Statistical Quantities and Definition of Terms for Cooling Channel Beam Optics

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Introduction

- MICE's stated goal: "to demonstrate that the volume occupied by a muon beam can be reduced"
- This cannot occur with non-interacting particles acted upon solely by electromagnetic fields
- Phase space volume is often characterized by emittances
- One way the lattice impacts cooling performance is via the beta function at the absorber
- I will discuss the precise meaning of these comcepts, and how they relate to what MICE is trying to accomplish

Introduction

- What we care about are beam sizes
  - Luminisity
  - Beam size for keeping inside apertures
- Magnets create a relationship between beam size and phase space volume: beta function
  - Limited based on magnet technology, physical constraints, beam dynamics
- Given these limitations (magnet technology,...), the only way to improve what we want (luminosity, beam size) is to reduce phase space volume

Outline

- Concepts from Hamiltonian Dynamics
  - Invariant quantites
  - Periodic systems
  - Normalization
  - Canonical, kinetic, and scaled momenta
- Particle distributions
  - Evolution of the distribution
  - Second moment matrix
  - Normalization and emittances
  - Linearization
  - Mismatch
  - Rotational invariance

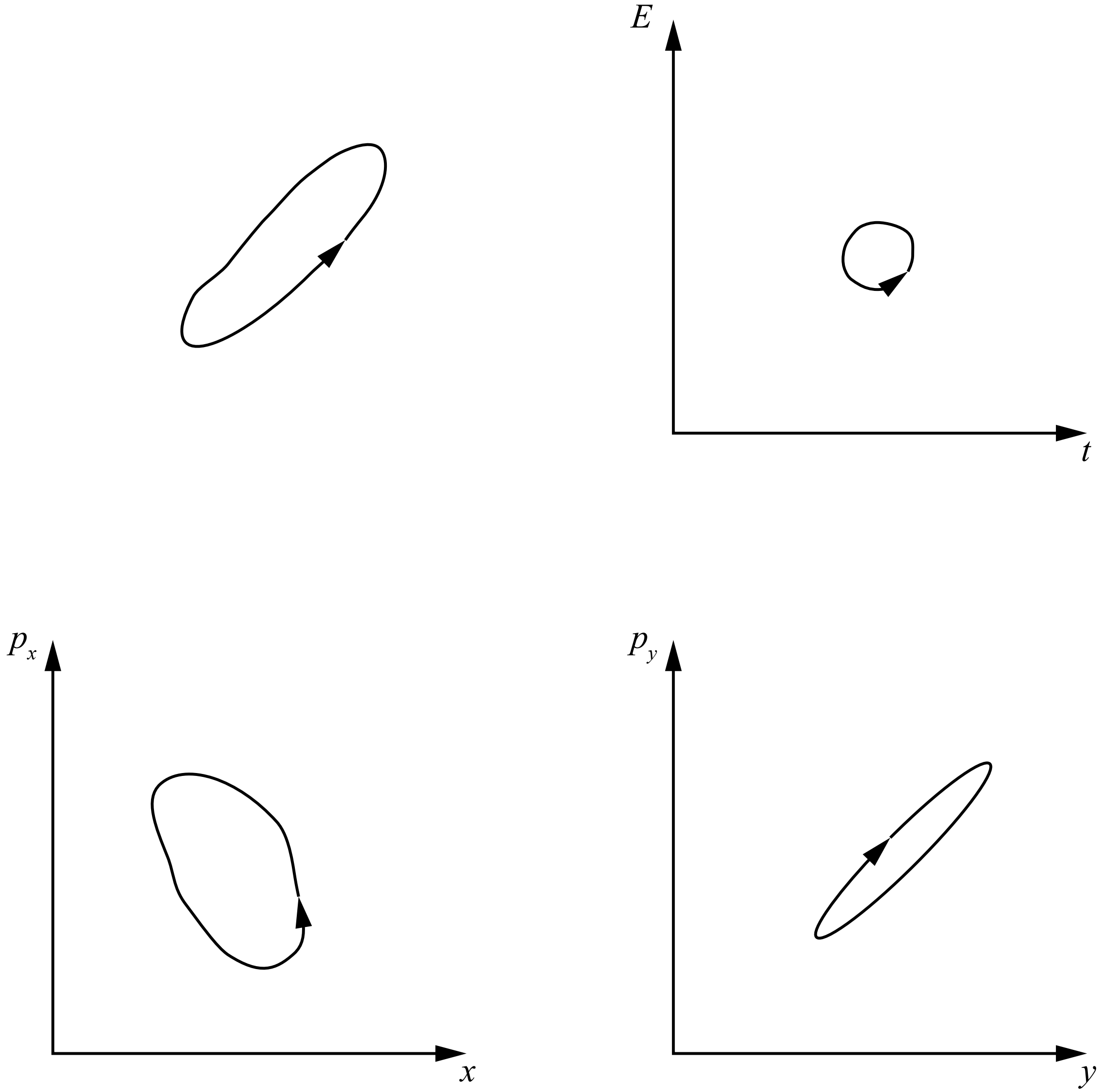
Outline

- Directly measuring phase space density

Poincaré's Integral Invariant

- Definition of the invariant:
  - Start with a closed curve in $2n$-dimensional phase space
