Chaos Removal in $R+qR^2$ gravity: the Mixmaster Model

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Table of contents

- A Brief view on the extended theories of gravity, in particular on the scalar-tensor picture.
- The introduction of homogeneous models, with a focus on the Bianchi IX model and its features.
- Create a Link between extended theories and homogeneus models, in particular by studying the Mixmaster model in the R^2-gravity case.
- For this model we show the point-Universe trajectories in order to analyse if a quadratic correction in Einstein-Hilbert Lagrangian modify the nature of the cosmological singularity.

f(R) Gravity

The f(R) theories of gravity are a generalization of the Einstein-Hilbert Lagrangian consisting in a replacement of the Ricci Scalar R by a general function f(R)

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R)$$

- Introduction of the field χ
- Variation respect to the new field imply $\chi = R$
- Field redefinition $\varphi = f'(\chi)$

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}[\varphi R - \chi(\varphi)\varphi + f(\chi(\varphi))]$$ Jordan Frame

- Conformal substitution $g_{\mu\nu} \to \tilde{g_{\mu\nu}} = \varphi g_{\mu\nu}$
- Scalar redefinition $\varphi \to \phi = \sqrt{\frac{3}{16\pi}}\ln f'(R)$

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - U(\phi)\right]$$ Einstein Frame

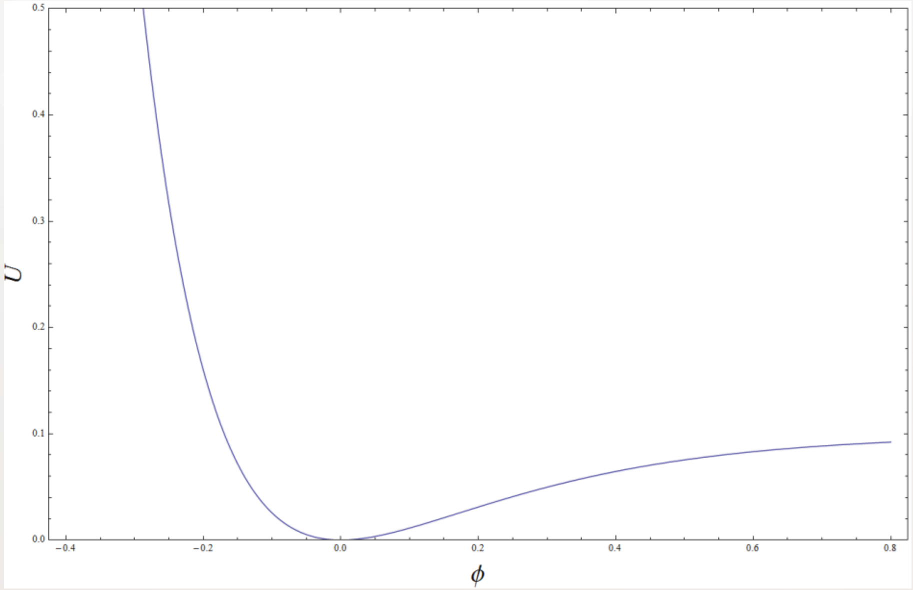
$$U(\phi) = \frac{Rf'(R) - f(R)}{16\pi(f'(R))^2}$$

f(R) Gravity

For small values of the Ricci scalar the first order correction to the Einstein-Hilbert Lagrangian is represented by a quadratic correction

$$f(R) = R + qR^2 \quad \longrightarrow \quad U(\phi) = \frac{1}{64\pi q}\left(1 - 2\exp^{-4\sqrt{\frac{\pi}{3}}\phi} + \exp^{-8\sqrt{\frac{\pi}{3}}\phi}\right)$$

This is the effective potential that emerges in the so called Starobinsky-inflation model



The shape of the scalar field polentіal term

Bianchi classification

The spatial line element is preserved under the transformation $x \rightarrow x'$

$$dl^2 = h_{\alpha\beta}(t,x)dx^\alpha dx^\beta = h_{\alpha\beta}(t,x')dx'^\alpha dx'^\beta = dl'^2$$

