Inflationary cosmology from higher-derivative gravity

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CHAOTIC INFLATION

$$\mathcal{L} = \left(\frac{R}{2\kappa^2} + \frac{g^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \sqrt{-g} \,.$$

Equations of motion are

$$\frac{3H^2}{\kappa^2} = \frac{\dot{\phi}^2}{2} + V(\phi) \equiv \rho_\phi \,, \quad -\frac{(3H^2 + 2\dot{H})}{\kappa^2} = \frac{\dot{\phi}^2}{2} - V(\phi) \equiv p_\phi \,,$$

with the conservation law

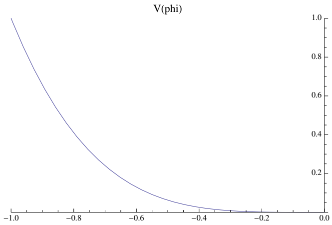
$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi) \quad (\rightarrow \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0) \,.$$

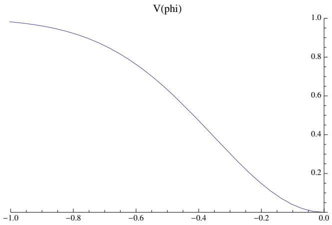
Inflation is realized by a (quasi) de Sitter solution $H \simeq H_0$ with $a(t) \sim \exp\left[H_0 t\right]$. It must be $\omega_\phi = p_\phi/\rho_\phi \simeq -1$,

$$\omega_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + V(\phi)} \simeq -1 \rightarrow |\dot{\phi}^2| \ll V(\phi) \,,$$

namely the "slow roll approximation" must be realized.

SLOW ROLL INFLATION





Inflation takes place for $\phi \to -\infty$ in the slow roll approximation

$$\epsilon = -\frac{\dot{H}}{H^2} \ll 1\,, \quad |\eta| = |-\frac{\dot{H}}{H^2} - \frac{\ddot{H}}{2H\dot{H}}| \ll 1\,,$$

such that

$$H_0^2 \simeq \frac{\kappa^2 V(\phi)}{3}\,.$$

Equations of motion in slow roll approximation:

$$\frac{3H^2}{\kappa^2} \simeq V(\phi)\,, \quad 3H\dot{\phi} \simeq -V'(\phi)\,,$$

and we can derive $\dot{\phi}$ and therefore $\dot{H}$. The slow roll paramters are given by:

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2\,, \quad \eta = \frac{1}{\kappa^2}\left(\frac{V''(\phi)}{V(\phi)}\right)\,,$$

and must be small during inflation. Inflation ends when $-\dot{H}/H^2 \simeq 1$ (it means $\ddot{a} \simeq 0$), namely at $\epsilon \simeq 1$. The amount of inflation is given by the $\mathcal{N}$-fold number

$$\mathcal{N} = \log\left[\frac{a(t_f)}{a(t_i)}\right] = \kappa^2 \int_{\phi_e}^{\phi_i} \frac{V(\phi)}{V'(\phi)} d\phi\,,$$

and to solve the problems of initial condition it must be $\dot{a}_{\mathrm{i}}/\dot{a}_0 < 10^{-5}$, where the anisotropy in our universe is $\sim 10^{-5}$. It follows (on dS solution) $76 < \mathcal{N}$; the last data say it is enough $\mathcal{N} \simeq 55$ to have thermalization of observable universe. Thypically, it is required $55 < \mathcal{N} < 65$.

VIABLE INFLATION

Some fact about inflation are well-known:

- Inflation is described by a quasi de Sitter solution where $R \simeq 12H^2 \sim M_{Pl}^2$;
- We need a graceful exit from inflation ($\dot{H} < 0$);
- The amount of inflation is measured by $\mathcal{N} \equiv \log\left[a_f/a_i\right]$ and $55 < \mathcal{N} < 65$ to solve problem of initial conditions;
- The fluctuations of energy density at the end of inflation are $\Delta^2_{\mathcal{R}} (\equiv |\delta\rho/\rho|) \simeq 10^{-9}$;
- The Hubble flow functions

$$\epsilon_1 = -\frac{\dot{H}}{H^2}\,, \quad \epsilon_2 = -\frac{2\dot{H}}{H^2} + \frac{\ddot{H}}{H\dot{H}} \equiv \frac{\dot{\epsilon}_1}{H\epsilon_1}\,,$$

must be small during inflation. The spectral index n_s and the tensor-to-scalar ratio r are given by (at the first order),

$$n_s = 1 - 2\epsilon_1 - \epsilon_2\,, \quad r = 16\epsilon_1\,,$$

and the very last Planck data constrain them as $n_{\mathrm{s}} = 0.9603 \pm 0.0073\,(68\%\,\mathrm{CL})$ and $r < 0.11\,(95\%\,\mathrm{CL})$.