  - Project the curve into the n coordinate-momentum planes
  - Add up the (oriented) areas in these planes
- Take all points on the curve, and evolve them according to Hamilton's equations of motion; the invariant doesn't change
- Change variables via a canonical transformation
  - This changes the definitions of coordinates and momenta, and therefore the planes
  - The invariant doesn't change

Poincaré's Integral Invariant



Poincaré’s Integral Invariant

- Poincaré’s integral invariant is the fundamental thing we’re trying to beat with cooling
- Poincaré’s integral invariant should be vaguely familiar
  - It has a similarity to some definitions of emittances, in terms of a phase space area
  - It is even closer to the definition of action (as in action-angle variables)
- Some parts of this seem a bit off
  - The curve is completely arbitrary
  - Adding the areas in the planes together
  - What about phase space density?

Liouville's Theorem

- Two equivalent forms
  - The integral form:
    - Pick a bounded region in phase space
    - Evolve every point in that region according to a Hamiltonian
    - The volume of that region doesn't change
  - The differential form:
    - Define a probability distribution in phase space, and choose a point within that distribution
    - Evolve the probability distribution and the point according to the Hamiltonian
    - The value of the probability distribution (i.e., the phase space density) at that point does not change
- Liouville's theorem follows directly from Poincaré's integral invariant (but not the reverse)