Vacuum Einstein's equations

$$-R_l^l = \frac{(\dot{a}bc)^\cdot}{abc} + \frac{1}{2(abc)^2}\left[\lambda_l^2 a^4 - (\lambda_m b^2 - \lambda_n c^2)^2\right] = 0$$

$$-R_m^m = \frac{(a\dot{b}c)^\cdot}{abc} + \frac{1}{2(abc)^2}\left[\lambda_m^2 b^4 - (\lambda_l a^2 - \lambda_n c^2)^2\right] = 0$$

$$-R_n^n = \frac{(ab\dot{c})^\cdot}{abc} + \frac{1}{2(abc)^2}\left[\lambda_n^2 c^4 - (\lambda_l a^2 - \lambda_m b^2)^2\right] = 0$$

$$-R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0$$

Type	**a**	$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{n_3}$
I	0	0	0	0
II	0	1	0	0
VII	0	1	1	0
VI	0	1	-1	0
IX	**0**	**1**	**1**	**1**
VIII	**0**	**1**	**1**	**-1**
V	1	0	0	0
IV	1	0	0	1
VII_a	a	0	1	1
III $(a=1)$ / VI_a $(a \neq 1)$	a	0	1	-1

The Bianchi Classification

Bianchi I and Bianchi IX models

$(\lambda_l, \lambda_m, \lambda_n) = (0, 0, 0)$

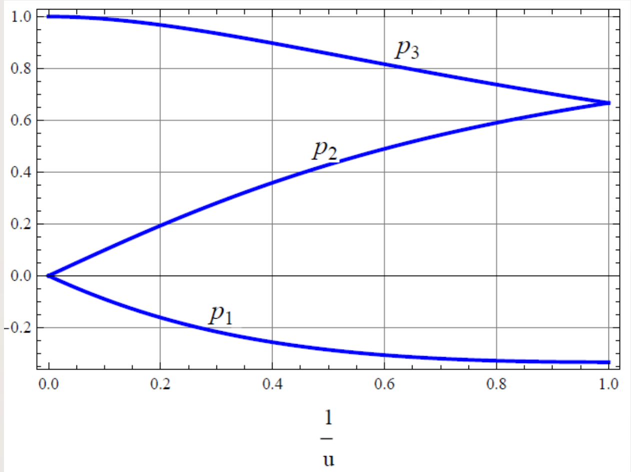
Spatial line element

$$dl^2 = t^{2p_1}(dx^1)^2 + t^{2p_2}(dx^2)^2 + t^{2p_3}(dx^3)^2$$

Kasner relations

$$p_1 + p_2 + p_3 = 1$$

$$p_1^2 + p_2^2 + p_3^2 = 1$$



Behaviour of Kasner indexes respect to the inverse of parameter u

$(\lambda_l, \lambda_m, \lambda_n) = (1, 1, 1)$

Logarithmic variables

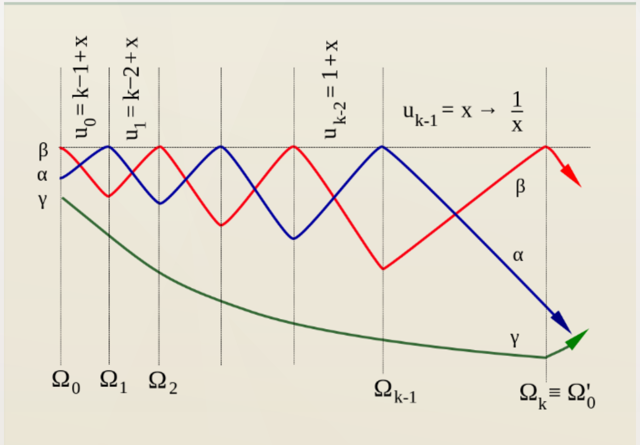
$$\alpha = \ln a \quad \beta = \ln b \quad \gamma = \ln c \quad dt = (abc)d\tau$$