MODIFIED GRAVITY THEORIES FOR INFLATION

$$I = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4 x \sqrt{-g} F(R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma}, R_{\mu\nu} R^{\mu\nu} ...) \, .$$

A simple example of modifie dgravity theory is given by $f(R)$:

$$I = \int_{\mathcal{M}} d^4 x \sqrt{-g} \left[\frac{R}{2\kappa^2} + f(R) \right] \, ,$$

where $f(R)$ is a generic function of the Ricci scalar only. the field equations read

$$F'(R) \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = \kappa^2 T_{\mu\nu}^{(\mathrm{matter})} + \left\{ \frac{1}{2} g_{\mu\nu} [F(R) - RF'(R)] \right.$$
$$\left. + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) F'(R) \right\} \, , \quad F(R) = R + 2\kappa^2 f(R)$$

This theories can be used to study inflation.

$F(R)$-FRIEDMANN-LIKE EQUATIONS

On flat FRW metric the field equations read

$$\frac{3H^2}{\kappa^2} = \rho_{\mathrm{eff}}\,, \quad -\frac{(3H^2 + 2\dot{H})}{\kappa^2} = p_{\mathrm{eff}}\,,$$

where

$$\begin{aligned} \rho_{\mathrm{eff}} &= \left[(Rf_R - f) - 6H\dot{f}_R - 6H^2 f_R\right], \\ p_{\mathrm{eff}} &= \left[(f - Rf_R) + 4H\dot{f}_R + 2\ddot{f}_R + (4\dot{H} + 6H^2)f_R\right]. \end{aligned}$$

In this case, the Hubble flow functions have to be replaced by the variables

$$\epsilon_1 = -\frac{\dot{H}}{H^2}\,, \quad \epsilon_3 = \frac{\kappa^2 \dot{f}_R}{H\,(1 + 2\kappa^2 f_R)}\,, \quad \epsilon_4 = \frac{\ddot{f}_R}{H\dot{f}_R}\,.$$

The spectral index and the tensor-to-scalar ratio are given by

$$n_s = 1 - 4\epsilon_1 + 2\epsilon_3 - 2\epsilon_4\,, \quad r = 48\epsilon_3^2\,,$$

or, in terms of the standard Hubble flow functions $\epsilon_1\,,\epsilon_2$,

$$n_s = 1 - 2\epsilon_2\,, \quad r = 48\epsilon_1^2\,.$$

CONFORMAL TRANSFORMATIONS IN $F(R)$-GRAVITY

The (Jordan frame) action of a $F(R)$-modified gravity theory,

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{F(R)}{2\kappa^2}\right],$$

can be rewritten in the Einstein frame as

$$I = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right),$$

where

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{-\sigma}g_{\mu\nu}\,, \quad \sigma = -\sqrt{\frac{3}{2\kappa^2}}\ln[F_R(R)]\,, \quad V(\sigma) = \frac{R}{F'(R)} - \frac{F(R)}{F'(R)^2}\,.$$

In this way, we can study inflation from modified gravity in the scalar field representation.

EXAMPLE: R^2-MODEL FOR INFLATION

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{R + \kappa^2 R^2/\gamma}{2\kappa^2}\right] .$$

In the scalar field representation we get

$$I = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right) ,$$

$$V(\sigma) = \frac{\gamma}{4\kappa^2}\left(1 - \mathrm{e}^{\sqrt{2\kappa^2/3}\sigma}\right)^2 .$$

The inflation is reproduced for large (and negative) values of $\sigma \equiv \sigma(t)$, such that $V(\sigma) \simeq \gamma/(4\kappa^2)$, and the FRW metric are

$$H^2 \simeq \frac{\gamma}{12} , \quad 3H\dot{\sigma} \simeq \left(\frac{\gamma}{\sqrt{6\kappa^2}}\right)\mathrm{e}^{\sqrt{2\kappa^2/3}\sigma} , \sigma \simeq -\sqrt{\frac{3}{2\kappa^2}}\ln\left[\frac{\sqrt{2\gamma}}{3^{\frac{3}{2}}}(t_0 - t)\right] .$$

Here, t_0 is bounded at the beginning of the inflation and $\gamma < M_{Pl}^2$. The field slowly rolls down toward the potential minimum at $\sigma \to 0^-$, $V(0) = -\gamma/(4\kappa^2) < 0$.

