Question of quantum equivalence between Jordan and Einstein frame

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Outline

1. Scalar-tensor theories in cosmology
2. Transition between Jordan frame and Einstein frame
3. Quantum (non-)equivalence
4. Geometric approach and unique effective action
5. Conclusion & outlook

Scalar-tensor theories in cosmology

- General scalar-tensor theory covers most single field inflationary models:

$$S[g,\varphi] = \int \mathrm{d}^4x\, g^{1/2} \left[U(\varphi) R(g) - \frac{1}{2} G(\varphi) (\nabla\varphi)^2 - V(\varphi) \right]$$

- Classically, geometric $f(R)$ theories can as well be reformulated as scalar-tensor theories of the above type with:

$$U = f_{,R}(R), \qquad G = 0, \qquad V = R(U)U - f(R(U))$$

- Non-minimal coupling $U(\varphi)\, R$ to gravity:
  1. $U(\varphi)$ incorporates concept of varying gravitational constant
  2. $U(\varphi)$ induced at 1-loop for minimally coupled self-interacting φ
  3. $U(\varphi)$ arises naturally in string theory inspired models

Example: Non-minimal Higgs inflation

- Generalized potentials for non-minimal Higgs inflation:

$$U(\varphi) = \frac{1}{2}\left(M_{\mathsf{P}}^2 + \xi\,\varphi^2\right), \quad V(\varphi) = \frac{\lambda}{4}(\varphi^2 - v^2)^2,$$
$$G(\varphi) = 1, \quad \varphi := |\Phi| = \sqrt{\Phi^a \Phi^b \delta_{ab}}, \quad a = 1, \ldots, 4\,, \quad v \simeq 246 \ \text{GeV}$$

- Standard Model: Particle masses $m_{\mathsf{part}}(\varphi) \propto \varphi$ via Higgs mechanism

- A minimally coupled scalar field would lead to too small Higgs masses:

$$\underbrace{(\Delta T/T)^2 \simeq 10^{-10} \propto \lambda}_{\text{cosmology}} \quad \leftarrow \lightning \rightarrow \quad \underbrace{M_{\mathsf{H}}^2 \propto \lambda\, v^2 \simeq 10^4 \ \text{GeV}}_{\text{standard model}}$$

- Strong non-minimal coupling to gravity $\xi\varphi^2 R$ ($\xi \simeq 10^4$) leads to:

$$(\Delta T/T)^2 \simeq 10^{-10} \propto \lambda/\xi^2 \quad \Rightarrow \text{field independent flat EF potential}$$

- Predictability: quantum corrections of heavy SM sector: q_{t}, $W^{\pm}$, Z

- Effective potential determines cosmological parameters: A_{s}, n_{s}, r,...

Predictions in non-minimal Higgs inflation[1–4]

- EW vacuum $v \sim 10^2$ GeV vs. energy scale of inflation $E_{\inf} \sim 10^{16}$ GeV
- Dependence of couplings on the energy scale essential: $g_i \to g_i(t)$
- Renormalization group flow governed by β functions: $\frac{\mathrm{d}g_i}{\mathrm{d}t} = \beta_{g_i}$
- β functions perturbatively extracted from off-shell quantum divergences
- Numerically, M_{H} comes very close to 126 GeV, but...

  ...what if quantum corrections are frame dependent?

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[1] Bezrukov, Magnin and Shaposhnikov (2009) *Phys. Lett. B* **675** 88-92.
[2] De Simone, Hertzberg and Wilczek (2009) *Phys. Lett. B* **678** 1-8.
[3] Barvinsky, Kamenshchik, Kiefer, Starobinsky and C.S. (2009) *JCAP* **12** 003.
[4] Bezrukov and Shaposhnikov (2009) *JHEP* **07** 089.