Einstein's Equations

$$2\alpha_{\tau\tau} = (b^2 - c^2)^2 - a^4$$

$$2\beta_{\tau\tau} = (a^2 - c^2)^2 - b^4$$

$$2\gamma_{\tau\tau} = (a^2 - b^2)^2 - c^4$$



Picture of oscillatory behaviour

Hamiltonian formulation

Misner variables $(\alpha, \beta_+, \beta_-)$

$$\alpha = \frac{1}{3}\ln t$$

$$\beta_+ = \frac{1-p_3}{6}\ln t = \frac{1-p_3}{2}\alpha$$

$$\beta_- = \frac{p_1 - p_2}{2\sqrt{3}}\ln t = \sqrt{3}(p_1 - p_2)\alpha$$

Bianchi IX Line element

$$ds^2 = N(t)^2 dt^2 - \eta_{ab}(\alpha, \beta_\pm)\omega^a\omega^b$$

Isotropic component of the Universe

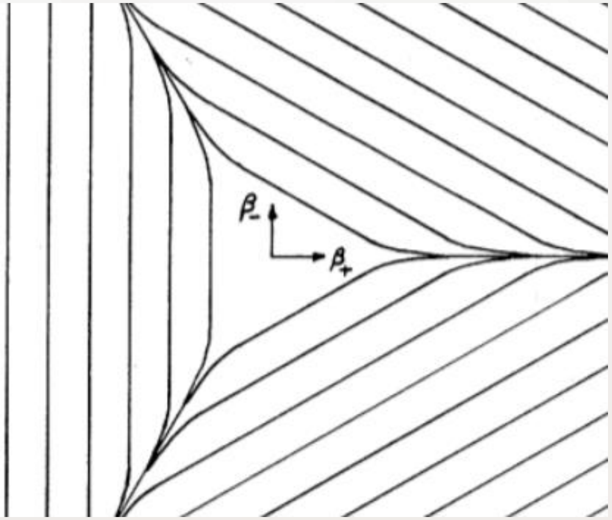
Anisotropies of the Universe

Bianchi IX SuperHamiltonian constrain

$$\mathcal{H}^s = -p_\alpha^2 + p_+^2 + p_-^2 + 12\pi^2 e^{4\alpha} V(\beta_\pm) = 0$$

Reduced Hamiltonian

$$-p_\alpha = H_{ADM} \equiv \sqrt{p_+^2 + p_-^2 + 12\pi^2 e^{4\alpha} V(\beta_\pm)}$$



Equipotential lines of V(β±)

Mixmaster dynamics

$$V(\beta_\pm) = e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+}\left[\cosh(4\sqrt{3}\beta_-) - 1)\right]$$

- $V(\beta_\pm) = 0$ (Bianchi I)

$$H_{ADM} = \sqrt{p_+^2 + p_-^2}$$

$$\beta'_\pm = \frac{\partial H_{ADM}}{\partial p_\pm} = \frac{p_\pm}{H_{ADM}}$$

$$p'_\pm = -\frac{\partial H_{ADM}}{\partial \beta_\pm} = 0$$

$$\beta' = \sqrt{(\beta'_+)^2 + (\beta'_-)^2} = 1$$

- $V(\beta_\pm) = e^{-8\beta_+}$ (Bianchi II)

$$H_{ADM} = \sqrt{p_+^2 + p_-^2 + 12\pi^2 e^{4\alpha - 8\beta_+}}$$

$$\begin{cases} \alpha \to -\infty \\ V >> 1 \end{cases} \Rightarrow H_{ADM}^{-2} e^{4\alpha - 8\beta_+} \simeq 1 \to \beta_+ \simeq \beta_{wall} = \frac{1}{2}\alpha - \frac{1}{8}\ln\left(\frac{H_{ADM}^2}{12\pi^2}\right)$$

$$|\beta'_{wall}| = \frac{1}{2}$$

The particle always collides on the walls. The oscillatory behaviour of the Bianchi IX model is map in a never-ending bouncing of the point-Universe against the walls