The slow roll paramters read

$$\epsilon = \frac{4}{3}\frac{1}{(2-\mathrm{e}^{-\sqrt{2\kappa^2/3}\sigma})^2}\,, \quad |\eta| = \frac{4}{3}\frac{1}{|2-\mathrm{e}^{-\sqrt{2\kappa^2/3}\sigma}|}\,.$$

For $\sigma \to -\infty$ this numbers are very small. The inflation ends when such parameters are of the order of unit, namely at $\sigma_e \simeq -0.17\sqrt{3/(2\kappa^2)}$. The e-folds number results to be

$$\mathcal{N} \simeq \frac{3\mathrm{e}^{-\sqrt{2\kappa^2/3}\sigma}}{4}\Big|_{\sigma_e}^{\sigma_i} \simeq \frac{1}{4}\sqrt{\frac{2\gamma}{3}t_0}\,, \quad |\sigma_i| \ll |\sigma_e|\,.$$

For example, in order to obtain $\mathcal{N}=60$, we must require $\sigma_i \simeq -\sqrt{3/(2\kappa^2)}4.38 \simeq 1.07 M_{Pl}$. We also have during inflation

$$\epsilon \simeq \frac{3}{4\mathcal{N}^2}\,, \quad |\eta| \simeq \frac{1}{\mathcal{N}}\,, \quad n_s \simeq 1-\frac{2}{\mathcal{N}}\,, \quad r \simeq \frac{12}{\mathcal{N}^2}\,.$$

Since $1 > n_s > 1-\sqrt{0.11/3} \simeq 0.809$ when $r < 0.11$, these indices are compatible with Planck data. For example, for $\mathcal{N}=60$, one has $n_s = 0.967$ and $r = 0.003$.

RECONSTRUCTION OF $F(R)$-GRAVITY FROM $V(\sigma)$

Since the relation between potential of scalar field in Einstein frame represantation and modified gravity in Jordan frame is given by (**)

$$V(\sigma) \equiv \frac{R}{F_R(R)} - \frac{F(R)}{F_R(R)^2} =$$

$$\frac{1}{2\kappa^2}\left\{ \mathrm{e}^{\left(\sqrt{2\kappa^2/3}\right)\sigma} R\left(\mathrm{e}^{-\left(\sqrt{2\kappa^2/3}\right)\sigma}\right) - \mathrm{e}^{2\left(\sqrt{2\kappa^2/3}\right)\sigma} F\left[R\left(\mathrm{e}^{-\left(\sqrt{2\kappa^2/3}\right)\sigma}\right)\right]\right\},$$

by taking the derivative with respect to R, we get

$$RF_R(R) = -2\kappa^2\sqrt{\frac{3}{2\kappa^2}}\frac{d}{d\sigma}\left(\frac{V(\sigma)}{e^{2\left(\sqrt{2\kappa^2/3}\right)\sigma}}\right).$$

As a result, giving the explicit form of the potential $V(\sigma)$, thanks to the relation $\sigma = -\sqrt{3/2(\kappa^2)}\ln[F_R(R)]$, we obtain an equation for $F_R(R)$, and therefore the $F(R)$-gravity model in the Jordan frame. In this process, one introduces an integration constant, which has to be fixed by requiring that (**) holds true.

$V(\sigma) \sim c_0 + c_1 \exp[\sigma] + c_2 \exp[2\sigma]$: $R + R^2 + \Lambda$-models

Generalization of R^2-model to the case with cosmological constant (viable inflation):

$$V(\sigma) = \left[c_0 + c_1 e^{\sqrt{2\kappa^2/3}\sigma} + c_2 e^{2\sqrt{2\kappa^2/3}\sigma}\right] .$$

It follows

$$F_R(R) = -\frac{c_1}{2c_0} + \frac{R}{2c_0} , \quad F(R) = -\frac{c_1}{2c_0} R + \frac{R^2}{4c_0} + \Lambda$$

with $\Lambda = c_1^2/(4c_0) - c_2$. Thus, by putting $-c_1/(2c_0) = 1$,

$$F(R) = R + \frac{R^2}{4c_0} + c_0 - c_2 .$$

Interesting cases:

- $c_2 = 0$: the model admits also the Reissner-Nordström-de Sitter solution with two free parameters.
- $c_0 = c_2$: we recover the Starobinsky model $F(R) = R + R^2/(4c_0)$.

