, Matiounine, Smith, Migneron, van Neerven '96; Bierenbaum, Blümlein, Klein '07, '08, '09]

Heavy flavor contributions to deep-inelastic scattering

Hadronic tensor: $W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} F_2(x, Q^2)$

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Heavy flavor
Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$

OMEs A_{ij} also essential to define the variable flavor number scheme
→ describe transition from n_f to $n_f + 1$ massless quarks

Wilson coefficients in terms of OMEs

2014
$$\begin{aligned}L^{\mathsf{NS}}_{q,(2,L)}(N_F+1) &= a_s^2\Big[A^{(2),\mathsf{NS}}_{qq,Q}(N_F+1)\ \delta_2+\hat{C}^{(2),\mathsf{NS}}_{q,(2,L)}(N_F)\Big]\\ &+ a_s^3\Big[A^{(3),\mathsf{NS}}_{qq,Q}(N_F+1)\ \delta_2+A^{(2),\mathsf{NS}}_{qq,Q}(N_F+1)C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)+\hat{C}^{(3),\mathsf{NS}}_{q,(2,L)}(N_F)\Big]\end{aligned}$$

2010
$$L^{\mathsf{PS}}_{q,(2,L)}(N_F+1) = a_s^3\Big[\ A^{(3),\mathsf{PS}}_{qq,Q}(N_F+1)\ \delta_2+A^{(2)}_{gq,Q}(N_F)\ \ N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+1)+N_F\hat{\tilde{C}}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F$$

2010
$$\begin{aligned}L^{\mathsf{S}}_{g,(2,L)}(N_F+1) &= a_s^2A^{(1)}_{gg,Q}(N_F+1)N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+1)+a_s^3\Big[\ A^{(3)}_{qg,Q}(N_F+1)\ \delta_2\\ &+A^{(1)}_{gg,Q}(N_F+1)\ \ N_F\tilde{C}^{(2)}_{g,(2,L)}(N_F+1)+A^{(2)}_{gg,Q}(N_F+1)\ \ N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+\\ &+\ A^{(1)}_{Qg}(N_F+1)\ \ N_F\tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)+N_F\hat{\tilde{C}}^{(3)}_{g,(2,L)}(N_F)\Big]\ ,\end{aligned}$$

2014
$$\begin{aligned}H^{\mathsf{PS}}_{q,(2,L)}(N_F+1) &= a_s^2\Big[\ A^{(2),\mathsf{PS}}_{Qq}(N_F+1)\ \delta_2+\ \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)\Big]+a_s^3\Big[\ A^{(3),\mathsf{PS}}_{Qq}(N_F+1)\ \delta_2\\ &+\ \tilde{C}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F+1)+A^{(2)}_{gq,Q}(N_F+1)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1)\\ &+A^{(2),\mathsf{PS}}_{Qq}(N_F+1)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)\Big]\ ,\end{aligned}$$

$$\begin{aligned}H^{\mathsf{S}}_{g,(2,L)}(N_F+1) &= a_s\Big[\ A^{(1)}_{Qg}(N_F+1)\ \delta_2+\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1)\Big]+a_s^2\Big[\ A^{(2)}_{Qg}(N_F+1)\ \delta_2\\ &+\ A^{(1)}_{Qg}(N_F+1)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)+\ A^{(1)}_{gg,Q}(N_F+1)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1)\\ &+\ \tilde{C}^{(2)}_{g,(2,L)}(N_F+1)\Big]+\ a_s^3\Big[\ A^{(3)}_{Qg}(N_F+1)\ \delta_2+\ A^{(2)}_{Qg}(N_F+1)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+\\ &+\ A^{(2)}_{gg,Q}(N_F+1)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1)+\ A^{(1)}_{Qg}(N_F+1)\Big\{C^{(2),\mathsf{NS}}_{q,(2,L)}(N_F+1)\\ &+\ \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)\Big\}+\ A^{(1)}_{gg,Q}(N_F+1)\ \tilde{C}^{(2)}_{g,(2,L)}(N_F+1)+\ \tilde{C}^{(3)}_{g,(2,L)}(N_F+1\end{aligned}$$

Status of Wilson coefficients and OMEs

Moments for F_2: $N = 2 \dots 10(14)$ ✓ [Bierenbaum, Blümlein, Klein, '09]

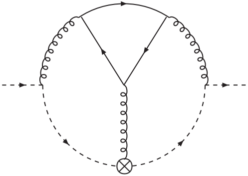
Massive operator matrix elements at NNLO

- A_{Qg} work in progress
- A_{gg} work in progress
- A_{Qq}^{PS} ✓ → this talk
- $A_{qq,Q}^{\mathsf{NS}}$ ✓ [Ablinger et al. '14]
- $A_{qq,Q}^{\mathsf{TR}}$ ✓ [Ablinger et al. '14]
- $A_{gq,Q}$ ✓ [Ablinger et al. '14]
- $A_{qg,Q}$ ✓ [Ablinger et al. '10]
- $A_{qq,Q}^{\mathsf{PS}}$ ✓ [Ablinger et al. '10]

Heavy flavor Wilson coefficients at NNLO

- $H_{g,2}^{\mathsf{S}}$ work in progress
- $H_{q,2}^{\mathsf{PS}}$ ✓ → this talk
- $L_{q,2}^{\mathsf{NS}}$ ✓ [Ablinger et al. '14]
- $L_{q,2}^{\mathsf{PS}}$ ✓ [Behring et al. '14]
- $L_{g,2}^{\mathsf{S}}$ ✓ [Behring et al. '14]

Calculating massive operator matrix elements



Definition of the OMEs A_{ij}

$$A_{ij} := \langle j|O_i|j\rangle$$

O_i: local operators

$|j\rangle$: partonic states (massless, on-shell)

Outline of the computation

- Generate diagrams (`QGRAF`) [P. Nogueira '93]
- Apply Feynman rules including operators; $(\Delta.p)^N \to \frac{1}{1-x\Delta.p}$
- Reduce to master integrals (extension of `Reduze 2`) [von Manteuffel, Studerus '10,'12]
- Solve the master integrals (→ next slide)
- Put everything together and create results in N- and x-space

Solving the master integrals

Solve the master integrals applying

- higher hypergeometric functions,
- Mellin-Barnes integrals,
- Almkvist-Zeilberger algorithm,
- difference equations.

Transform resulting sum representations using

- `Sigma` [Schneider '05-], `HarmonicSums` [Ablinger, Blümlein, Schneider '10,'13],
- `EvaluateMultiSums` & `SumProduction` [Ablinger, Blümlein, Hasselhuhn, Schneider '10-].