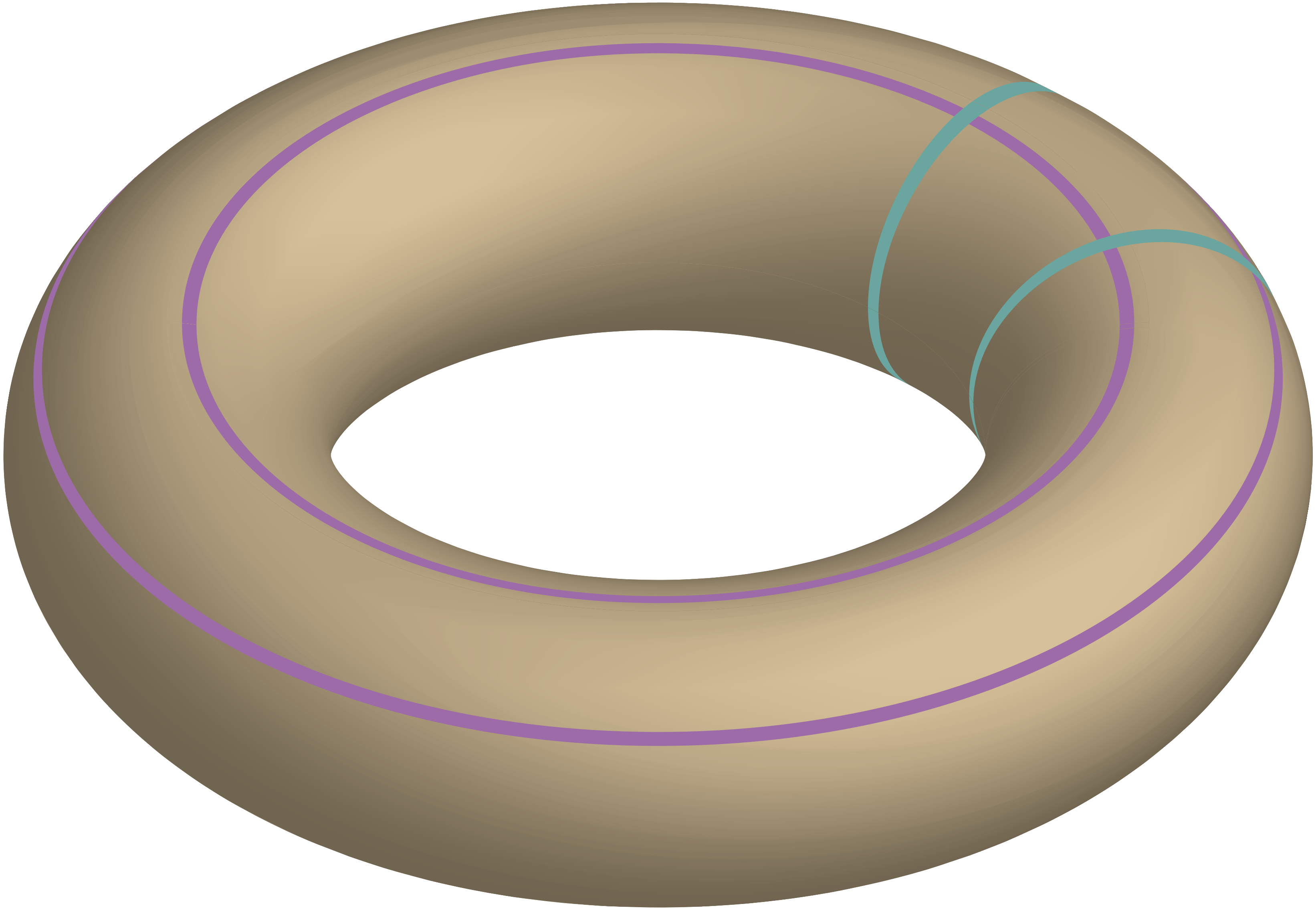
Periodic Systems

- Definition: the Hamiltonian is the same at $s + L$ as it is at s for any s
- Create an object in phase space as follows:
  - Start at some initial phase space point
  - Evolve to one period later, add the point
  - Continue adding points after each period
- The resulting object is an n-dimensional surface in $2n$-dimensional phase space
  - Only for stable, integrable systems
  - Accelerators are close enough for this to be useful

Periodic Systems

- Draw a closed curve on that surface
  - This curve has a poincare integral invariant
  - Slide the curve any way you like on the surface, the poincare integral invariant doesn't change
  - There are n independent curves that can't be distorted into each other or collapsed to a point
  - Value of the integral invariant for curve i is J_i, the action
- Summary: any particle in a periodic system can be assigned n invariant action values (to the extent that the system is integrable)

Periodic Systems



Periodic Systems: Normalization

- Define a transformation from “real” phase space $(\vec{x}, \vec{p})$ to “normalized phase space $(\vec{x}_N, \vec{p}_N)$
  - Transformation defined by $x_{N,i}^2 + p_{N,i}^2 = 2J_i$
  - In these new variables, motion is on circles in each phase space plane
- The transformation itself defines the relationship between the invariant action and the bounds of the motion in physical space
  - A magnetic lattice cannot modify the action, but it can modify the transformation

Periodic Systems: Normalization

- Familiar example: decoupled, linear motion. Transformation is

$$x = \sqrt{\beta}x_N \qquad p_x = (p_{x,N} - \alpha x_N)/\sqrt{\beta}$$

- β and α are the Courant-Snyder lattice functions
- β relates the beam size to the oscillation amplitude