$$V(\sigma) \sim \gamma \exp[-n\sigma]\,, n > 0\text{: } c_0 R^{\frac{n+2}{n+1}}\text{-models}$$

For the potential (**)

$$V(\sigma) = \frac{\alpha}{\kappa^2}\left(1 - \mathrm{e}^{\sqrt{2\kappa^2/3}\sigma}\right) + \frac{\gamma}{\kappa^2}\mathrm{e}^{-n\sqrt{2\kappa^2/3}\sigma}\,, \quad \gamma, n > 0\,,$$

by putting $\alpha = -\gamma(n+2)$, $\gamma(\sim M_{Pl}^2) \gg 1$, we obtain at the perturbative level ($c_1 \propto \gamma^{-1}\,, c_2 \propto \gamma^{-2}$...)

$$F_R(R \ll \gamma) \simeq 1 + c_1 R + c_2 R^2 + \ldots \quad F_R(R \gg \gamma) \simeq \left(\frac{1}{4(n+2)}\right)^{\frac{1}{1+n}}\left(\frac{R}{\gamma}\right)^{\frac{1}{n+1}}\,.$$

Note: if in the Einstein frame inflation is realized at $R_{EF} \sim \gamma$, in the Jordan frame $R_{JF} \simeq \mathrm{e}^{-\sqrt{2\kappa^2/3}\sigma} R_{EF} \gg \gamma$. Thus, the potential (**) is realized by a model which returns GR at small curvatures and $F(R) = c_0 R^{\frac{n+2}{n+1}}$ at high curvatures.
The potential (**) has a minimum at $\sigma \to 0^-$ when inflation ends; for $\sigma \to -\infty$ the slow roll conditions must be satisfied.

For $\sigma \to -\infty$,

$$V(\sigma) \simeq \frac{\gamma}{\kappa^2} e^{-n\sqrt{2\kappa^2/3}\sigma}\,, \quad H^2 \simeq \frac{\gamma}{3} e^{-n\sqrt{2\kappa^2/3}\sigma}\,, \quad \epsilon \simeq \frac{n^2}{3}\,, \quad |\eta| \simeq \frac{2n^2}{3}\,.$$

It means, $0 < n \ll 1$. Then,

$$3H\dot{\sigma} \simeq \sqrt{\frac{2}{3\kappa^2}}\gamma\, n e^{-n\sqrt{2\kappa^2/3}\sigma}\,, \quad \sigma = \frac{2}{n}\sqrt{\frac{3}{2\kappa^2}} \ln\left[\frac{n^2}{3}\sqrt{\frac{\gamma}{3}}(t_0 + t)\right]\,,$$

and

$$H = \frac{3}{n^2(t_0 + t)}\,, \quad \frac{\ddot{a}}{a} = H^2 + \dot{H} = \frac{3}{n^2(t_0 + t)^2}\left(\frac{3}{n^2} - 1\right) > 0\,, \quad (n < \sqrt{3})\,.$$

We have an acceleration as soon as $\epsilon < 1$.

$$N \simeq -\frac{1}{n}\sqrt{\frac{3\kappa^2}{2}}\sigma\Big|_{\sigma_e}^{\sigma_i} \simeq -\frac{3}{n^2} \ln\left[\frac{n^2}{3}\sqrt{\frac{\gamma}{3}}t_0\right]\,, \quad n_s \simeq 1 - \frac{2n^2}{3}\,, \quad r \simeq \frac{16n^2}{3}\,.$$

The result suggests that only the models close to R^2-gravity are able to produce this kind of inflation, since n must be extremely close to 0^+ (we remember $F(R) = c_0 R^{\frac{n+2}{n+1}}$).

OTHER EXAMPLES

- From the potential

$$V(\sigma) = \frac{3\gamma}{4\kappa^2} - \frac{\gamma}{\kappa^2} e^{\sqrt{2\kappa^2/3}\sigma/2}\,, \quad 1 \ll \gamma \sim M_{Pl}^2\,,$$

we get

$$F(R) = \frac{R}{2} + \frac{R^2}{6\gamma} + \frac{\sqrt{3}}{36}\left(4R/\gamma + 3\right)^{3/2} + \frac{\gamma}{4}\,.$$

Note that $F(R \ll \gamma) \simeq R + \gamma/2$. We can set the cosmological constant equal to zero, adding a suitable term in the potential proportional to $\exp\left[2\sqrt{2\kappa^2/3}\sigma\right]$, which does not change the dynamics of inflation. The model reproduces a viable inflation with

$$\epsilon \simeq \frac{3}{N^2}\,, \quad |\eta| \simeq \frac{1}{N}\,, \quad n_s \simeq 1 - \frac{1}{N}\,, \quad r \simeq \frac{48}{N^2}\,.$$