Transition between Jordan frame and Einstein frame

- Explicit transformations between the frames $(g_{\mu\nu}, \varphi) \to (\hat{g}_{\mu\nu}, \hat{\varphi})$:

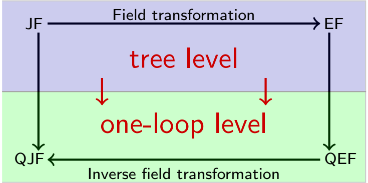
$$S_{\rm J}[g,\varphi] = \int d^4x\, g^{1/2} \Big[U(\varphi)\, R(g) - \tfrac{1}{2}\, G(\varphi)\, g^{\mu\nu}\, \nabla_\mu\varphi\, \nabla_\nu\varphi - V(\varphi) \Big]$$

$$\downarrow \quad g_{\mu\nu} = \tfrac{U_0}{U}\, \hat{g}_{\mu\nu}, \qquad U\sqrt{g}\, R = U_0 \sqrt{\hat{g}} \Big[R(\hat{g}) - \tfrac{3}{2} \left(\tfrac{U'}{U}\right)^2 (\hat{\nabla}\varphi)^2 \Big] \quad \downarrow$$

$$\downarrow \quad \left(\tfrac{\partial\hat{\varphi}}{\partial\varphi}\right)^2 = \left(\tfrac{U_0}{U}\right) \left(\tfrac{G\,U + 3\,(U')^2}{U} \right), \qquad \hat{V}(\hat{\varphi}) = U_0^2\, \tfrac{V(\varphi)}{U^2(\varphi)} \Big|_{\varphi = \varphi(\hat{\varphi})} \quad \downarrow$$

$$S_{\rm E}[\hat{g},\hat{\varphi}] = \int d^4x\, \hat{g}^{1/2} \Big[U_0\, R(\hat{g}) - \tfrac{1}{2}\, \hat{g}^{\mu\nu}\, \hat{\nabla}_\mu\hat{\varphi}\, \hat{\nabla}_\nu\hat{\varphi} - \hat{V}(\hat{\varphi}) \Big]$$

- Tree-level actions are equivalent, but what about quantum equivalence?



One-loop calculations in the Einstein frame

- Effective action: $\Gamma^{\mathrm{div}}_{1-\text{loop}} \propto \int \mathrm{d}^4x \frac{g^{1/2}}{\epsilon} \sum_i \alpha^i_{\text{1-loop}}(\varphi)\, \mathcal{O}_i[R, \nabla\varphi]$

- Effective potential: $\mathcal{O}_i = \mathbf{1}$ might serve as indicator for frame depedence

- One-loop effective potential calculated in the Einstein frame[1,2]:

$$\hat{g}^{1/2}\, \hat{V}^{\mathrm{div,\,E}}_{1-\mathrm{loop}} = \hat{g}^{1/2} \left\{ \frac{1}{2} (\partial^2_{\hat{\varphi}} \hat{V})^2 - \frac{2}{U_0} (\partial_{\hat{\varphi}} \hat{V})^2 + \frac{5}{U_0^2} \hat{V}^2 \right\}$$

- Inverse transformation back to Jordan frame:

$$\hat{g}^{1/2} = \frac{U^2}{U_0^2}\, g^{1/2}, \qquad \hat{V} = \frac{U_0^2}{U^2}\, V, \qquad \frac{\partial \hat{V}}{\partial \hat{\varphi}} = \frac{U_0^{3/2}}{U^2}\, \frac{(V'\,U - 2\,U'\,V)}{\sqrt{G\,U + 3\,U'^2}},$$

$$\frac{\partial^2 \hat{V}}{\partial \hat{\varphi}^2} = \frac{U_0}{U^2 \left[G\,U + 3\,(U)'^2\right]^2} \Big[12\,V\,(U')^4 - 9\,U\,(U')^3\,V' - 3\,U^2\,U'\,V'\,U'' + 3\,U^2\,(U')^2\,V''$$
$$+ G\left(5\,U\,V\,(U')^2 - \tfrac{7}{2}\,U^2\,U'\,V' - 2\,U^2\,V\,U'' + U^3\,V''\right) + G'\left(U^2\,U'\,V - \tfrac{1}{2}\,U^3\,V'\right)\Big]$$

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1 Barvinsky, Kamenshchik and Karmazin (1993) *Phys.Rev. D* **48** 3677-3694.

2 Kamenshchik and C. S. (2015) *Phys.Rev. D* **91** 8, 084033.

Einstein frame effective potential in the Jordan frame

One-loop effective potential calculated in the Einstein frame and expressed in terms of the Jordan frame parametrization[1]:

$$
\begin{aligned}
g^{1/2}\, V^{\text{div, E}}_{1-\text{loop}} = g^{1/2} \Big\{ V^2 \Big[& s^4 \Big(\frac{6G'(U')^3 U''}{U^3} - \frac{3G'(U')^5}{U^4} + \frac{(G')^2 (U')^2}{2U^2} - \frac{18 (U')^6 U''}{U^5} + \frac{18 (U')^4 (U'')^2}{U^4} \\
& + \frac{9(U')^8}{2U^6} \Big) + s^3 \Big(-\frac{2G'U'U''}{U^2} + \frac{5G'(U')^3}{U^3} + \frac{36(U')^4 U''}{U^4} - \frac{12(U')^2(U'')^2}{U^3} - \frac{15(U')^6}{U^5} \Big) \\
& + s^2 \Big(-\frac{10(U')^2 U''}{U^3} + \frac{25(U')^4}{2U^4} + \frac{2(U'')^2}{U^2} \Big) - s\, \frac{8(U')^2}{U^3} + \frac{5}{U^2} \Big] + VV' \Big[s^4 \Big(\frac{-6G'(U')^2 U''}{U^2} \\
& + \frac{3G'(U')^4}{U^3} - \frac{(G')^2 U'}{2\,U} + \frac{18(U')^5 U''}{U^4} - \frac{18(U')^3(U'')^2}{U^2} - \frac{9(U')^7}{2\,U^5} \Big) + s^3 \Big(\frac{G'U''}{U} - \frac{6G'(U')^2}{U^2} \\
& - \frac{39(U')^3 U''}{U^3} + \frac{6U'(U'')^2}{U^2} + \frac{18(U')^5}{U^4} \Big) + s^2 \Big(\frac{7U'U''}{U^2} - \frac{35(U')^3}{2\,U^3} \Big) \Big] + 8\, s\, \frac{U'}{U^2} + VV'' \Big[s^3 \Big(\frac{G'U'}{U} \\
& + 6\frac{(U')^2 U''}{U^2} - \frac{3(U')^4}{U^3} \Big) + s^2 \Big(\frac{5(U')^2}{U^2} - \frac{2U''}{U} \Big) \Big] + (V')^2 \Big[s^4 \Big(\frac{3G'U'U''}{2U} - \frac{3G'(U')^3}{4U^2} + \frac{(G')^2}{8} \\
& - \frac{9(U')^4 U''}{2U^3} + \frac{9(U')^2(U'')^2}{2U^2} + \frac{9(U')^6}{8U^4} \Big) + s^3 \Big(\frac{7G'U'}{4U} + \frac{21(U')^2 U''}{2U^2} - \frac{21(U')^4}{4U^3} \Big) + \frac{49 s^2 (U')^2}{8U^2} - s\, \frac{2}{U} \Big] \\
& + V'V'' \Big[s^3 \Big(-\frac{G'}{2} - \frac{3U'U''}{U} + \frac{3(U')^3}{2U^2} \Big) - s^2\, \frac{7U'}{2U} \Big] + s^2\, \frac{1}{2}\, (V'')^2 \Big\}
\end{aligned}
$$

[1] Kamenshchik and C. S. (2015) *Phys.Rev. D* **91** 8, 084033.

Direct one-loop Jordan frame calculations and comparison

- One-loop effective potential calculated directly in the Jordan frame[1]:

$$g^{1/2}V^{\text{div, J}}_{1\text{-loop}} = g^{1/2}\Big\{V^2\left[\frac{2\,s^2\,(U')^4}{U^4}-\frac{2\,s\,(U')^2}{U^3}+\frac{5}{U^2}\right]+V\,V'\left[-\frac{8\,s^2\,(U')^3}{U^3}+\frac{4\,s\,U'}{U^2}\right]$$
$$+2\,V\,V''\,\frac{s^2\,(U')^2}{U^2}+(V')^2\left[\frac{8\,s^2\,(U')^2}{U^2}-\frac{2\,s}{U}\right]-4\,V'V''\,\frac{s^2\,U'}{U}+\frac{1}{2}(V'')^2\,s^2\Big\}$$

- "Suppression function" $s(\varphi)=\frac{U}{GU+3(U')^2}$

- Comparison yields: $\Delta\Gamma^{\text{div, off-shell}}_{1-\text{loop}} := \left[\Gamma^{\text{div, E}}_{1\text{-loop}} - \Gamma^{\text{div, J}}_{1\text{-loop}}\right] \neq 0$

- $\Rightarrow$ off-shell one-loop divergences are frame-dependent[2,3]

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[1] Kamenshchik and C. S. (2011) *Phys. Rev. D* **84** 024026.
[2] Kamenshchik and C. S. (2012) *AIP Conf. Proc.* **1514** 161-164.
[3] Kamenshchik and C. S. (2015) *Phys.Rev. D* **91** 8, 084033.