Occurring mathematical structures (N-space):

- Harmonic sums [Vermaseren '99; Blümlein, Kurth '99]
- Gen. harmonic sums (S-sums) [Moch, Uwer, Weinzierl '02; Ablinger, Blümlein, Schneider '13]
- (Gen.) cyclotomic sums [Ablinger, Blümlein, Schneider '11]
- Binomially weighted sums [Ablinger, Blümlein, Raab, Schneider '14]

Constant part of the pure-singlet OME $a_{Qq}^{\mathrm{PS},(3)}$

$$
\begin{aligned}
& a_{Qq}^{(3),\mathrm{PS}}(N) = \\
& C_F T_F^2 \Bigg[\frac{32}{27(N-1)(N+3)(N+4)(N+5)} \Bigg(\frac{P_{15}}{N^3(N+1)^2(N+2)^2} S_2 \\
& -\frac{P_{19}}{N^3(N+1)^3(N+2)^2} S_1^2 + \frac{2P_{28}}{3N^4(N+1)^4(N+2)^3} S_1 - \frac{2P_{32}}{9N^5(N+1)^4(N+2)^4} \Bigg) \\
& -\frac{32P_3}{9(N-1)N^3(N+1)^2(N+2)^2}\zeta_2 + \Bigg(\frac{32}{27}S_1^3 - \frac{160}{9}S_1S_2 - \frac{512}{27}S_3 + \frac{128}{3}S_{2,1} \\
& +\frac{32}{3}S_1\zeta_2 - \frac{1024}{9}\zeta_3 \Bigg)F \Bigg] + C_F N_F T_F^2 \Bigg[\frac{16P_7}{27(N-1)N^3(N+1)^3(N+2)^2} S_1^2 \\
& +\frac{208P_7}{27(N-1)N^3(N+1)^3(N+2)^2} S_2 - \frac{32P_{21}}{81(N-1)N^4(N+1)^4(N+2)^3} S_1 \\
& +\frac{32P_{29}}{243(N-1)N^5(N+1)^5(N+2)^4} + \Bigg(-\frac{16}{27}S_1^3 - \frac{208}{9}S_1S_2 - \frac{1760}{27}S_3 - \frac{16}{3}S_1\zeta_2 \\
& +\frac{224}{9}\zeta_3 \Bigg)F + \frac{1}{(N-1)N^3(N+1)^3(N+2)^2}\frac{16P_7}{9}\zeta_2 \Bigg] \\
& +C_F^2 T_F \Bigg[\frac{32P_9}{3(N-1)N^3(N+1)^3(N+2)^2} S_{2,1} - \frac{16P_{14}}{9(N-1)N^3(N+1)^3(N+2)^2} S_3 \\
& -\frac{4P_{17}}{3(N-1)N^4(N+1)^4(N+2)^3} S_1^2 + \frac{4P_{23}}{3(N-1)N^4(N+1)^4(N+2)^3} S_2 \\
& +\frac{4P_{31}}{3(N-1)N^6(N+1)^6(N+2)^4} + \Bigg(\Bigg(\frac{2P_5}{N^2(N+1)^2} - \frac{4P_1}{N(N+1)}S_1 \Bigg)\zeta_2 \\
& -\frac{4P_1}{9N(N+1)}S_1^3 \Bigg)G + \Bigg(\Bigg(\frac{80}{9}S_3 - 64S_{2,1} \Bigg)S_1 - \frac{2}{9}S_1^4 - \frac{20}{3}S_1^2S_2 + \frac{46}{3}S_2^2 + \frac{124}{3}S_4 \\
& +\frac{416}{3}S_{2,1,1} + 64\Bigg(\Bigg(S_3(2) - S_{1,2}(2,1) + S_{2,1}(2,1) - S_{1,1,1}(2,1,1) \Bigg) S_1\left(\frac{1}{2}\right) \\
& -S_{1,3}\left(2,\frac{1}{2}\right) + S_{2,2}\left(2,\frac{1}{2}\right) - S_{3,1}\left(2,\frac{1}{2}\right) + S_{1,1,2}\left(2,\frac{1}{2},1\right) - S_{1,1,2}\left(2,1,\frac{1}{2}\right) \\
& -S_{1,2,1}\left(2,\frac{1}{2},1\right) + S_{1,2,1}\left(2,1,\frac{1}{2}\right) - S_{2,1,1}\left(2,\frac{1}{2},1\right) - S_{2,1,1}\left(2,1,\frac{1}{2}\right) \\
& +S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right) + S_{1,1,1,1}\left(2,1,\frac{1}{2},1\right) + S_{1,1,1,1}\left(2,1,1,\frac{1}{2}\right) \Bigg) + \Bigg(12S_2 - 4S_1^2 \Bigg)\zeta_2 \\
& +\Bigg(\frac{112}{3}S_1 - 448S_1\left(\frac{1}{2}\right) \Bigg)\zeta_3 + 144\zeta_4 - 32B_4 \Bigg)F + \frac{32P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Bigg(-S_3(2) \\
& +S_{1,2}(2,1) - S_{2,1}(2,1) + S_{1,1,1}(2,1,1) + 7\zeta_3 \Bigg) + \Bigg(-\frac{4P_8}{3(N-1)N^3(N+1)^3(N+2)^2} S_2 \\
& +\frac{8P_{27}}{3(N-1)N^5(N+1)^5(N+2)^4} \Bigg)S_1 + \frac{1}{(N-1)N^3(N+1)^3(N+2)^2}\frac{4P_{16}}{3}\zeta_3 \Bigg] \\
& +C_A C_F T_F \Bigg[-\frac{8P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2} S_{2,1} + \frac{8P_{12}}{3(N-1)N^3(N+1)^3(N+2)^2} S_{-3} \\
& +\frac{16P_{13}}{3(N-1)N^3(N+1)^3(N+2)^2} S_{-2,1} + \frac{8P_{22}}{27(N-1)^2N^3(N+1)^3(N+2)^2} S_3 \\
& -\frac{4P_{24}}{27(N-1)^2N^4(N+1)^4(N+2)^3} S_1^2 - \frac{4P_{26}}{27(N-1)^2N^4(N+1)^4(N+2)^3} S_2 \\
& -\frac{8P_{33}}{243(N-1)^2N^6(N+1)^6(N+2)^5} + \Bigg(\frac{4P_4}{27(N-1)N(N+1)(N+2)} S_1^3 \\
& +\Bigg(\frac{8}{9}(137N^2+137N+334)S_3 - \frac{16}{3}(35N^2+35N+18)S_{-2,1} \Bigg)S_1 \\
& +\frac{8}{3}(69N^2+69N+94)S_{-3}S_1 + \frac{64}{3}(7N^2+7N+13)S_{-2}S_2 + \frac{2}{3}(29N^2+29N+74)S_2^2 \\
& +\frac{4}{3}(143N^2+143N+310)S_4 - \frac{16}{3}(3N^2+3N-2)S_{-2}^2 + \frac{16}{3}(31N^2+31N+50)S_{-4} \\
& -8(7N^2+7N+26)S_{3,1} - 64(3N^2+3N+2)S_{-2,2} - \frac{32}{3}(23N^2+23N+22)S_{-3,1} \\
& +\frac{64}{3}(13N^2+13N+2)S_{-2,1,1} + \frac{4P_4}{3(N-1)N(N+1)(N+2)}S_1\zeta_2 \\
& -\frac{8}{3}(11N^2+11N+10)S_1\zeta_3 \Bigg)G + \Bigg(\frac{112}{3}S_{-2}S_1^2 + \frac{2}{9}S_1^4 + \frac{68}{3}S_1^2S_2 - \frac{80}{3}S_{2,1,1} \\
& +32\Bigg(\Bigg(-S_3(2) + S_{1,2}(2,1) - S_{2,1}(2,1) + S_{1,1,1}(2,1,1) \Bigg) S_1\left(\frac{1}{2}\right) + S_{1,3}\left(2,\frac{1}{2}\right) \\
& -S_{2,2}\left(2,\frac{1}{2}\right) + S_{3,1}\left(2,\frac{1}{2}\right) - S_{1,1,2}\left(2,\frac{1}{2},1\right) + S_{1,1,2}\left(2,1,\frac{1}{2}\right) + S_{1,2,1}\left(2,\frac{1}{2},1\right) \\
& -S_{1,2,1}\left(2,1,\frac{1}{2}\right) + S_{2,1,1}\left(2,\frac{1}{2},1\right) + S_{2,1,1}\left(2,1,\frac{1}{2}\right) - S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right) \\
& -S_{1,1,1,1}\left(2,1,\frac{1}{2},1\right) - S_{1,1,1,1}\left(2,1,1,\frac{1}{2}\right) \Bigg) + \Bigg(4S_1^2 + 12S_2 + 24S_{-2} \Bigg)\zeta_2 \\
& +224S_1\left(\frac{1}{2}\right)\zeta_3 - 144\zeta_4 + 16B_4 \Bigg)F + \frac{16P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Bigg(S_3(2) - S_{1,2}(2,1) \\
& +S_{2,1}(2,1) - S_{1,1,1}(2,1,1) - 7\zeta_3 \Bigg) + \Bigg(\frac{4P_{11}}{9(N-1)^2N^3(N+1)^3(N+2)^2} S_2 \\
& +\frac{4P_{30}}{81(N-1)^2N^5(N+1)^5(N+2)^4} \Bigg)S_1 + \Bigg(\frac{32P_6}{3(N-1)N^3(N+1)^3(N+2)^2} S_1 \\
& -\frac{8P_{18}}{3(N-1)N^4(N+1)^4(N+2)^3} \Bigg)S_{-2} - \frac{4P_{25}}{9(N-1)^2N^4(N+1)^4(N+2)^3}\zeta_2 \\
& -\frac{8P_{20}}{9(N-1)^2N^3(N+1)^3(N+2)^2}\zeta_3 \Bigg].
\end{aligned}
$$