Misner-Cithrè variables

$$V(\beta_\pm) = e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+}\left[\cosh(4\sqrt{3}\beta_-) - 1)\right]$$

Misner variables

$$(\alpha, \beta_+, \beta_-)$$

$$\alpha = -e^\tau \xi$$
$$\beta_+ = e^\tau \sqrt{\xi^2 - 1}\cos\theta$$
$$\beta_- = e^\tau \sqrt{\xi^2 - 1}\sin\theta$$

Misner-Cithrè variables

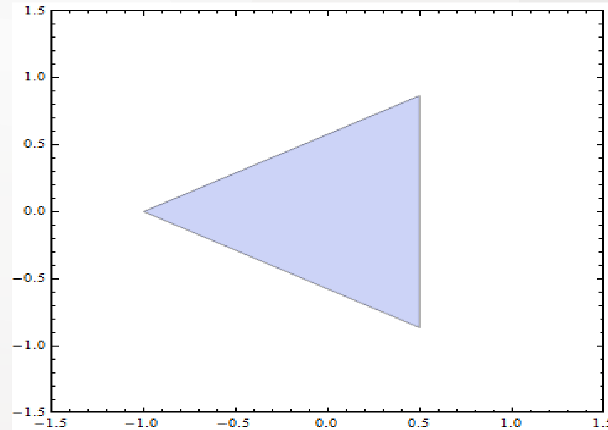
$$(\tau, \xi, \theta)$$

Anisotropy parameters

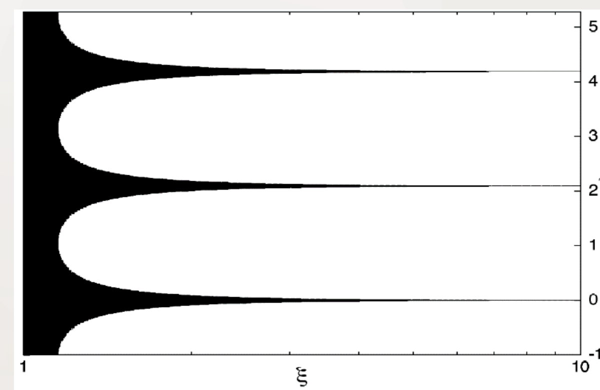
$$Q_1 = \frac{1}{3} + \frac{\beta_+ + \sqrt{3}\beta_-}{3\alpha}$$
$$Q_2 = \frac{1}{3} + \frac{\beta_+ - \sqrt{3}\beta_-}{3\alpha}$$
$$Q_3 = \frac{1}{3} - \frac{2\beta_+}{3\alpha}$$

Anisotropy parameters

$$Q_1 = \frac{1}{3}\left[1 - \frac{\sqrt{\xi^2-1}}{\xi}(\cos\theta + \sqrt{3}\sin\theta)\right]$$
$$Q_2 = \frac{1}{3}\left[1 - \frac{\sqrt{\xi^2-1}}{\xi}(\cos\theta - \sqrt{3}\sin\theta)\right]$$
$$Q_3 = \frac{1}{3}\left[1 + 2\frac{\sqrt{\xi^2-1}}{\xi}\cos\theta\right]$$



