The infrared structure of gauge theory amplitudes

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with **Einan Gardi, Mark Harley**, **Lorenzo Magnea, Chris White**

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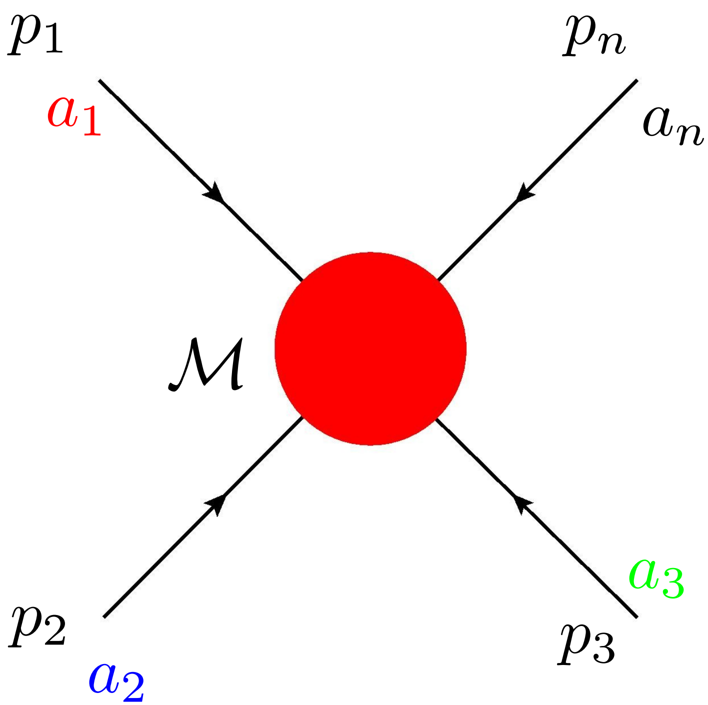
Gauge theory scattering amplitudes

We consider **fixed-angle** scattering amplitudes in SU(N) gauge theory:



- Fixed-angle:
  all the kinematic invariants of the same order

$$p_i \cdot p_j = \mathcal{O}\left(Q^2\right)$$
$$Q^2 \gg \Lambda^2_{QCD}$$

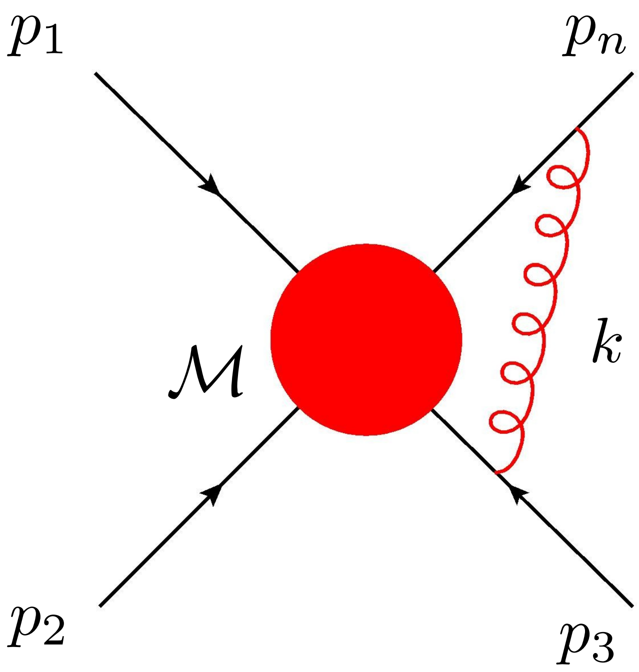
- **Vector** in colour space

The amplitude is a tensor with n indices, belonging to arbitrary representations of SU(N).
We decompose it on a basis of colour tensors with the same structure of indices

$$\mathcal{M}_{a_1,a_2,a_3,a_4} = \sum_{[i]} \mathcal{M}^{[i]} \quad c^{[i]}_{a_1,a_2,a_3,a_4}$$

Infrared singularities

The amplitude diverges when a subset of propagators goes on shell



- Soft gauge bosons

$$k^\mu = \mathcal{O}(\lambda Q) \qquad \mu = 1, .., 4$$

$$\lambda \ll 1$$

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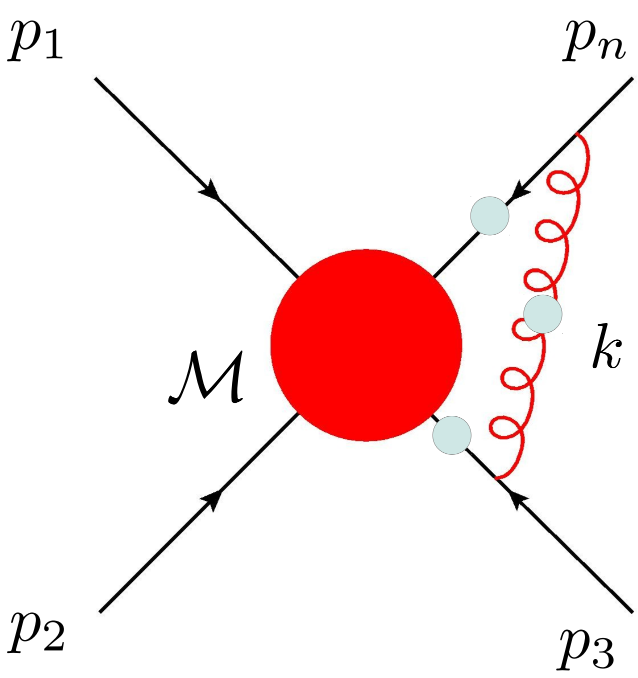