[Ablinger et al. '14]

Constant part of the pure-singlet OME $a_{Qq}^{\mathrm{PS},(3)}$

$$a_{Qq}^{(3),\mathrm{PS}}(N) =$$
$$C_F T_F^2 \Bigg[\frac{32}{27(N-1)(N+3)(N+4)(N+5)} \Bigg(\frac{P_{15}}{N^3(N+1)^2(N+2)^2} S_2$$
$$- \frac{P_{19}}{N^3(N+1)^3(N+2)^2}S_1^2 + \frac{2P_{28}}{3N^4(N+1)^4(N+2)^3}S_1 - \frac{2P_{32}}{9N^5(N+1)^4(N+2)^4}\Bigg)$$
$$- \frac{32P_3}{9(N-1)N^3(N+1)^2(N+2)^2}\zeta_2 + \Bigg(\frac{32}{27}S_1^3 - \frac{160}{9}S_1S_2 - \frac{512}{27}S_3 + \frac{128}{3}S_{2,1}$$
$$+ \frac{32}{3}S_1\zeta_2 - \frac{1024}{9}\zeta_3\Bigg)F\Bigg] + C_F N_F T_F^2 \Bigg[\frac{16P_7}{27(N-1)N^3(N+1)^3(N+2)^2}S_1^2$$
$$+ \frac{208P_7}{27(N-1)N^3(N+1)^3(N+2)^2}S_2 - \frac{32P_{21}}{81(N-1)N^4(N+1)^4(N+2)^3}S_1$$
$$+ \frac{32P_{29}}{243(N-1)N^5(N+1)^5(N+2)^4} + \Bigg(-\frac{16}{27}S_1^3 - \frac{208}{9}S_1S_2 - \frac{1760}{27}S_3 - \frac{16}{3}S_1\zeta_2$$

$$+C_A C_F T_F \Bigg[-\frac{8P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{2,1} + \frac{8P_{12}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{-3}$$
$$+ \frac{16P_{13}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{-2,1} + \frac{8P_{22}}{27(N-1)^2N^3(N+1)^3(N+2)^2}S_3$$
$$- \frac{4P_{24}}{27(N-1)^2N^4(N+1)^4(N+2)^3}S_1^2 - \frac{4P_{26}}{27(N-1)^2N^4(N+1)^4(N+2)^3}S_2$$
$$- \frac{8P_{33}}{243(N-1)^2N^6(N+1)^6(N+2)^5} + \Bigg(\frac{4P_4}{27(N-1)N(N+1)(N+2)}S_1^3$$
$$+ \Bigg(\frac{8}{9}(137N^2+137N+334)S_3 - \frac{16}{3}(35N^2+35N+18)\boxed{S_{-2,1}}\Bigg)S_1$$
$$+ \frac{8}{3}(69N^2+69N+94)S_{-3}S_1 + \frac{64}{3}(7N^2+7N+13)S_{-2}S_2 + \frac{2}{3}(29N^2+29N+74)S_2^2$$
$$+ \frac{4}{3}(143N^2+143N+310)S_4 - \frac{16}{3}(3N^2+3N-2)S_{-2}^2 + \frac{16}{3}(31N^2+31N+50)S_{-4}$$