Point universe living domain in Misner variables



Point universe living domain in Misner-Cithrè variables

Poincarè half-plane

Poincarè variables

$$\xi = \frac{1+u+u^2+v^2}{\sqrt{3}v}$$

$$\theta = -\arctan\left(\frac{\sqrt{3}(1+2u)}{-1+2u+2u^2+2v^2}\right)$$

Anisotropy parameters

$$Q_1 = \frac{-u}{1+u+u^2+v^2}$$

$$Q_2 = \frac{1+u}{1+u+u^2+v^2}$$

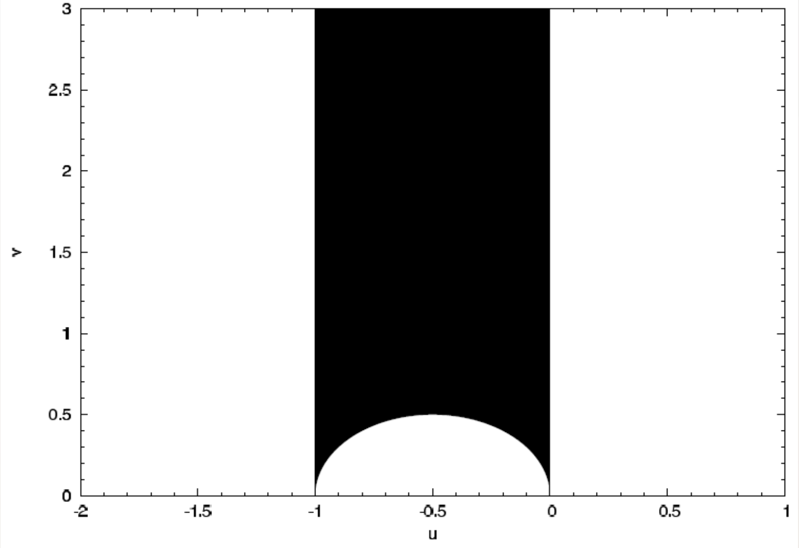
$$Q_3 = \frac{u(u+1)+v^2}{1+u+u^2+v^2}$$

SuperHamiltonian constrain

$$\mathcal{H}_{IX} = -p_\tau^2 + v^2(p_u^2+p_v^2) + 12\pi^2 e^{2\tau} e^{-4e^\tau \xi(u,v)} V(u,v,\tau) = 0$$

Reduced Hamiltonian

$$-p_\tau = H_{ADM} \equiv \sqrt{v^2(p_u^2+p_v^2) + 12\pi^2 e^{2\tau} e^{-4e^\tau \xi(u,v)} V(u,v,\tau)}$$



Point universe living domain in Poincare variables

Mixmaster Universe in the R^2 Gravity

Bianchi IX action in the Scalar-Tensor framework

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - U(\phi)\right]$$

Bianchi IX superHamiltonian constraint in the Scalar-Tensor framework

$$\mathcal{H} = -p_\alpha^2 + p_+^2 + p_-^2 + p_\phi^2 + 12\pi^2 e^{4\alpha}V(\beta_\pm) + 4e^{6\alpha}U(\phi) = 0$$

Bianchi IX reduced Hamiltonian in the Scalar-Tensor framework

$$-p_\alpha = H \equiv \sqrt{p_+^2 + p_-^2 + p_\phi^2 + 12\pi^2 e^{4\alpha}V(\beta_\pm) + 4e^{6\alpha}U(\phi)}$$

$$f(R) = R + qR^2$$

$$U(\phi) = \frac{1}{64\pi q}\left(1 - 2\exp^{-4\sqrt{\frac{\pi}{3}}\phi} + \exp^{-8\sqrt{\frac{\pi}{3}}\phi}\right)$$

Mixmaster Universe in the R^2 Gravity

Misner-Cithrè-like variables $(\tau, \xi, \theta, \delta)$

$$\alpha = -e^{\tau}\xi$$
$$\beta_+ = e^{\tau}\sqrt{\xi^2 - 1}\cos\theta$$
$$\beta_- = e^{\tau}\sqrt{\xi^2 - 1}\sin\theta\cos\delta$$
$$\phi = e^{\tau}\sqrt{\xi^2 - 1}\sin\theta\sin\delta$$

$\delta \to 0$

Misner-Cithrè variables (τ, ξ, θ)

$$\alpha = -e^{\tau}\xi$$
$$\beta_+ = e^{\tau}\sqrt{\xi^2 - 1}\cos\theta$$
$$\beta_- = e^{\tau}\sqrt{\xi^2 - 1}\sin\theta$$