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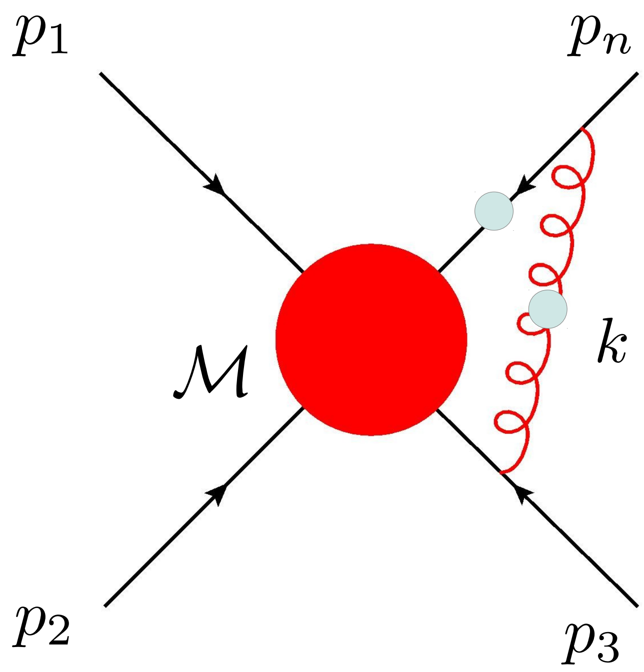

- Collinear splitting

$$p_n^\mu \simeq Q\,\delta^\mu{}_+ \qquad p_n^2 = 0$$

$$k^+ = \mathcal{O}\left(Q\right)$$
$$k^- = \mathcal{O}\left(\lambda Q\right)$$
$$k_\perp = \mathcal{O}\left(\sqrt{\lambda}Q\right)$$

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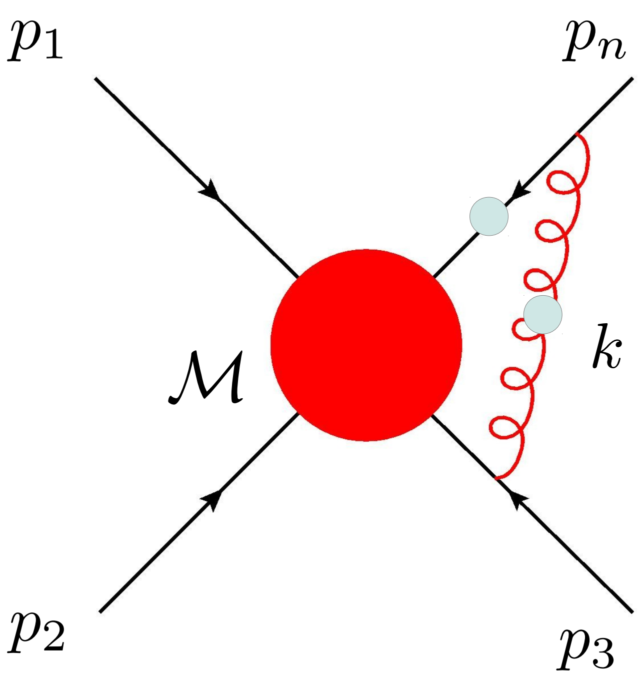

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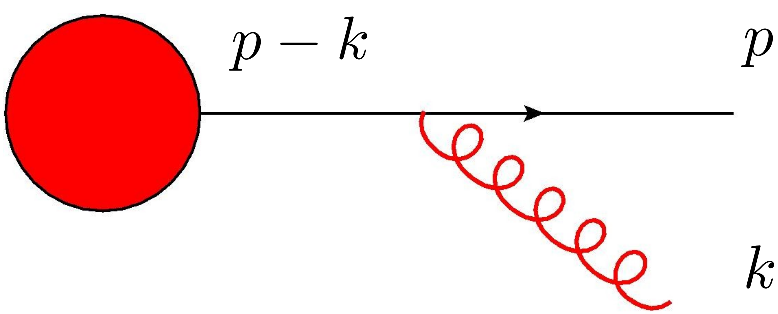
$$k_\perp = \mathcal{O}\left(\sqrt{\lambda} Q\right)$$

- Long distance singularities, they are not renormalised with local counterterms.
- It is necessary for phenomenological applications to study the pattern of these singularities.

Eikonal approximation

We focus on the soft divergences of scattering amplitudes by using the *eikonal approximation*



$$k \ll 1$$

$$\overline{u}(p)\,(ig_s\gamma^\mu T^a)\ \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}$$

$$\overline{u}(p) \times \boxed{g_s T^a\ \frac{\beta^\mu}{\beta\cdot k}} \qquad p^\mu = Q\beta^\mu$$