$$\boxed{S_{-2,1}}$$

Harmonic Sums

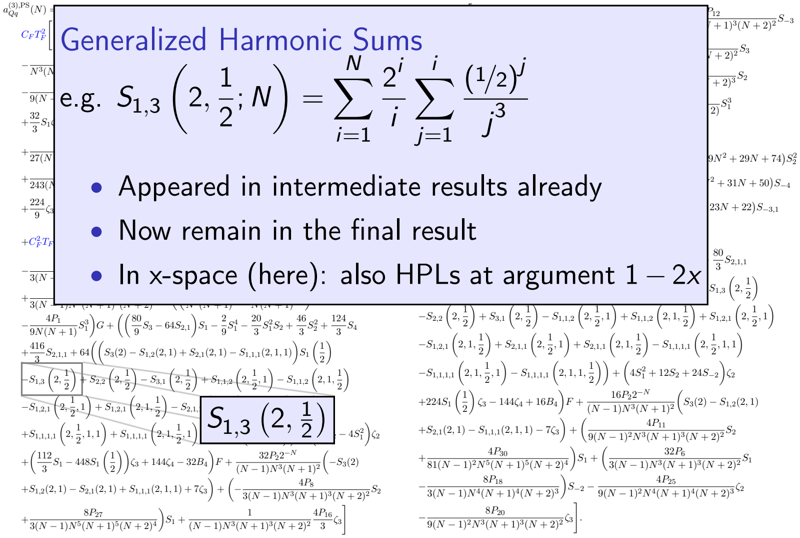
$$\text{e.g. } S_{-2,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i}{i^2} \sum_{j=1}^{i} \frac{1}{j}$$

- Translate to harmonic polylogarithms (HPLs) in x-space
- Until recently sufficient for completed OMEs

$$+ \frac{8P_{27}}{3(N-1)N^5(N+1)^5(N+2)^4}\Bigg)S_1 + \frac{1}{(N-1)N^3(N+1)^3(N+2)^2}\frac{4P_{16}}{3}\zeta_3\Bigg]$$
$$- \frac{8P_{20}}{9(N-1)^2N^3(N+1)^3(N+2)^2}\zeta_3\Bigg].$$

[Ablinger et al. '14]

Constant part of the pure-singlet OME $a_{Qq}^{\mathrm{PS},(3)}$



Generalized Harmonic Sums

$$\text{e.g. } S_{1,3}\left(2,\frac{1}{2};N\right)=\sum_{i=1}^{N}\frac{2^i}{i}\sum_{j=1}^{i}\frac{(1/2)^j}{j^3}$$

- Appeared in intermediate results already
- Now remain in the final result
- In x-space (here): also HPLs at argument $1-2x$

$$-\frac{4P_1}{9N(N+1)}S_1^3\Big)G+\Bigg(\Big(\frac{80}{9}S_3-64S_{2,1}\Big)S_1-\frac{2}{9}S_1^4-\frac{20}{3}S_1^2S_2+\frac{46}{3}S_2^2+\frac{124}{3}S_4$$

$$+\frac{416}{3}S_{2,1,1}+64\Bigg(\Big(S_3(2)-S_{1,2}(2,1)+S_{2,1}(2,1)-S_{1,1,1}(2,1,1)\Big)S_1\left(\frac{1}{2}\right)$$

$$\boxed{-S_{1,3}\left(2,\frac{1}{2}\right)}+S_{2,2}\left(2,\frac{1}{2}\right)-S_{3,1}\left(2,\frac{1}{2}\right)+S_{1,1,2}\left(2,\frac{1}{2},1\right)-S_{1,1,2}\left(2,1,\frac{1}{2}\right)$$

$$S_{1,3}\left(2,\tfrac{1}{2}\right)$$

$$+\left(\frac{112}{3}S_1-448S_1\left(\frac{1}{2}\right)\right)\zeta_3+144\zeta_4-32B_4\Bigg)F+\frac{32P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Bigg(-S_3(2)$$

$$+S_{1,2}(2,1)-S_{2,1}(2,1)+S_{1,1,1}(2,1,1)+7\zeta_3\Bigg)+\Bigg(-\frac{4P_8}{3(N-1)N^3(N+1)^3(N+2)^2}S_2$$

$$+\frac{8P_{27}}{3(N-1)N^5(N+1)^5(N+2)^4}\Bigg)S_1+\frac{1}{(N-1)N^3(N+1)^3(N+2)^2}\frac{4P_{16}}{3}\zeta_3\Bigg]$$

$$-S_{2,2}\left(2,\frac{1}{2}\right)+S_{3,1}\left(2,\frac{1}{2}\right)-S_{1,1,2}\left(2,\frac{1}{2},1\right)+S_{1,1,2}\left(2,1,\frac{1}{2}\right)+S_{1,2,1}\left(2,\frac{1}{2},1\right)$$

$$-S_{1,2,1}\left(2,1,\frac{1}{2}\right)+S_{2,1,1}\left(2,\frac{1}{2},1\right)+S_{2,1,1}\left(2,1,\frac{1}{2}\right)-S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)$$

$$-S_{1,1,1,1}\left(2,1,\frac{1}{2},1\right)-S_{1,1,1,1}\left(2,1,1,\frac{1}{2}\right)\Bigg)+\Big(4S_1^2+12S_2+24S_{-2}\Big)\zeta_2$$

$$+224S_1\left(\frac{1}{2}\right)\zeta_3-144\zeta_4+16B_4\Bigg)F+\frac{16P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Bigg(S_3(2)-S_{1,2}(2,1)$$

$$+S_{2,1}(2,1)-S_{1,1,1}(2,1,1)-7\zeta_3\Bigg)+\Bigg(\frac{4P_{11}}{9(N-1)^2N^3(N+1)^3(N+2)^2}S_2$$

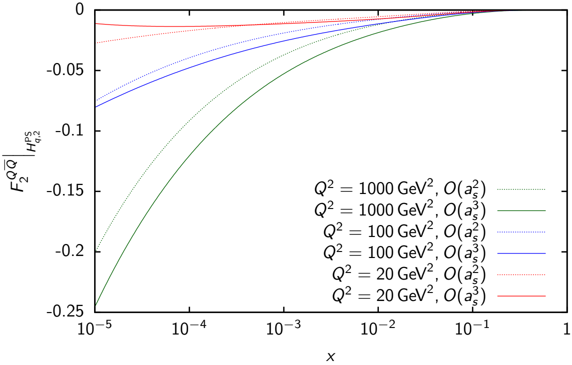
$$+\frac{4P_{30}}{81(N-1)^2N^5(N+1)^5(N+2)^4}\Bigg)S_1+\Bigg(\frac{32P_6}{3(N-1)N^3(N+1)^3(N+2)^2}S_1$$

$$-\frac{8P_{18}}{3(N-1)N^4(N+1)^4(N+2)^3}\Bigg)S_{-2}-\frac{4P_{25}}{9(N-1)^2N^4(N+1)^4(N+2)^3}\zeta_2$$

$$-\frac{8P_{20}}{9(N-1)^2N^3(N+1)^3(N+2)^2}\zeta_3\Bigg].$$

[Ablinger et al. '14]