Poincare variables transformations

$$\{\xi, \theta, \delta\} \rightarrow \{u, v, \delta\}$$

$$\xi = \frac{1 + u + u^2 + v^2}{\sqrt{3}v}$$

$$\theta = -\arctan\left(\frac{\sqrt{3}(1 + 2u)}{-1 + 2u + 2u^2 + 2v^2}\right)$$

$$-\infty < \tau < +\infty$$
$$-\infty < u < +\infty$$
$$0 < v < +\infty$$
$$0 < \delta < 2\pi$$

Mixmaster Universe in the R^2 Gravity

SuperHamiltonian constrain

$$\mathcal{H} = -p_\tau^2 + v^2 \left[p_u^2 + p_v^2 + 4\frac{p_\delta^2}{(1+2u)^2} \right] + e^{2\tau}\mathcal{V}(u,v,\delta,\tau) = 0$$

Reduced Hamiltonian

$$H = \sqrt{v^2 \left[p_u^2 + p_v^2 + 4\frac{p_\delta^2}{(1+2u)^2} \right] + e^{2\tau}\mathcal{V}(u,v,\delta,\tau)}$$

Curvature and scalar field potential terms

$$e^{2\tau}\mathcal{V} = e^{2\tau}[12\pi^2 e^{-4e^\tau \xi(u,v)} V(u,v,\delta,\tau) + 4e^{6e^\tau \xi(u,v)} U(u,v,\delta,\tau)] =$$
$$= 12\pi^2 e^{2\tau}\left(e^{-\frac{12e^\tau}{\sqrt{3}v}(u+u^2+v^2)} + e^{-\frac{6e^\tau}{\sqrt{3}v}(1+(1+2u)\cos\delta)} + e^{-\frac{6e^\tau}{\sqrt{3}v}(1-(1+2u)\cos\delta)}\right) +$$
$$+\frac{e^{2\tau}}{8\pi q}\left(e^{-\frac{12e^\tau}{\sqrt{3}v}(1+u+u^2+v^2)} - 2e^{-\frac{6e^\tau}{\sqrt{3}v}(1+u+u^2+v^2-2\sqrt{2\pi^3}(1+2u)\sin\delta)} + e^{-\frac{6e^\tau}{\sqrt{3}v}(1+u+u^2+v^2-4\sqrt{2\pi^3}(1+2u)\sin\delta)}\right)$$

point-Universe living domain

$$\begin{cases} 1+(1+2u)\cos\delta > 0 \\ 1-(1+2u)\cos\delta > 0 \\ u(u+1)+v^2 > 0 \\ 1+u+u^2+v^2-4\sqrt{2\pi^3}(1+2u)\sin\delta > 0 \end{cases}$$

Free potential case

- $\mathcal{V} = 0$ Bianchi I model with a massless scalar field

$$H = v\sqrt{p_u^2 + p_v^2 + 4\frac{p_\delta^2}{(1+2u)^2}}$$

$$\tau \to \infty \quad , \quad \frac{\partial H}{\partial \tau} = 0$$

$$H \simeq \epsilon = const.$$

Line Element for the configuration variables $\{u, v, \delta\}$

$$ds^2 = \frac{\epsilon}{v^2}\left[du^2 + dv^2 + \frac{(1+2u)^2}{4}d\delta^2\right]$$

- Negative constant curvature $R = -\frac{6}{\epsilon}$
- The presence of the singular values $u = -\frac{1}{2}$ and $v = 0$ allows to restrict the domain of the configuration space to analyse the point-Universe trajectories

$$-\tfrac{1}{2} < u < +\infty \ , \ 0 < v < +\infty \ , \ 0 < \delta < 2\pi$$

Free potential Hamiltonian equations

$$\dot{u} = \frac{\partial H}{\partial p_u} = \frac{v^2}{\epsilon}p_u \quad , \quad \dot{p_u} = -\frac{\partial H}{\partial u} = \frac{8v^2}{\epsilon}\frac{p_\delta^2}{(1+2u)^3}$$

$$\dot{v} = \frac{\partial H}{\partial p_v} = \frac{v^2}{\epsilon}p_v \quad , \quad \dot{p_v} = -\frac{\partial H}{\partial v} = -\frac{\epsilon}{v}$$