On-shell comparison

- Formal equivalence theorem[1−2]: S-matrix is parametrization independent

- Jordan frame equations of motion $\delta S_{\rm J}/\delta g^{\alpha\beta} = 0$ and $\delta S_{\rm J}/\delta\varphi = 0$:

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = \left(\frac{G + 2\,U''}{2\,U}\right)\varphi_{,\alpha}\varphi_{,\beta} - \left[\left(\frac{G+4\,U''}{4\,U}\right)(\nabla\varphi)^2 - \frac{U'}{U}\Box\varphi - \frac{1}{2}\,\frac{V}{U}\right] g_{\alpha\beta} + \frac{U'}{U}\,\varphi_{;\alpha\beta}$$

$$\Box\varphi = -\frac{U'}{G}\,R - \frac{1}{2}\,\frac{G'}{G}\,(\nabla\varphi)^2 + \frac{V'}{G}$$

- Constant scalar background field $\nabla\varphi = 0$ leads to on-shell identities[3]:

$$V \triangleq U^2, \qquad R \triangleq 2\,U, \qquad R_{\mu\nu}R^{\mu\nu} \triangleq U^2$$

- Non-trivial cancellation ensures explicit on-shell equivalence[3]

$$\Gamma^{\rm div,\,E}_{1-{\rm loop}}[g,\varphi] \triangleq \frac{1}{32\,\pi^2\,\epsilon}\int {\rm d}^4x\, g^{1/2}\left(-\frac{57}{20}\,U^2\right) \triangleq \Gamma^{\rm div,\,J}_{1-{\rm loop}}[g,\varphi]$$

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[1] Chisholm (1961) *Nucl. Phys.* **26** 469.

[2] Kallosh and Tyutin (1973) *Yad. Fiz.* **17** 190-209.

[3] Kamenshchik and C. S. (2015) *Phys.Rev. D* **91** 8, 084033.

Origin of frame dependence and covariant formalism

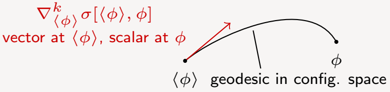
- Ordinary definition of the effective action (DeWitt notation)

$$\exp \frac{i}{\hbar} \Gamma[\langle\phi\rangle] = \int \mathcal{D}\phi \exp \frac{i}{\hbar} \left\{ S[\phi] + \frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle^k} \left(\langle\phi\rangle^k - \phi^k \right) \right\}$$

- Consider configuration space $\mathcal{C}$ as differentiable manifold

  Vilkovisky's geometric proposal[1]: $(\langle\phi\rangle^k - \phi^k) \to \nabla^k_{\langle\phi\rangle} \sigma[\langle\phi\rangle, \phi]$

"world function" $\sigma[\langle\phi\rangle, \phi] = \frac{1}{2}(\text{geodesic distance between } \langle\phi\rangle \text{ and } \phi)^2$



- One-loop: $\Gamma^{\text{ordinary}}_{1-\text{loop}} \sim \frac{1}{2} \text{Tr} \ln S_{,\,ij}$ not a true scalar, since $S_{,ij}$ not a tensor
- Make it covariant: $S_{,\,ij} \to \nabla_i \nabla_j S = S_{,\,ij} - \Gamma^k{}_{ij}\, S_{,\,k}$
- Consistent with on-shell equivalence $S_{,k} = 0$: $\Gamma^{\text{unique}}_{1-\text{loop}} = \Gamma^{\text{ordinary}}_{1-\text{loop}}$

[1] Vilkovisky (1984) *Nucl. Phys. B* **234** 125-137.

Conclusion & Outlook

- Off-shell quantum divergences are parametrization dependent
- On-shell result in agreement with equivalence theorem
- Geometric approach can explain the origin of the off-shell quantum frame ambiguity and at the same time might suggest a natural resolution
- Implications of the covariant approach for the cosmological debate:
  - Jordan and Einstein frames are just two particular coordinate systems in field space (among infinitely many others)
  - Quantum ambiguity arises due to the non-covariance of the formalism. There is no "physically preferred" frame
- Not just academic: parametrization independent observables required
- Choice of parametrization should have no physical meaning

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Kamenshchik and C. S. (2015) *Phys.Rev. D* **91** 8, 084033.

Moss (2014) arXiv:1409.2108v2, Moss (2015) arXiv:1509.03554v1