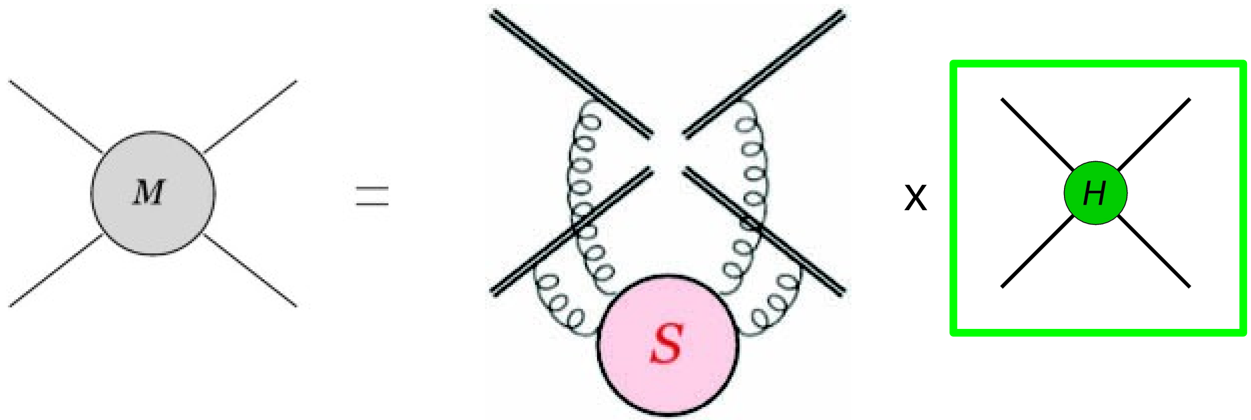
- Independent of the **energy** and the **spin** of the hard parton
- The soft gluon is sensitive only to the direction and colour charge

External fields are replaced by Wilson lines

$$\Phi_\beta(\infty,0) = \mathcal{P}\exp\left[ig_s T^a\int_0^\infty d\sigma\,\beta^\mu\,A_\mu(\sigma\beta)\right]$$

Factorisation

- Amplitude with **massive** external partons: the singularity structure is depicted as



Finite hard vector encoding the **short distance** dynamics

$$S\left(\gamma_{ij}, \alpha_s(\mu^2), \mu, \epsilon\right) = \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) \, |0\rangle \qquad \gamma_{ij} = \frac{\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}}$$

Generates the singularities of the amplitude

For massless external partons we must introduce jet functions to describe

- Renormalisation group equation

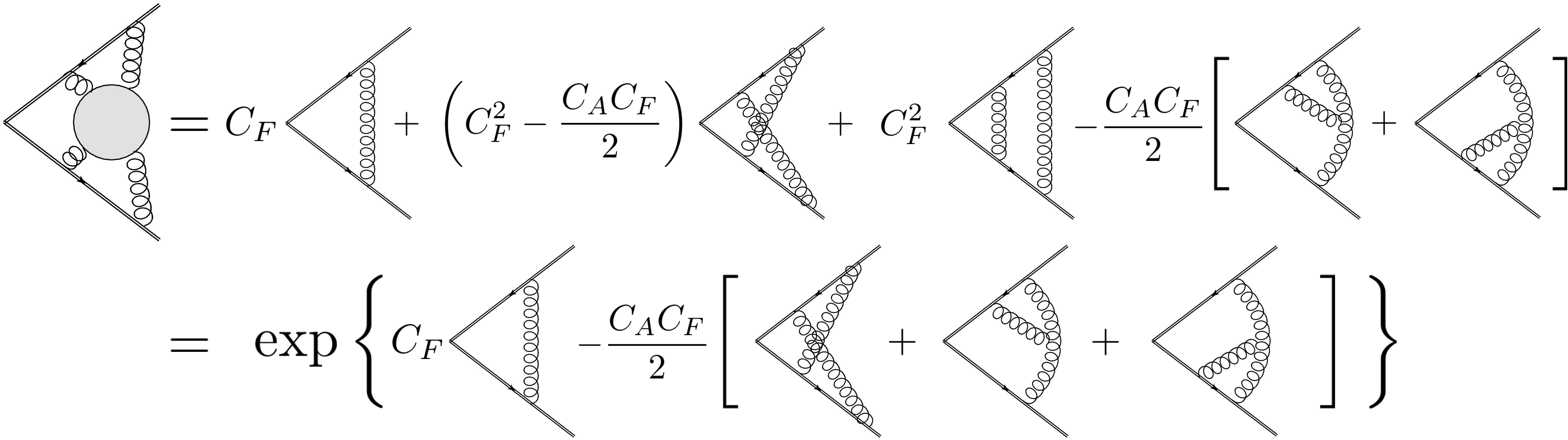
$$\mu \frac{d}{d\mu} S\left(\gamma_{ij}, \alpha_s, \mu, \epsilon\right) = -S\left(\gamma_{ij}, \alpha_s, \mu, \epsilon\right) \times \boxed{\Gamma\left(\gamma_{ij}, \alpha_s\right)}$$

Soft anomalous dimension matrix

The **soft anomalous dimension** governs the pattern of infrared divergences to all orders

Webs and non abelian exponentiation

Correlators of Wilson lines are obtained by exponentiating a subset of Feynman diagrams with modified colour factors, called webs: the simplest example is with two lines



(Sterman '81, Gatheral, Frenkel, Taylor '83)

The proof of exponentiation in the multi line case is more recent (Gardi, Laenen, Stavenga, White '10, Mitov, Sterman, Sung '10)

$$S(\beta_i \cdot \beta_j, \alpha_s(\mu^2)) = \langle 0| \prod_{i=1}^{n} \mathcal{W}_{\beta_i}(\infty, 0) |0\rangle \equiv \exp\left[w\right] = \exp\left[\sum_{D,D'} \mathcal{F}(D)\, \mathcal{R}_{DD'}\, \mathcal{C}(D^{'})\right]$$

Kinematic factor of diagram D

Mixing matrix of
combinatoric origin

Colour factor of diagram D'

The soft anomalous dimension

We can compute the soft anomalous dimension directly in terms of webs
(Gardi, Smillie, White '11)

$$S = \exp\left[w\right] = \exp\left[\sum_{n=0}^{\infty}\sum_{k=-n}^{\infty} w^{(n,k)}\right]$$