- From the potential (**)

$$V(\sigma) = \frac{\gamma(2-n)}{2\kappa^2} - \frac{\gamma}{\kappa^2} e^{n\sqrt{2\kappa^2/3}\sigma}\,, \quad 1 \ll \gamma \sim M_{Pl}^2\,, \quad 0 < n < 2\,,$$

we get ($c_1 \propto \gamma^{-1}\,, c_2 \propto \gamma^{-2}, c_3 \propto \gamma^{-3}...$)

$$F_R(R \ll \gamma) \simeq 1 + c_1 R + c_2 R^2 + c_3 R^3 + ...$$

$$F_R(R \gg \gamma) \simeq \left(\frac{R}{2\gamma(2-n)}\right) + \left(\frac{R}{2\gamma(2-n)}\right)^{1-n}\,,$$

such that models like F(R)$=b_1 R^2 + b_2 R^{2-n}$ realize the potential (**) at high curvatures. The model reproduces a viable inflation with

$$\epsilon \simeq \frac{3}{4n^2 N^2}\,, \quad |\eta| \simeq \frac{1}{N}\,, \quad n_s \simeq 1 - \frac{1}{N}\,, \quad r \simeq \frac{12}{n^2 N^2}\,.$$

Since $1 > n_s > 1 - (\sqrt{0.11/12})n \simeq 1 - (0.0957)n$ when $r < 0.11$, one has that

$$0.3386 < n < 1\,,$$

in order to make the spectral indexes compatible with the Planck data (i.e. $F(R \gg M_{Pl}) \simeq c_1 R^2 + c_2 R^\zeta$, $1 < \zeta < 5/3$).

CONFORMAL ANOMALY

The renormalizized procedure of the matter field lagrangian at the Planck scale leads to the so-called conformal anomaly,

$$\langle T^{\mu}_{\mu} \rangle = \alpha \left(W + \frac{2}{3} \Box R \right) - \beta \mathcal{G} + \xi \Box R \,,$$

$W = C^{\xi\sigma\mu\nu} C_{\xi\sigma\mu\nu}$ being the'square" of the Weyl tensor and $\mathcal{G}$ the Gauss-Bonnet topological invariant. Moreover, $\alpha\,, \beta$ and ξ are related to the number of conformal fields present in the theory. For $\mathcal{N}_{\rm super} = 4$ SU(N) super Yang-Mills (SYM) theory,

$$\alpha = \beta = \frac{N^2}{64\pi^2} \,, \quad \xi = -\frac{N^2}{96\pi^2} \,, \quad N \gg 1$$

and $\frac{2}{3}\alpha + \xi = 0$, but in principle the contribution of the $\Box R$ term could be reintroduced via R^2-term in the action, such that we redefine

$$\langle T^{\mu}_{\mu} \rangle = \left(\alpha W - \beta G + \delta \Box R \right) \,, \quad 0 < \alpha \,, \beta \,, \quad \delta < 0 \,.$$

CONFORMAL ANOMALY IN $F(R)$-GRAVITY

The action reads

$$I = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\,\left[R + 2\kappa^2\tilde{\gamma}R^2 + f(R) + 2\kappa^2\mathcal{L}_{\rm QC}\right]\,, \quad \tilde{\gamma} \equiv \frac{\gamma N^2}{192\pi^2} > 0.$$

The trace of field equations is $(\alpha, \beta > 0)$

$$R = -\kappa^2\left(\alpha W - \beta G + \delta\Box R\right) - 2f(R) + Rf_R(R) + 3\Box f_R(R)\,, \quad \delta \equiv -12\tilde{\gamma} < 0\,.$$

On FRW metric we must have

$$\frac{3}{\kappa^2}H^2 = \rho_{QC} + \rho_{MG} \equiv \rho_{\rm eff}\,, \quad -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) = p_{QC} + p_{MG} \equiv p_{\rm eff}\,,$$

with the continuity equation

$$\dot{\rho}_{\rm eff} + 3H\left(\rho_{\rm eff} + p_{\rm eff}\right) = 0\,.$$

Note that on FRW metric $W = 0$.