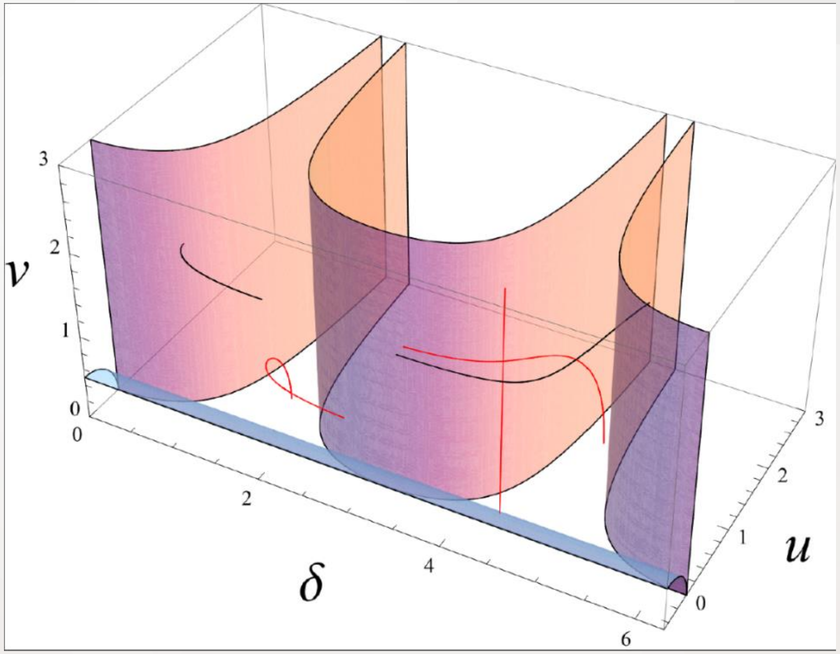
$$\dot{\delta} = \frac{\partial H}{\partial p_\delta} = \frac{4v^2}{\epsilon}\frac{p_\delta}{(1+2u)^2} \quad , \quad \dot{p_\delta} = -\frac{\partial H}{\partial \delta} = 0$$

Intermediate potential case

- $\mathcal{V} \simeq 12\pi^2 e^{-4e^{\tau}\xi(u,v)} V(u, v, \delta, \tau)$ Bianchi IX model with a massless scalar field

point-Universe living domain

$$\begin{cases} 1 + (1+2u)\cos\delta > 0 \\ 1 - (1+2u)\cos\delta > 0 \\ u(u+1) + v^2 > 0 \end{cases}$$