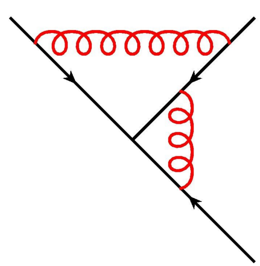
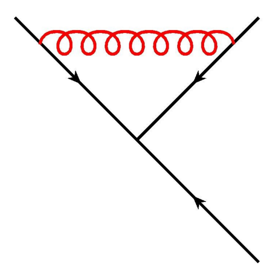
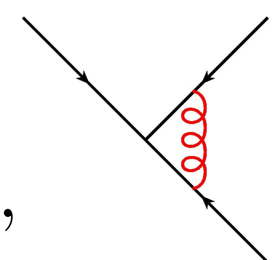
$$\Gamma^{(1)} = -2w^{(1,-1)}$$

$$\Gamma^{(2)} = -4w^{(2,-1)} - 2\left[w^{(1,-1)}, w^{(1,0)}\right]$$

$$\Gamma^{(3)} = -6w^{(3,-1)} + \frac{3}{2}b_0\left[w^{(1,-1)}, w^{(1,1)}\right] + 3\left[w^{(1,0)}, w^{(2,-1)}\right] + 3\left[w^{(2,0)}, w^{(1,-1)}\right] + \left[w^{(1,0)}, \left[w^{(1,-1)}, w^{(1,0)}\right]\right] + \left[w^{(1,-1)}, \left[w^{(1,-1)}, w^{(1,0)}\right]\right]$$

We define **subtracted webs** which directly include the contribution of the commutators of the decompositions (Gardi '13)

$$\boxed{\Gamma^{(n)} \equiv -2n\,\overline{w}^{(n,-1)}}$$

$\overline{w}^{(2,-1)} =$  $+\frac{1}{2}\Big[$  ,  $\Big]$

Multiple gluon exchange webs

We focus on the webs involving only multiple exchanges of single gluon propagators (MGEWs). Three-loop MGEWs depicted below recently computed (Gardi '13)

- Simplest contributions to the correlator of 4 lines, definite procedure for the integrals of the kinematic coefficients (Gardi, Smillie, White '11, Gardi '13)
- Interesting contribution to the anomalous dimension, involving multi-parton correlations
- Relevant physical information in the single pole of **subtracted webs** (Gardi '13)

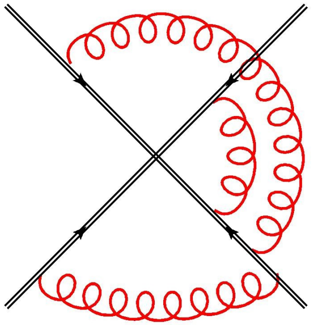
$$\overline{w}^{(n)} = \sum_j F_{w,j}^{(n)}\left(\gamma_{ij}\right)\, c^{[j]}$$

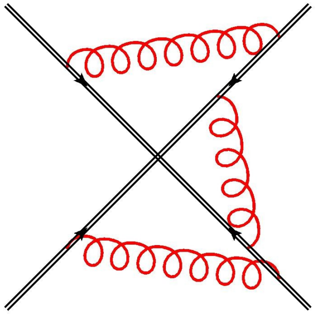
Subtracted web kernel

$$F_{w,j}^{(n)}\left(\gamma_{ij}\right) = \int_0^1 \left[\prod_{k=1}^{n} dx_k\, p_0(x_k, \alpha_k)\right] \boxed{\mathcal{G}_{W,\,j}^{(n)}\Big(x_i, q(x_i, \alpha_i)\Big)}$$

$$p_0(x,\alpha) \equiv \left(q(x,\alpha)\right)^{-1} = \left[x^2 + (1-x)^2 + \left(\alpha + \frac{1}{\alpha}\right)\right]^{-1}$$

$$\gamma_k = -\left(\alpha_k + \frac{1}{\alpha_k}\right)$$





Structure of MGEWs

The web kernels are purely logarithmic

$$\mathcal{G}_k\left(x_i,\alpha_i\right)=\mathcal{G}_k\Bigg(\log(x_i),\log\left(q(x_j,\alpha_j)\right)\Bigg)$$

The integrals factorise and the functions in the kinematic coefficients depend on a **single cusp angle**

- **Factorisation conjecture**: the result is a sum of products of polylogarithms involving a single cusp angle (Gardi '13)

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The analytic dependence is encoded by the variable α_i

$$\frac{2\beta_i \cdot \beta_j + i0}{\sqrt{\beta_i^2 - i0}\sqrt{\beta_i^2 - i0}} = -\left(\alpha + \frac{1}{\alpha}\right)$$

$\alpha \to 1 = \text{Straight line limit}$

$\alpha \to 0 = \text{Massless limit}$

$\alpha \to -1 = \text{Threshold}$

$\alpha \to -\alpha$ Crossing symmetry from spacelike kinematics to timelike kinematics
Discontinuity of the functions