Pure-singlet Wilson coefficient $H_{q,2}^{\mathrm{PS}}$



[Ablinger et al. '14]

For comparison (HERA kinematics): [Alekhin, Blümlein, Moch '13]

x	10^{-4}	10^{-3}	10^{-2}	10^{-1}
$F_2(x, Q^2 = 20\,\mathrm{GeV}^2)$	1.94	1.14	0.64	0.40
$F_2(x, Q^2 = 100\,\mathrm{GeV}^2)$		1.70	0.83	0.41
$F_2(x, Q^2 = 1000\,\mathrm{GeV}^2)$			1.04	0.41

Variable flavor number scheme (VFNS)

- Transition from scheme with n_f massless and 1 massive flavor to scheme with $n_f + 1$ effectively massless flavors
- Massive OMEs appear in the matching conditions of the PDFs

NNLO matching condition for non-singlet case:
[Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09, '09]

$$\begin{aligned} & f_k(n_f+1,\mu^2) + \bar{f}_k(n_f+1,\mu^2) \\ &\quad = A_{qq,Q}^{\mathrm{NS}} \otimes [f_k(n_f,\mu^2) + \bar{f}_k(n_f,\mu^2)] \\ &\quad + \frac{1}{n_f}\left(A_{qq,Q}^{\mathrm{PS}} \otimes \Sigma(n_f,\mu^2) + A_{qg,Q} \otimes G(n_f,\mu^2)\right) \end{aligned}$$

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- Ingredients at NNLO are now complete for the above relation
- $A_{qq,Q}^{\mathrm{PS}}$ and $A_{qg,Q}$ start at NNLO

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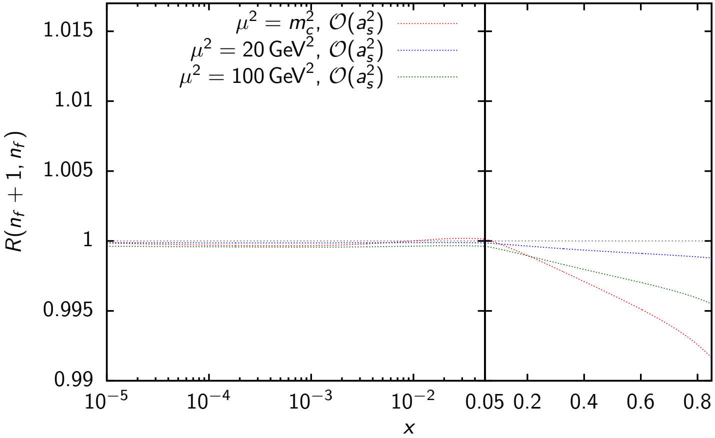
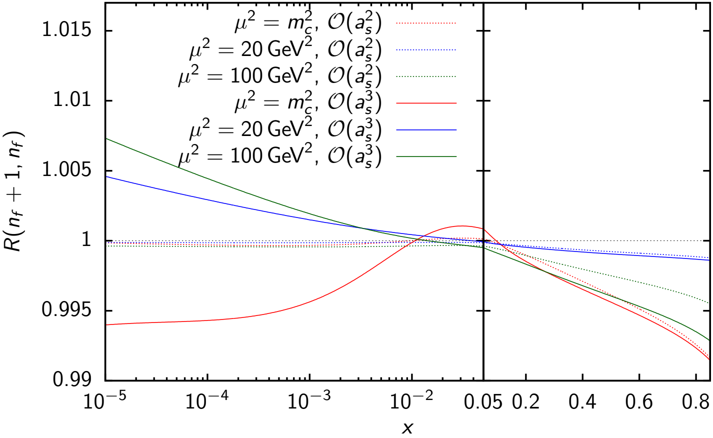
Variable flavor number scheme (VFNS)



[Ablinger et al. '14]

$$R(n_f+1, n_f) = \frac{u(n_f+1,\mu^2) + \bar{u}(n_f+1,\mu^2)}{u(n_f,\mu^2) + \bar{u}(n_f,\mu^2)}, \quad \text{here } n_f = 3$$

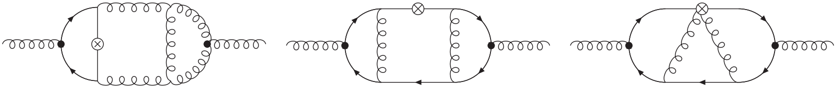
Variable flavor number scheme (VFNS)



[Ablinger et al. '14]

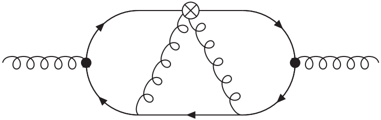
$$R(n_f+1, n_f) = \frac{u(n_f+1, \mu^2) + \bar{u}(n_f+1, \mu^2)}{u(n_f, \mu^2) + \bar{u}(n_f, \mu^2)}, \quad \text{here } n_f = 3$$

$A_{Qg}^{(3)}$: Ladder and V diagrams



- Ladder topologies enter $A_{Qg}^{(3)}$, $A_{gg,Q}^{(3)}$ and $A_{qg,Q}^{(3)}$
- Scalar prototypes (without numerators) of $A_{Qg}^{(3)}$ ladder diagrams were calculated in 2012 [Ablinger et al. '12]
  $\Rightarrow$ Calculation of complete scalar graphs
- Now: Calculation of the physical $A_{Qg}^{(3)}$ diagrams (with numerators) using master integrals

Example: V diagram 12



- Particularly difficult diagram: V topology with 5 massive lines
- Operator introduces “non-planarity” into planar diagram
- Reduction using integration-by-parts identities requires 92 master integrals
- Calculation of master integral is very demanding ($\mathcal{O}$(*weeks*) of computation time)
- Required new computer algebra tools for the solution of coupled systems of difference equations and handling of binomially weighted sums
  → Extension of `OreSys` [Gerhold '02]