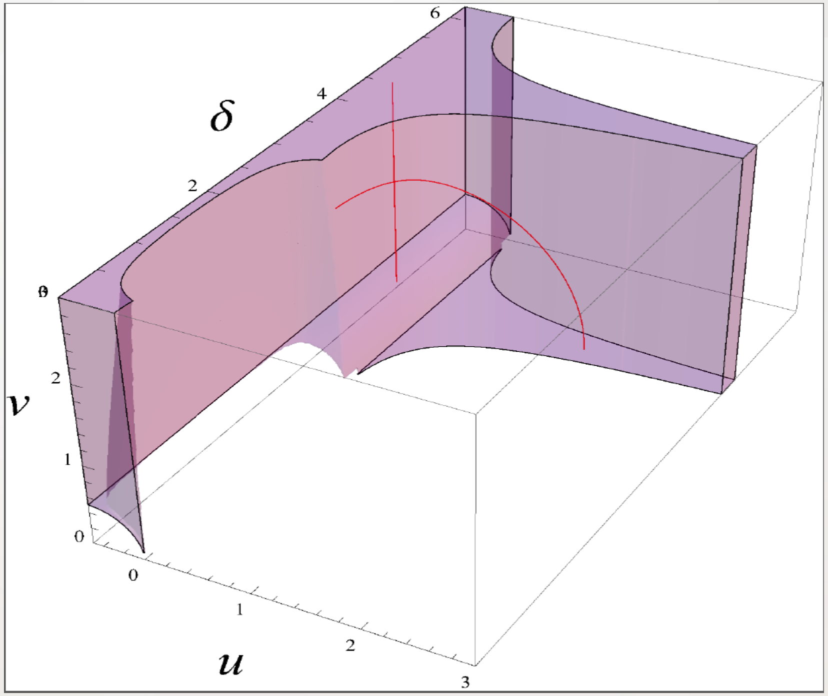
The point-Universe is able to approach the "absolute" for $v \to 0, \infty$ with no other successive bounces. It shows, in a new graphic way, the known feature about the removal of the oscillatory behaviour of the Mixmaster model coupled with a massless scalar field

Full potential case

- $\mathcal{V} = 12\pi^2 e^{-4e^{\tau}\xi(u,v)} V_{IX}(u,v,\delta,\tau) + 4e^{6e^{\tau}\xi(u,v)} U(u,v,\delta,\tau)$

point-Universe living domain

$$\begin{cases} 1+(1+2u)\cos\delta > 0 \\ 1-(1+2u)\cos\delta > 0 \\ u(u+1)+v^2 > 0 \\ 1+u+u^2+v^2-4\sqrt{2\pi^3}(1+2u)\sin\delta > 0 \end{cases}$$



The point-Universe trajectories that are able to approach the "absolute" already exist. A quadratic correction in the Ricci scalar is able to remove the typical never-ending bounces of the point-Universe against the walls, i.e. the chaos removal.

Conclusions

- In the Scalar-Tensor version of the R^2-gravity the scalar field potential term has an exponential profile comparable respect to the curvature potential.
- We individuate a natural parametrization of the configuration variables, very useful for future attempts to quantize the system.
- Since the classical evolution is expected to be predictive up to a finite value of the Universe volume, for sufficiently small coupling constant q values, the present model can be considered as the quadratic Taylor expansion of a generic f(R) theory.
- The result we derived in the homogeneous cosmological setting, can be naturally extended to a generic inhomogeneous Universe, simply following the line of investigation related to the BKL conjecture.

Thank You!

Bianchi IX with a massless scalar field

SuperHamiltonian constrain

$$\mathcal{H}^s = -p_\alpha^2 + p_+^2 + p_-^2 + p_\phi^2 + \frac{3(4\pi)^4}{k^2} e^{4\alpha} V(\beta_\pm) = 0$$

$$\begin{cases} \beta_\pm \propto \frac{p_\pm}{p_\alpha}\alpha = \pi_\pm \alpha \\ \phi \propto \frac{p_\phi}{p_\alpha}\alpha = \pi_\phi \alpha \end{cases} \qquad \pi_+^2 + \pi_-^2 + \pi_\phi^2 = 1$$

$$p_1 + p_2 + p_3 = 1$$

$$p_1^2 + p_2^2 + p_3^2 = 1 + \frac{2}{3}(1 - \pi_\phi^2) = 1 - q^2$$

$$p_1 = \frac{-u}{1 + u + u^2}$$

$$p_2 = \frac{1 + u}{1 + u + u^2}\left[u - \frac{u-1}{2}\left(1 - \sqrt{1 - \gamma^2}\right)\right]$$

$$p_3 = \frac{1 + u}{1 + u + u^2}\left[1 + \frac{u-1}{2}\left(1 - \sqrt{1 - \gamma^2}\right)\right]$$

$$\gamma^2 = \frac{2(1 + u + u^2)q^2}{(u^2 - 1)^2}$$