- **Alphabet conjecture**: the **symbol** of each function appearing in the subtracted web have a **restricted alphabet** (Gardi '13)

$$\left\{\alpha_k, \eta_k \equiv 1 - \alpha_k^2\right\}$$

A basis of functions

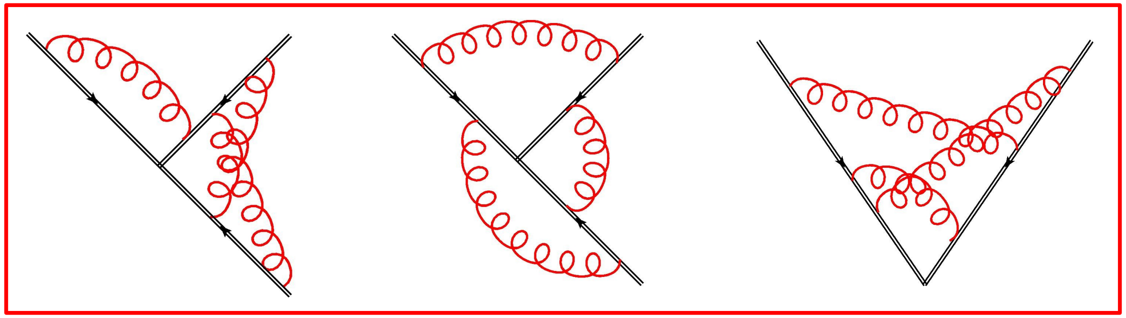
Basis of functions involving only one cusp angle, which respects the alphabet conjecture (Gardi, Harley, Magnea, White, GF '14)

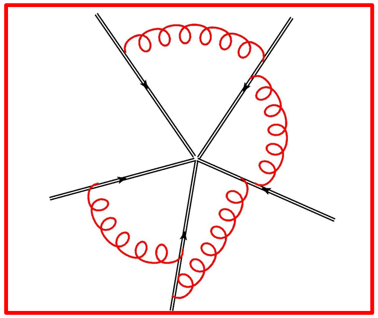
$$M_{k,l,n}(\alpha) = \frac{1}{r(\alpha)} \int_0^1 dx\, p_0(x,\alpha)\, \log^k\left(\frac{q(x,\alpha)}{x^2}\right) \log^l\left(\frac{x}{1-x}\right) \log^n\left(\tilde{q}(x,\alpha)\right)$$

$$r(\alpha) = \frac{1+\alpha^2}{1-\alpha^2}\ ,\ \tilde{q}(x,\alpha) = \log\frac{1-(1-\alpha)x}{1+\frac{1-\alpha}{\alpha}x}$$

Alphabet: $\left\{\alpha, \frac{\alpha}{1-\alpha^2}\right\}$

All the MGEWs at two and three loops are written in terms of products of these functions. We also checked a four loop web connecting five lines.

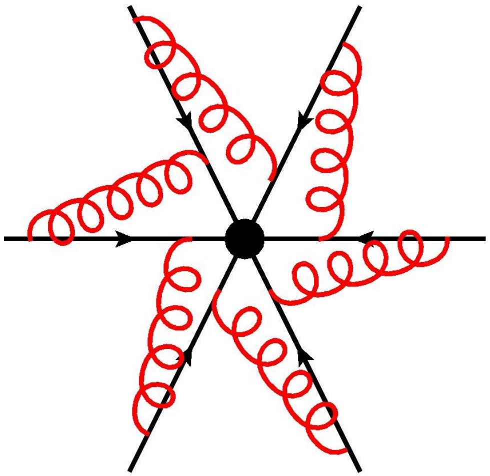




(Gardi, Harley, Magnea, White, GF '14)

Towards MGEWs to all orders

The simplicity of MGEWs allows to explore highly symmetric configurations at higher order.



“Escher staircase” diagram: enters in the MGEW connecting n legs at n loops

$$\mathcal{G}_S(x_i) = \frac{1}{(n-1)!} \left[\log\Big(S_n(x_i)\Big)\right]^{n-1} \theta(S_n(x_i)-1)$$

$$S_n(x_i) = \prod_{i=1}^{n} \log\left(\frac{x_i}{1-x_i}\right)$$

It combines with the “anti-staircase”, having opposite chirality $\mathcal{G}_{AS}(x_i) = (-1)^{n-1}\,\mathcal{G}_S(x_i)$
Their contribution to antisymmetric colour structure of the web vanishes to all orders.

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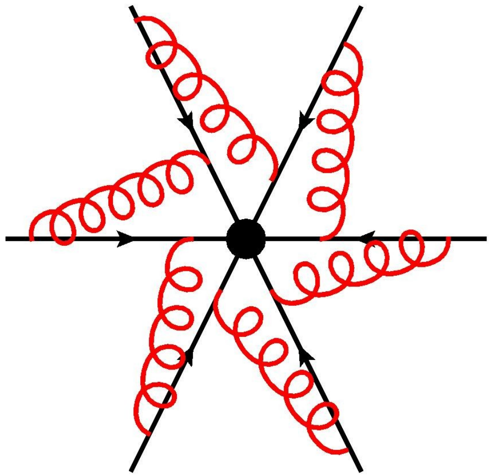

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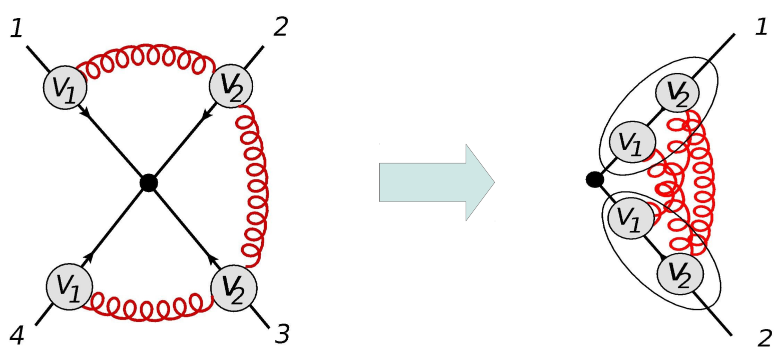
Their contribution to antisymmetric colour structure of the web vanishes to all orders.