Example: V diagram 12 - the result

$$D_{5b} = -\frac{(N+1)P_{11}}{P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}\binom{2i_1}{i_1}}{i_1^3}-\frac{2(N+1)P_{11}}{P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_1\big(i_1\big)}{i_1^2}$$

$$+\frac{3(N+1)P_{11}}{P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_2\big(i_1\big)}{i_1}+\frac{2(N+1)P_{11}}{P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}\binom{2i_1}{i_1}S_2\big(i_1\big)}{i_1}$$

$$+\frac{3(N+1)P_{11}}{P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_{-2}\big(i_1\big)}{i_1}+\frac{6(N+1)P_{11}}{P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}\binom{2i_1}{i_1}S_{-2}\big(i_1\big)}{i_1}$$

$$+\frac{3(N+1)P_{11}}{P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_{1,1}\big(i_1\big)}{i_1}-\frac{16P_{13}}{P_{49}}S_{3,1}-\frac{16P_{14}}{P_{49}}S_{2,1,1}+\frac{P_{18}}{P_{49}}S_2^2+\frac{2P_{19}}{P_{49}}S_4$$

$$-\frac{3P_{21}}{(N+1)P_{48}\binom{2N}{N}}\left[\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1;i_1\right)+3\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},-1;i_1\right)\right]$$

$$-\frac{(-1)^N2^{N+2}P_{35}}{P_{50}}S_{1,2}\left(\frac{1}{2},1\right)-\frac{(-1)^N3\cdot 2^{N+2}P_{35}}{P_{50}}S_{1,2}\left(\frac{1}{2},-1\right)+\frac{2P_{43}}{P_{51}}$$

$$-\frac{288\big(N^2+10N+23\big)}{(N+3)(N+4)(N+5)}\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1;i_1\right)\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2^2}$$

$$+\frac{384(6N+13)}{(N+3)(N+4)(N+5)}\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1;i_1\right)\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2}$$

$$-\frac{864\big(N^2+10N+23\big)}{(N+3)(N+4)(N+5)}\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},-1;i_1\right)\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2^2}$$

$$+\frac{1152(6N+13)}{(N+3)(N+4)(N+5)}\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},-1;i_1\right)\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2}$$

$$+\frac{\big(N^2+10N+23\big)}{(N+3)(N+4)(N+5)}\left[\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\left[288S_{1,2}\left(\frac{1}{2},1;i_1\right)+864S_{1,2}\left(\frac{1}{2},-1;i_1\right)\right]\right.$$

$$\times\sum_{i_1=1}^{N}\frac{1}{\binom{2i_1}{i_1}i_1^2}-\frac{6N+13}{(N+3)(N+4)(N+5)}\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\Bigg[384S_{1,2}\left(\frac{1}{2},1;i_1\right)$$

$$+1152S_{1,2}\left(\frac{1}{2},-1;i_1\right)\Bigg]\sum_{i_1=1}^{N}\frac{1}{\binom{2i_1}{i_1}i_1}-\frac{48\big(N^2+10N+23\big)}{(N+3)(N+4)(N+5)}\Bigg[\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}}{i_2^3}}{\binom{2i_1}{i_1}\big(1+i_1\big)}$$

$$+2\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_1\big(i_2\big)}{i_2^2}}{\binom{2i_1}{i_1}\big(1+i_1\big)}+3\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_2\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+i_1\big)}+2\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_2\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+i_1\big)}$$

$$+3\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{-2}\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+i_1\big)}+6\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_{-2}\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+i_1\big)}+3\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{1,1}\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+i_1\big)}\Bigg]$$

$$+\frac{32\big(3N^2+18N+43\big)}{(N+3)(N+4)(N+5)}\Bigg[\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}}{i_2^3}}{\binom{2i_1}{i_1}\big(1+2i_1\big)}+2\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_1\big(i_2\big)}{i_2^2}}{\binom{2i_1}{i_1}\big(1+2i_1\big)}$$

$$-3\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_2\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+2i_1\big)}-2\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_2\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+2i_1\big)}-3\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{-2}\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+2i_1\big)}$$

$$-6\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_{-2}\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+2i_1\big)}-3\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{1,1}\big(i_2\big)}{i_2}}{\binom{2i_1}{i_1}\big(1+2i_1\big)}\Bigg]-\frac{12P_{30}}{P_{52}}\zeta_2$$

$$+\Bigg\{-\frac{68}{(N+4)(N+5)}S_1^2+\frac{384(6N+13)}{(N+3)(N+4)(N+5)}S_1(-2)$$

$$-\frac{384\big(N^2+10N+23\big)}{(N+3)(N+4)(N+5)}S_2(-2)-\frac{3P_{21}}{(N+1)P_{48}\binom{2N}{N}}\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}-\frac{8P_{24}}{P_{53}}$$

$$+\frac{288\big(N^2+10N+23\big)}{(N+3)(N+4)(N+5)}\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}(-2)^{i_2}\binom{2i_2}{i_2}}{\binom{2i_1}{i_1}i_1^2}-\frac{384(6N+13)}{(N+3)(N+4)(N+5)}$$

$$\times\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}(-2)^{i_2}\binom{2i_2}{i_2}}{\binom{2i_1}{i_1}i_1}+\left[\frac{192(-1)^N}{(N+4)^2}+\frac{8P_1}{(N+1)(N+2)(N+3)(N+4)^2(N+5)}\right]S_1$$

$$+\left[-\frac{204}{(N+4)(N+5)}+\frac{96(-1)^N(2N+9)}{(N+4)(N+5)}\right]S_2+\Bigg[\frac{192(-1)^N(2N+9)}{(N+4)(N+5)}$$

$$+\frac{16\big(6N^2+55N+123\big)}{(N+3)(N+4)(N+5)}\Bigg]S_{-2}\Bigg\}\zeta_3+(-1)^N\left[\frac{2P_{42}}{P_{51}}+\frac{12P_{22}}{P_{53}(N+5)}\zeta_2\right]$$

$$+\Bigg\{\frac{16}{N+3)(N+4)(N+5)}\Bigg[(11N+31)(13N+64)S_{3,1}-2\big(5N^2+29N+52\big)S_{-2,2}$$

$$+2\big(7N^2+31N+44\big)S_{-2,1,1}-\big(13N^2+85N+164\big)S_{-3,1}$$

$$-2\big(28N^2+223N+409\big)S_{2,1,1}-\frac{1}{8}\big(412N^2+2933N+4827\big)S_4\Bigg]$$

$$+\frac{32P_9(N+5)}{P_{49}}S_{2,1}-\frac{8P_{17}}{3P_{49}}S_3-\frac{2P_{41}}{P_{54}}+\frac{12P_{23}(N+4)(N+5)}{P_{55}}\zeta_2$$

$$+\frac{\big(288N^2+2487N+4805\big)}{(N+3)(N+4)(N+5)}S_2^2+\left[-\frac{96(-1)^NP_{20}}{P_{55}}+\frac{4P_{33}}{P_{55}(N+3)}-\frac{54\zeta_2}{(N+4)(N+5)}\zeta_2\right]S_2$$

$$+\left[\frac{1344(-1)^N}{(N+4)^2}+\frac{32P_5(N+4)(N+5)}{P_{49}}\right]S_{-2,1}\Bigg\}S_1+\Bigg[\frac{64\big(3N^2+26N+49\big)}{(N+3)(N+4)(N+5)}S_{2,1}$$