On FRW metric the trace of field equations leads to an expression for $p_{\rm eff}$,

$$-\rho_{\rm eff}+3p_{\rm eff}=-\frac{R}{\kappa^2}\equiv-\beta G+\delta\Box R+\frac{1}{\kappa^2}\left(2f(R)-Rf_R(R)-3\Box f_R(R)\right).$$

Now we can eliminate the pressure in the conservation law, such that

$$\frac{d}{dt}\left(\rho_{\rm eff}a^4\right)=-\dot{a}a^3\left(-\rho_{\rm eff}+3p_{\rm eff}\right)=$$

$$=a^3\dot{a}\left[24\beta\frac{\dot{a}^2\ddot{a}}{a^3}+\delta\left(\ddot{R}+3H\dot{R}\right)\right]-\frac{\dot{a}a^3}{\kappa^2}\left(2f(R)-Rf_R(R)-3\Box f_R(R)\right),$$

whose integration leads to

$$\rho_{\rm eff}=\frac{\rho_0}{a^4}+6\beta H^4+\delta\left(18H^2\dot{H}+6\ddot{H}H-3\dot{H}^2\right)+\rho_{MG},$$

where

$$\rho_{MG}=\frac{1}{2\kappa^2}\left(Rf_R(R)-f(R)-6H^2f_R(R)-6H\dot{f}_R(R)\right).$$

Given $\rho_{\rm eff}$, we also have $p_{\rm eff}$ from the continuity equation. We will put $\rho_0=0$.

APPLICATION TO INFLATION

$$f(R) = -2\Lambda_{\rm eff}\left[1 - \exp\left(-\frac{R}{R_0}\right)\right]\,, \quad R_0\,, \Lambda_{\rm eff} > 0\,,$$

namely $f(R) \simeq -2\Lambda_{\rm eff}\theta\,(R - R_0)$. At hight curvatures $R_0 \ll R$, from the first Friedmann-like equation, by taking into account the conformal anomaly, we get

$$\frac{3}{\kappa^2}H^2 \simeq 6\beta H^4 + \delta\left(18H^2\dot{H} + 6\ddot{H}H - 3\dot{H}^2\right) + \frac{\Lambda_{\rm eff}}{\kappa^2}\,,$$

such that

$$H^2_{\rm dS\pm} = \frac{1}{4\beta\kappa^2}\left[1 \pm \sqrt{1 - \frac{8\Lambda_{\rm eff}\beta\kappa^2}{3}}\right]\,,$$

By perturbing the equation above, it is possible to show that the only (real) unstable de Sitter solution able to describe the inflation is H_+ with $R_0 < H_+^2$, such that we require also $H_-^2 < R$. The De Sitter solution is determined by the $\mathcal{G}$-contribution (β coefficient). Stability/instability depends on $\delta < 0$ (R^2 constribution to conformal anomaly).

The dynamics of inflation depends on modified gravity. By pertubing the EOMs with respect to $R_{\rm dS} \equiv 12 H_{\rm dS+}^2$ one has for exponential model,

$$-\delta\left(\Delta\ddot{H}(t) + 3H_{\rm dS}\Delta\dot{H}(t)\right) + \Delta H(t)\left(\frac{1}{\kappa^2} - 4H_{\rm dS}^2\beta\right) = -\frac{{\rm e}^{-R_{\rm dS}/R_0}\Lambda_{\rm eff}}{12 H_{\rm dS}\kappa^2}\left(\frac{R_{\rm dS}}{R_0} + 2\right) .$$

with the solution

$$\Delta H(t) = A_0 {\rm e}^{\lambda t} - \frac{{\rm e}^{-R_{\rm dS}/R_0}\Lambda_{\rm eff}}{12 H_{\rm dS}\kappa^2}\left(\frac{R_{\rm dS}}{R_0} + 2\right)\left(\frac{1}{\kappa^2} - 4H_{\rm dS}^2\beta\right)^{-1} ,$$

$$\lambda_{1,2} = \frac{-3H_{\rm dS} \pm \sqrt{9H_{\rm dS}^2 + \frac{4}{\delta}\left(\frac{1}{\kappa^2} - 4H_{\rm dS}^2\beta\right)}}{2} ,$$

where λ_1 is real and positive for our unstable de Sitter solution and A_0 is determined by putting $\Delta H(t=0)$=0 at the beginning of inflation. One finds $A_0 < 0$, such that R slowly decreases during inflation.