Collinear reduction: by computing collinear limits of a web, we get the contributions to webs spanning fewer Wilson lines. Below, we identify lines 1,3 and 2,4.



Only a **restricted number** of **webs** contains the relevant information to compute webs to a given perturbative order.

Conclusion

- Correlators of Wilson lines are an essential tool to understand the pattern of infrared divergences of gauge theory scattering amplitudes.

- The non-Abelian exponentiation theorem organizes the calculation in terms of **webs**. We focus on a specific class of webs (MGEWs), with simple analytic structure, encoded by the **factorisation** and **alphabet conjectures**.

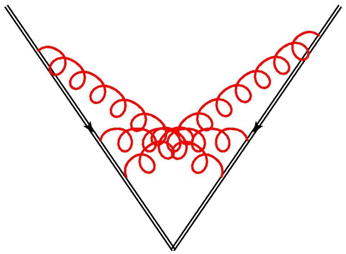
- It is possible to identify a **basis of functions**, which is sufficient to write all the known MGEWs up to three loops and one web at four loops.

- Does the same basis hold for generic MGEWs? Is it possible to prove the two conjectures?

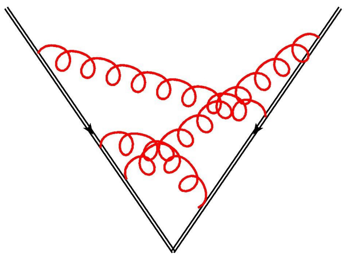
- The combined use of the basis and of the collinear reduction procedure suggests that the problem of computing MGEWs to all orders could be solved at some point.

Back up slides

Results for three-loop webs



$$\mathcal{G}^{(3)}_{(3,3),\,a}\,(x,y,z) \;=\; -\frac{4}{3}\ln^2\left(\frac{x}{1-x}\,\frac{1-z}{z}\right)\theta(z-x)\,\theta(y-z)\,,$$



$$\mathcal{G}^{(3)}_{(3,3),\,b}\,(x,y,z) \;=\; -\frac{4}{3}\ln\left(\frac{x}{1-x}\,\frac{1-y}{y}\right)\ln\left(\frac{y}{1-y}\,\frac{1-z}{z}\right)\theta(y-x)\,\theta(z-y)\,.$$

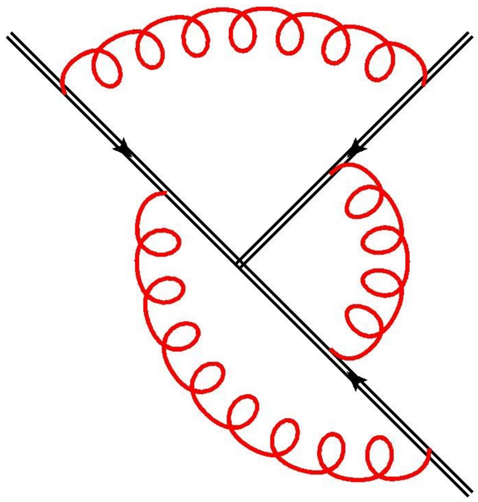
It's convenient to group these diagrams according to a particular colour basis

$$\begin{aligned} F^{(3)}_{(V_1V_2)_+(V_1V_2)_+}(\alpha) &= 2\,F^{(3)}_{(3,3),a}(\alpha) + F^{(3)}_{(3,3),b}(\alpha)\,, \\ F^{(3)}_{V_3V_3}(\alpha) &= F^{(3)}_{(3,3),a}(\alpha) + \tfrac{3}{2}F^{(3)}_{(3,3),b}(\alpha)\,, \end{aligned}$$

Giving the final result

$$\begin{aligned} F^{(3)}_{(V_1V_2)_+(V_1V_2)_+}(\alpha) &= -\frac{2}{3}\,r^3(\alpha)\,M_{0,2,0}(\alpha)\,M^2_{0,0,0}(\alpha) \\ F^{(3)}_{V_3V_3}(\alpha) &= -\frac{4}{3}\,r^3(\alpha)\left[\frac{1}{4}M^2_{0,0,0}(\alpha)M_{2,0,0}(\alpha) - \frac{1}{4}M_{0,0,0}(\alpha)M^2_{1,0,0}(\alpha) + M_{0,0,0}(\alpha)M_{1,1,1}(\alpha)\right. \\ &\left. - M_{0,1,1}(\alpha)M_{1,0,0}(\alpha) + \frac{3}{2}M_{0,2,2}(\alpha) - \frac{1}{4}M^2_{0,0,0}(\alpha)M_{0,2,0}(\alpha) + \frac{1}{48}M^5_{0,0,0}(\alpha)\right]. \end{aligned}$$