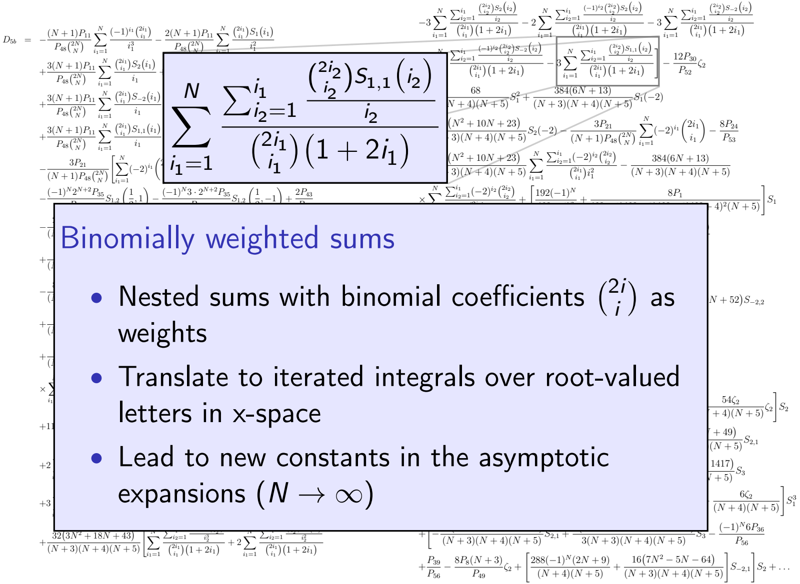
$$-\frac{16\big(7N^2+31N+44\big)}{(N+3)(N+4)(N+5)}S_{-2,1}-\frac{2P_{16}}{P_{49}}S_2+\frac{P_{38}}{P_{56}}-\frac{4\big(94N^2+777N+1417\big)}{(N+3)(N+4)(N+5)}S_3$$

$$-\frac{6P_2(N+3)}{P_{49}}\zeta_2\Bigg]S_1^2+\left[\frac{2\big(28N^2+123N+173\big)}{3(N+3)(N+4)(N+5)}S_2-\frac{4P_{28}}{3P_{55}(N+3)}+\frac{6\zeta_2}{(N+4)(N+5)}\right]S_1^3$$

$$+\Bigg[-\frac{32\big(35N^2+285N+542\big)}{(N+3)(N+4)(N+5)}S_{2,1}+\frac{4\big(1054N^2+7569N+12401\big)}{3(N+3)(N+4)(N+5)}S_3-\frac{(-1)^N6P_{36}}{P_{56}}$$

$$+\frac{P_{39}}{P_{56}}-\frac{8P_8(N+3)}{P_{49}}\zeta_2+\left[\frac{288(-1)^N(2N+9)}{(N+4)(N+5)}+\frac{16\big(7N^2-5N-64\big)}{(N+3)(N+4)(N+5)}\right]S_{-2,1}\Bigg]S_2+\ldots$$

Example: V diagram 12 - the result



$$\sum_{i_1=1}^{N} \frac{\sum_{i_2=1}^{i_1} \frac{\binom{2i_2}{i_2} S_{1,1}\left(i_2\right)}{i_2}}{\binom{2i_1}{i_1}\left(1+2i_1\right)}$$

Binomially weighted sums

- Nested sums with binomial coefficients $\binom{2i}{i}$ as weights
- Translate to iterated integrals over root-valued letters in x-space
- Lead to new constants in the asymptotic expansions ($N \to \infty$)

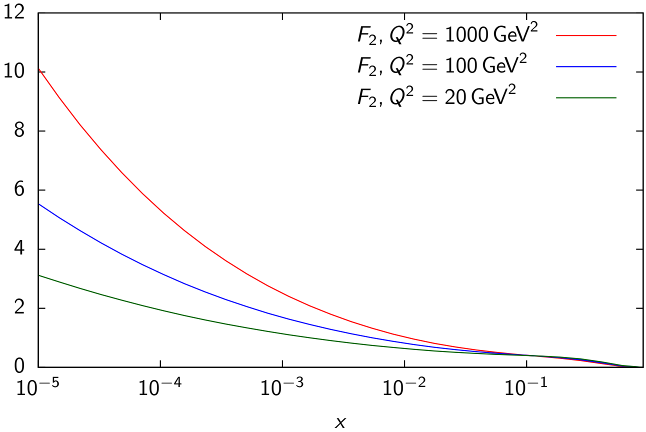
Conclusions

- Heavy quark corrections yield important contributions to DIS
  → essential for precision measurements
  of α_s (1%) and m_c (3%). [Alekhin et al. '12]
- New mathematical and computer-algebraic methods required for analytic calculation of the 3-loop corrections
  → includes new classes of higher transcendental functions and function spaces
- Completed Wilson coefficients and massive OMEs:
  - $L_{q,2}^{\mathrm{PS}}$, $L_{g,2}^{\mathrm{S}}$, $L_{q,2}^{\mathrm{NS}}$, $H_{q,2}^{\mathrm{PS}}$
  - $A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{qq,Q}^{\mathrm{TR}}$, A_{Qq}^{PS}, $A_{gq,Q}$
- Ingredients for first matching relation of the VFNS are complete.
- Physical ladder and V diagrams for A_{Qg} have been calculated.
- Calculation of the remaining massive OMEs A_{Qg} and $A_{gg,Q}$ and Wilson coefficient $H_{g,2}^{\mathrm{S}}$ is in progress.