$\mathcal{N}$-FOLDS, EoS PARAMETER AND SPECTRAL INDEX

The $\mathcal{N}$-folds follows from the fact $H_{\mathrm{dS}}|\Delta H(t)| \simeq H_{\mathrm{dS}}^2$ at the end of inflation. In our case,

$$\mathcal{N} = \frac{2b}{3}\left[-1+\sqrt{1+\frac{4}{9\gamma}\left(\frac{\sqrt{1-\frac{8}{3}\zeta}}{1+\sqrt{1-\frac{8}{3}\zeta}}\right)}\right]^{-1}, \quad \Lambda_{\mathrm{eff}} = \frac{\zeta}{\beta\kappa^2},$$

where γ comes from the R^2-term in the action (or $\Box R$ of conformal anomaly). Here, we have put $R_{\mathrm{dS}} = bR_0$ in the model $f(R) = -2\Lambda_{\mathrm{eff}}\left(1-\mathrm{e}^{-R/R_0}\right)$ under the condition

$$1 \ll b \leq \frac{1+\sqrt{1-\frac{8}{3}\zeta}}{1-\sqrt{1-\frac{8}{3}\zeta}}.$$

As an example, for $\Lambda_{\mathrm{eff}} = 1/(8\beta\kappa^2)$ and $b = 3$, this condition is satisfied and only the unstable de Sitter solution $R_{\mathrm{dS}} \equiv R_{\mathrm{dS}+}$ can be realized. In this case, to obtain $\mathcal{N} > 76$, γ has to meet the relation $\gamma > 3.8$.

By taking into account the perturbations on the de Sitter solution, one has that during the inflation the effective EoS-parameter reads

$$-1 < w_{\text{eff}} \left(\equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}} \right) < -1 + \frac{2R_{\text{dS}}}{3R_0\mathcal{N}} \,,$$

such that the universe is in a quintessence phase (R slowly decreases).

The slow roll paramters and spectral index are given by

$$\epsilon \simeq \frac{b^2}{\mathcal{N}^2} \frac{\mathrm{e}^{-b}\zeta \left(b+2\right)}{\left(1-\frac{8}{3}\zeta\right)} \ll 1 \,, \quad |\eta| \simeq \left| -\frac{b}{2\mathcal{N}} \right| \ll 1 \,,$$

$$n_{\mathrm{s}} = 1 - \frac{b}{\mathcal{N}} - \frac{6b^2}{\mathcal{N}^2} \frac{\mathrm{e}^{-b}\zeta \left(b+2\right)}{\left(1-\frac{8}{3}\zeta\right)} \,, \quad r = \frac{16b^2}{\mathcal{N}^2} \frac{\mathrm{e}^{-b}\zeta \left(b+2\right)}{\left(1-\frac{8}{3}\zeta\right)} \,.$$

This expressions may also satisfy the last BICEP2 results, where $r = 0.20^{+0.07}_{-0.05}\,(68\%\,\mathrm{CL})$. For example, if $\mathcal{N} = 76$, for $b = 2$, 3 and 4, we acquire $r = 0.22$, 0.23, and 0.18, respectively.

END OF INFLATION AND GRAVITATIONAL COUPLING

In the limit $R/R_0 \ll 1$, the model is $f(R) = 2\Lambda_{\text{eff}}(1 - \mathrm{e}^{-R/R_0}) \simeq 0$. By perturbing the EOMs we find

$$\Delta H(t) = c_0 \cos\left(B_0 t\right)^2 \ , \quad B_0 = \frac{1}{2}\sqrt{\frac{R_0(R_0 - 2\Lambda_{\text{eff}})}{6\Lambda_{\text{eff}} - \delta R_0^2 \kappa^2}} \, ,$$

such that, since $\delta < 0$, we must require $2\Lambda_{\text{eff}} < R_0 \equiv R_{\text{dS}}/b$.
The effective gravitational coupling $G_{\text{eff}} = G_N / \left(1 + f_R(R)\right)$, has to be positive to correctly describe the interactions between matter and gravity and in particular $G_{\text{eff}} \simeq G_N$ when $R/R_0 \ll 1$. This is true if $2\Lambda_{\text{eff}} \ll R_0$.
In general, the smaller the effective cosmological constant of the exponential gravity model is, the better all the viability conditions of inflation are met. In this way, our model tends to the original Starobinsky model, which takes into account the conformal anomaly only (Starobinsky, 1980). However, we have shown how the presence of a modification of gravity can realize the perturbations to drive a viable inflation.