Results for three-loop webs



Three colour structures are present in this web. The web kernels are written in terms of

$$L_{ij} \equiv \log\left(\frac{q(x_i,\alpha_{ij})}{x_i^2}\right) ; \qquad R_i \equiv \log\left(\frac{x_i}{1-x_i}\right) .$$

$$\begin{aligned}
\mathcal{G}^{(3)}_{(2,2,2),\,1} &= \frac{1}{3}\left[R_2^2 - \frac{1}{4}L_{23}^2 + \frac{1}{8}L_{12}^2 + \frac{1}{8}L_{31}^2 + \frac{1}{4}L_{12}L_{23} - \frac{1}{2}L_{31}L_{12} + \frac{1}{4}L_{23}L_{31}\right], \\
\mathcal{G}^{(3)}_{(2,2,2),\,2} &= \frac{1}{3}\left[R_3^2 - \frac{1}{4}L_{31}^2 + \frac{1}{8}L_{23}^2 + \frac{1}{8}L_{12}^2 + \frac{1}{4}L_{23}L_{31} - \frac{1}{2}L_{12}L_{23} + \frac{1}{4}L_{31}L_{12}\right], \\
\mathcal{G}^{(3)}_{(2,2,2),\,3} &= -\frac{1}{3}\left[R_1^2 - \frac{1}{4}L_{12}^2 + \frac{1}{8}L_{23}^2 + \frac{1}{8}L_{31}^2 + \frac{1}{4}L_{31}L_{12} - \frac{1}{2}L_{23}L_{31} + \frac{1}{4}L_{12}L_{23}\right].
\end{aligned}$$

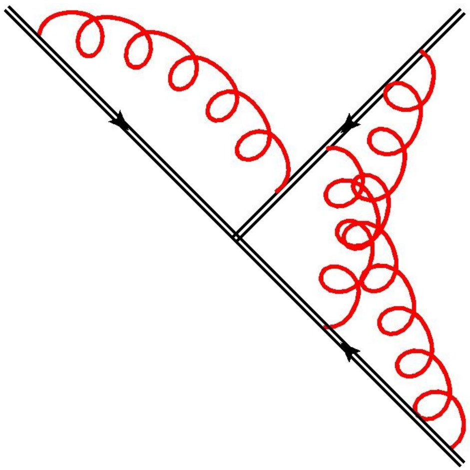
The integrated result is

$$
\begin{aligned}
F^{(3)}_{(2,2,2),\,1}(\alpha_{12},\alpha_{23},\alpha_{13}) \quad = \quad & \frac{1}{3}\, r(\alpha_{12}) r(\alpha_{23}) r(\alpha_{13}) \ \times \\
& \Bigg[-\, M_{0,0,0}(\alpha_{12}) M_{0,0,0}(\alpha_{13}) \left(\frac{1}{4}\, M_{2,0,0}(\alpha_{23}) -\, M_{0,2,0}(\alpha_{23}) \right) \\
& + \frac{1}{8}\, M_{0,0,0}(\alpha_{13}) M_{0,0,0}(\alpha_{23}) M_{2,0,0}(\alpha_{12}) \\
& + \frac{1}{8}\, M_{0,0,0}(\alpha_{12}) M_{0,0,0}(\alpha_{23}) M_{2,0,0}(\alpha_{13}) - \frac{1}{2}\, M_{0,0,0}(\alpha_{23}) M_{1,0,0}(\alpha_{12}) M_{1,0,0}(\alpha_{13}) \\
& + \frac{1}{4}\, M_{0,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{12}) M_{1,0,0}(\alpha_{23}) + \frac{1}{4}\, M_{0,0,0}(\alpha_{12}) M_{1,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{23}) \Bigg] .
\end{aligned}
$$

Where the remaining colour structures are obtained by symmetry

$$
\begin{aligned}
F^{(3)}_{(2,2,2),\,2}(\alpha_{12},\alpha_{23},\alpha_{13}) \quad &= \quad F^{(3)}_{(2,2,2),\,1}(\alpha_{23},\alpha_{13},\alpha_{12}) \ , \\
F^{(3)}_{(2,2,2),\,3}(\alpha_{12},\alpha_{23},\alpha_{13}) \quad &= \quad -\,F^{(3)}_{(2,2,2),\,1}(\alpha_{13},\alpha_{12},\alpha_{23}) \ ,
\end{aligned}
$$

Results for three-loop webs



Web kernels for the three colour structures.

$$\mathcal{G}^{(3)}_{(123),\,2} = \frac{1}{3}\left[\frac{1}{8}L_{13}^2 - \frac{1}{8}L_{23}^2 + \frac{1}{4}L_{23}L_{32} - \frac{1}{4}L_{13}L_{23}\right],$$

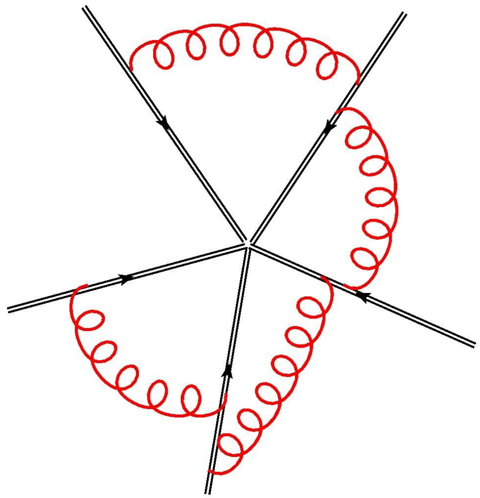
$$\mathcal{G}^{(3)}_{(123),\,3} = -\frac{1}{3}\left[\frac{1}{4}L_{23}L_{13} - \frac{1}{4}L_{23}L_{32} + \frac{1}{8}L_{23}^2 - \frac{1}{8}L_{13}^2 - R_2^2\right],$$

$$\mathcal{G}^{(3)}_{(1,2,3),4} = \frac{2}{3}\,\theta(x_2 - x_3)\Bigg[2\,L_{13}R_2 + L_{23}\,(R_3 - R_2) - \log^2\left(\frac{x_2}{x_3}\right) + \log\left(\frac{x_2}{x_3}\right)\log\left(\frac{1-x_2}{1-x_3}\right)\Bigg],$$