Backup

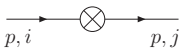
- The structure function $F_2 \rightarrow 17$
- Feynman rules for local operators $\rightarrow$ 18
- Non-planarity of diagram 12 $\rightarrow$ 19
- Operator matrix element $a_{Qq}^{\mathrm{PS},(3)} \rightarrow 20$

The structure function F_2

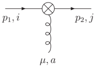


Kinematic limit at HERA: $x \geq \frac{Q^2}{yS}$, $S \sim 10^5\,\mathrm{GeV}^2$, $y \in [0, 1]$

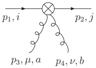
Feynman rules for local operators



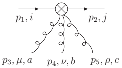
$$\delta^{ij}\not{\Delta}\gamma_\pm(\Delta\cdot p)^{N-1}\ , \quad N\geq 1$$



$$g t^a_{ji}\Delta^\mu\not{\Delta}\gamma_\pm \textstyle\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\ , \quad N\geq 2$$



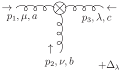
$$\begin{gathered}g^2\Delta^\mu\Delta^\nu\not{\Delta}\gamma_\pm \textstyle\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}\\ \Big[(t^a t^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^b t^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\Big]\ ,\\ N\geq 3\end{gathered}$$



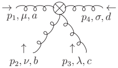
$$\begin{gathered}g^3\Delta^\mu\Delta^\nu\Delta^\rho\not{\Delta}\gamma_\pm \textstyle\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}\\ \Big[\ (t^a t^b t^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}\\ +(t^a t^c t^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}\\ +(t^b t^a t^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}\\ +(t^b t^c t^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\\ +(t^c t^a t^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}\\ +(t^c t^b t^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\ ,\\ N\geq 4\end{gathered}$$

$\gamma_+=1\ ,\quad \gamma_-=\gamma_5\ .$ For transversity, one has to replace: $\not{\Delta}\gamma_\pm\to\ \sigma^{\mu\nu}\Delta_\nu$.

$$\begin{gathered}\tfrac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}\\ \Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\ , \quad N\geq 2\end{gathered}$$



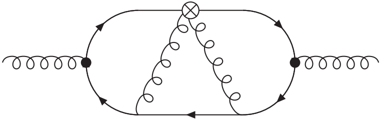
$$\begin{gathered}-ig\tfrac{1+(-1)^N}{2}f^{abc}\Bigg(\\ \Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}\\ +\Delta_\lambda\Big[\Delta\cdot p_1 p_{2,\mu}\Delta_\nu+\Delta\cdot p_2 p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2 g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]\\ \times\textstyle\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}\\ +\left\{\substack{p_1\to p_2\to p_3\to p_1\\ \mu\to\nu\to\lambda\to\mu}\right\}+\left\{\substack{p_1\to p_3\to p_2\to p_1\\ \mu\to\lambda\to\nu\to\mu}\right\}\Bigg)\ , \quad N\geq 2\end{gathered}$$



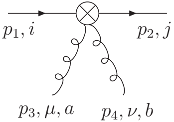
$$\begin{gathered}g^2\tfrac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)\\ +f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\ ,\\ O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}\\ +[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4 g_{\mu\sigma}]\textstyle\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}\\ -[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 g_{\mu\sigma}]\textstyle\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}\\ +[\Delta\cdot p_1\Delta\cdot p_4 g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4 p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 p_{4,\mu}\Delta_\sigma]\\ \times\textstyle\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\}\\ -\left\{\substack{p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu}\right\}-\left\{\substack{p_3\leftrightarrow p_4\\ \lambda\leftrightarrow\sigma}\right\}+\left\{\substack{p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4\\ \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma}\right\}\ , \quad N\geq 2\end{gathered}$$

Non-planarity of diagram 12

- The diagram is planar



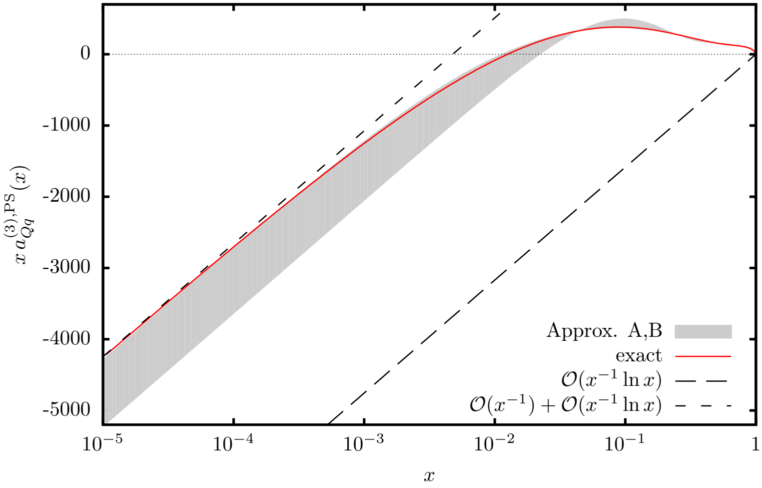
- The Feynman rule for the operator insertion reads



$$g^2\Delta^\mu\Delta^\nu\not{\Delta}\gamma_\pm \sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}$$
$$\left[(t^at^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^bt^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\right],$$
$$N \geq 3$$

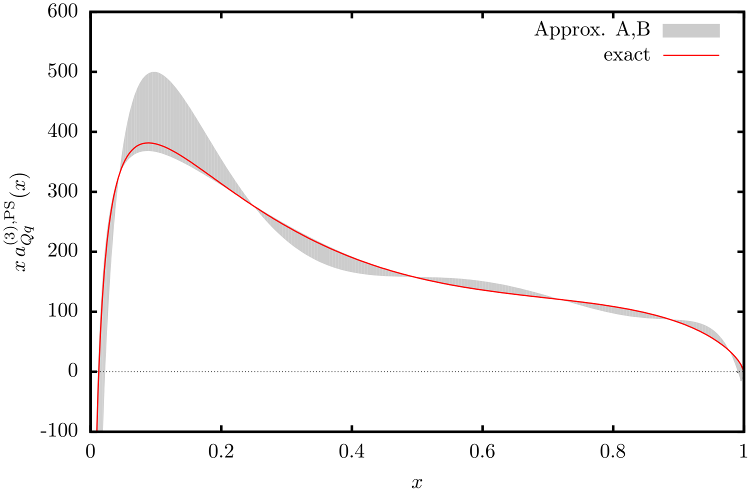
- $(p_1 + p_4)^{(l-j-1)}$ part mixes momenta from two different loops
- Shifting the momenta allows to put these mixed momenta into the operator or into one of the propagators, but one cannot get rid of this structure
- Effective non-planarity arises

Operator matrix element $a_{Qq}^{\mathrm{PS},(3)}$



Plot: [Ablinger et al. '14], approximate solution (grey band): [Kawamura et al. '12]

Operator matrix element $a_{Qq}^{\mathrm{PS},(3)}$



Plot: [Ablinger et al. '14], approximate solution (grey band): [Kawamura et al. '12]