HIGHER DERIVATIVE QUANTUM GRAVITY

The starting action of higher-derivative gravity has the following form,

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R}{\kappa^2} - \Lambda + aR_{\mu\nu}R^{\mu\nu} + bR^2 + cR_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} + d\Box R\right) ,$$

where $a\,,b\,,c\,,d$ are running coupling constants. If we introduce the Gauss Bonnet G and the "square" of the Weyl tensor W such that $R_{\mu\nu}R^{\mu\nu} = \frac{C^2}{2} - \frac{G}{2} + \frac{R^2}{3}$ and $R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} = 2C^2 - G + \frac{R^2}{3}$, we obtain

$$I = \int_{\mathcal{M}} d^4\sqrt{-g}\left[\frac{R}{\kappa^2(t')} - \frac{\omega(t')}{3\lambda(t')}R^2 + \frac{1}{\lambda(t')}C^2 - \gamma(t')G + \zeta(t')\Box R - \Lambda(t')\right] ,$$

where $t' = (t'_0/2)\log\left[R/R_0\right]^2$, t_0 is a number and $R_0 = 4\Lambda$ is the curvature of today universe, Λ being the cosmological constant.

The one-loop running coupling constants $\lambda(t')\,,\omega(t')\,,\kappa^2(t')\,,\Lambda(t')\,,\gamma(t')$ and $\zeta(t')$ are found from higher-derivative quantum gravity. They can be written as

$$\lambda(t') = \frac{\lambda(0)}{(1+\lambda(0)\beta_2 t')}\,, \quad \omega(t') = \omega_1\,,$$

$$\kappa^2(t') = \kappa_0^2(1+\lambda(0)\beta_2 t')^{0.77}\,, \quad \Lambda(t') = \frac{\Lambda_0}{(1+\lambda(0)\beta_2 t')^{0.55}}\,,$$

with $\beta_2 = 133/10$, $\omega_1 = -0.02$, $\kappa_0^2 = 16\pi/M_{Pl}^2$, $\Lambda_0 = 2\Lambda$. The expressions for $\omega(t')\,,\kappa^2(t')$ and $\Lambda(t')$ are derived by investigating the asymptotic behaviour of the running constants at high curvature. However, the derivatives of the coupling constants obey to a set of renormalization group equations. Here, $\lambda(0)$ is a number related to the bound of inflation (large curvature), such that $\lambda(t'=0) = \lambda(0)$. The form of $\gamma(t')$ and $\zeta(t')$, namely the coefficients in front of G and $\Box R$, is given by

$$\gamma(t') = \gamma_0(1+c_1 t')\,, \quad \zeta(t') = \zeta_0(1+c_2 t')\,, \quad c_1\gamma_0 < 0\,,$$

$\gamma_0\,,\zeta_0$ and $c_{1,2}$ constants. Condition $c_1\gamma_0 <$ will be necessary to reproduce Planck data.

The model posses a de Sitter solution at

$$H_{\mathrm{dS}}^2 \kappa_0^2 \simeq \frac{322.762}{(22085.2 - 34725.2(d\gamma/dt'))\, t_0' (\lambda(0) t')^{0.77}} \,,$$

where $(d\gamma/dt') < 0$, and the solution is always unstable for inflationary scenario, but, what it is more important, it is able to reproduce the spectral index and the tensor-to-scalar ratio of the Planck data thanks to the contribution of the Gauss Bonnet throught the coefficient γ_0. Given the running coupling constant in front of the GB $\gamma(t') = \gamma_0(1 + c_1 t')$, one has that the model reproduces the correct spectral index (tensor-to-scalar ratio is very close to zero) when

$$-4.61 < \gamma_0 c_1 < -2.98 \,.$$

For example, for $c_1 = 1$ and $\gamma_0 = -3$ we find $n_s \simeq 0.96740$.

VIABLE INFLATION IN MODIFIED GRAVITY

- Models $R/(2\kappa^2) + \gamma R^n + \text{const}$ can reproduce inflation only if n is close to $n = 2$.
- Other interesting extensions may be constructed by starting from quantum effects: for example, we can study inflation in the presence of trace anomaly,

$$\langle T^{\mu}_{\mu} \rangle = \alpha \left(W + \frac{2}{3}\Box R \right) - \beta \mathcal{G} + \xi \Box R \,,$$

$W\,, \mathcal{G}$ being the "square" of the Weyl tensor and the Gauss-Bonnet topological invariant in four dimensions. An other interesting application is the inflation from higher derivative quantum gravity in the form

$$\mathcal{L} = \sqrt{-g} \left(\frac{R}{\kappa^2} - \Lambda + aR_{\mu\nu}R^{\mu\nu} + bR^2 + cR_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} + d\Box R \right) \,,$$

where $\kappa^2 > 0$, $\Lambda\,, a\,, b\,, c$ and d are running coupling constants depending on the Ricci scalar.