Again, the result is expressed in terms of the basis

$$
\begin{aligned}
F^{(3)}_{(1,2,3),\,2}(\alpha_{13},\alpha_{23}) \quad &= \quad \frac{1}{12}\, r(\alpha_{13}) r^2(\alpha_{23}) \Bigg[\frac{1}{2}\, M_{2,0,0}(\alpha_{13}) M^2_{0,0,0}(\alpha_{23}) \\
&- \frac{1}{2}\, M_{2,0,0}(\alpha_{23}) M_{0,0,0}(\alpha_{13}) M_{0,0,0}(\alpha_{23}) + M_{0,0,0}(\alpha_{13}) M^2_{1,0,0}(\alpha_{23}) \\
&- M_{0,0,0}(\alpha_{23}) M_{1,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{23}) \Bigg]\,, \\
F^{(3)}_{(1,2,3),\,3}(\alpha_{13},\alpha_{23}) \quad &= \quad -\frac{1}{12}\, r(\alpha_{13}) r^2(\alpha_{23}) \Bigg[-\frac{1}{2}\, M_{2,0,0}(\alpha_{13}) M^2_{0,0,0}(\alpha_{23}) \\
&+ \frac{1}{2} \Big(M_{2,0,0}(\alpha_{23}) - 8 M_{0,2,0}(\alpha_{23}) \Big) M_{0,0,0}(\alpha_{13}) M_{0,0,0}(\alpha_{23}) \\
&- M_{0,0,0}(\alpha_{13}) M^2_{1,0,0}(\alpha_{23}) + M_{0,0,0}(\alpha_{23}) M_{1,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{23}) \Bigg]\,, \\
F^{(3)}_{(1,2,3),\,4}(\alpha_{13},\alpha_{23}) \quad &= \quad \frac{4}{3}\, r(\alpha_{13}) r^2(\alpha_{23}) \Bigg[M_{0,1,1}(\alpha_{23}) M_{1,0,0}(\alpha_{13}) \\
&+ \frac{1}{8} \Big(M^2_{1,0,0}(\alpha_{23}) - M_{0,0,0}(\alpha_{23}) M_{2,0,0}(\alpha_{23}) - \frac{1}{12}\, M^4_{0,0,0}(\alpha_{23}) \\
&+ 2\, M_{0,0,0}(\alpha_{23}) M_{0,2,0}(\alpha_{23}) \Big) M_{0,0,0}(\alpha_{13}) \Bigg]\,.
\end{aligned}
$$

Result of a four-loop web



Only one colour structure: the corresponding web kernel is

$$
\begin{aligned}
\mathcal{G}^{(4)}_{(1,2,2,2,1),\,1}\big(x_i, q(x_i,\alpha_i)\big) \quad = \quad & -\frac{1}{144}\Bigg\{ L_{12}^3 - 3L_{23}^3 + 3L_{34}^3 - L_{45}^3 \\
& + 3L_{12}^2\Big[L_{23} + L_{34} - 3L_{45}\Big] - 3L_{45}^2\Big[L_{23} + L_{34} - 3L_{12}\Big] \\
& + 3L_{23}^2\Big[L_{12} - 3L_{34} + 5L_{45}\Big] - 3L_{34}^2\Big[L_{45} - 3L_{23} + 5L_{12}\Big] \\
& + 6\Big[L_{12}L_{23}L_{34} - 3L_{12}L_{23}L_{45} + 3L_{12}L_{34}L_{45} - L_{23}L_{34}L_{45}\Big] \\
& + 24\Big[R_2^2\Big(L_{12} + L_{23} + L_{34} - 3L_{45}\Big) - R_3^2\Big(L_{23} + L_{34} + L_{45} - 3L_{12}\Big)\Big]\Bigg\}\,.
\end{aligned}
$$

Four-loop web result

$$
\begin{aligned}
F^{(4)}_{(1,2,2,2,1),1}(\alpha_{ij}) = &-\frac{1}{144}\, r(\alpha_{12})r(\alpha_{23})r(\alpha_{34})r(\alpha_{45}) \times \\
\times\Bigg\{ \Bigg[6\bigg(& M_{1,0,0}(\alpha_{12})M_{1,0,0}(\alpha_{23})M_{1,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{45}) \\
&- 3M_{1,0,0}(\alpha_{12})M_{1,0,0}(\alpha_{23})M_{1,0,0}(\alpha_{45})M_{0,0,0}(\alpha_{34})\bigg) \\
&+ \bigg(M_{3,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) - 9M_{2,0,0}(\alpha_{12})M_{1,0,0}(\alpha_{45})\bigg)M_{0,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34}) \\
&- 3\bigg(M_{3,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34}) + 3M_{2,0,0}(\alpha_{23})M_{1,0,0}(\alpha_{34})\bigg)M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) \\
&+ 3M_{2,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45})\bigg(M_{1,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34}) + M_{1,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{23})\bigg) \\
&+ 3M_{2,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34})\bigg(M_{1,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) + 5M_{1,0,0}(\alpha_{45})M_{0,0,0}(\alpha_{12})\bigg) \\
&+ 24M_{0,2,0}(\alpha_{23})\bigg(M_{1,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{45}) \\
&+M_{1,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) - 3M_{1,0,0}(\alpha_{45})M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{34})\bigg) + \\
&24M_{1,2,0}(\alpha_{23})M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{45})\Bigg] \\
&- \Big[(\alpha_{12} \leftrightarrow \alpha_{45}), (\alpha_{23} \leftrightarrow \alpha_{34})\Big]\Bigg\}.
\end{aligned}
$$