Periodic Systems: Examples

- A constant solenoid field
- A repeating lattice cell
- A ring

Canonical and Kinetic Momentum

- Kinetic momenta are what we know and love

$$\vec{p}_K = \gamma m \frac{d\vec{x}}{dt} \quad E_K = \gamma m c^2 \quad E_K^2 = |\vec{p}_K|^2 c^2 + (mc^2)^2$$

- With electromagnetic fields, these momenta (with geometric coordinates and clock time) do not obey Hamilton's equations of motion
- Defime canonical momenta which do obey Hamilton's equations of motion

$$\vec{p}_C = \vec{p}_K - q\vec{A} \qquad E_C = E_K - q\Phi$$

- Only canonical variables have the behavior I've described

Canonical and Kinetic Momentum

- In accelerators, we use longitudinal position as the independent variable
  - Coordinates are x, y, and t
  - Momenta are p_x, p_y, and E
- Generally no scalar potential
- Kinetic and canonical momenta different only in the presence of transverse vector potentials
- Transverse vector potentials rarely important:
  - Edge focusing from dipole magnets
  - End fields of specialized magnets (IR quads)
  - Solenoids (oops…)

Scaled Momenta

- At fixed energy, it is convenient to scale all momenta (and the Hamiltonian) by a reference mometum
  - Energy is a bit trickier…
  - There are other conventions, such as total momentum, longitudinal kinetic momentum, etc.
  - Difficulties when energy changes
- These momenta are now, to lowest order, angles
- Can scale canonical or kinetic momenta
- This is the common usage in accelerator physics
  - Beta functions (as defined above) are in m for these variables, not for real momenta

Distributions

- Deterministic and stochastic contributions to particle motion
- Deterministic equation for phase space coordinates

$$\frac{d\vec{z}}{ds} = \vec{f}(\vec{z}, s)$$

- Stochastic: probability distribution $\rho(\vec{x}, \vec{z}, s)$ such that $\rho(\vec{x}, \vec{z}, s)d\vec{x}\,ds$ is probability that
  - For a particle at $\vec{z}$
  - In the interval $[s, s + ds)$
  - Particle is displaced somewhere in the phase space volume element $[\vec{x}, \vec{x} + d\vec{x})$
  - Again, vectors are phase space

Distributions

- Probability distribution $\psi(\vec{z}, s)$ in phase space
- Write the continuity equation for ψ

$$\frac{\partial\psi}{\partial s} + \vec{\nabla} \cdot [\psi(\vec{z}, s)\vec{f}(\vec{z}, s)] = \int \psi(\vec{z} - \vec{x}, s)\rho(\vec{x}, \vec{z} - \vec{x}, s)d\vec{x} - \psi(\vec{z}, s) \int \rho(\vec{x}, \vec{z}, s)d\vec{x}$$

Moments

- Define first and second moments

$$\vec{a}(s) = \int \vec{z}\,\psi(\vec{z}, s)\,d\vec{z}$$

$$\Sigma(s) = \int [\vec{z} - \vec{a}(s)][\vec{z} - \vec{a}(s)]^T \psi(\vec{z}, s)\,d\vec{z}$$

- Define an “average” deterministic vector field containing average effect of stochastics

$$\vec{g}(\vec{z}, s) = \vec{f}(\vec{z}, s) + \int \vec{x}\,\rho(\vec{x}, \vec{z}, s)\,d\vec{x}$$

Moments

- Use continuity equation to write equations for moments: still exact

$$\frac{d\vec{a}}{ds} = \int \vec{g}(\vec{z},s)\psi(\vec{z},s)\,d\vec{z}$$

$$\frac{d\Sigma}{ds} = \int [\vec{z}-\vec{a}(s)]\vec{g}(\vec{z},s)^T\psi(\vec{z},s)\,d\vec{z} + \int \vec{g}(\vec{z},s)[\vec{z}-\vec{a}(s)]^T\psi(\vec{z},s)\,d\vec{z} + \int \vec{x}\vec{x}^T\rho(\vec{x},\vec{z},s)\psi(\vec{z},s)\,d\vec{x}\,d\vec{z}$$

- Σ not on the right hand side (yet…)

Normalization of Σ

- $\Sigma(s)$ can be diagonalized by a symplectic matrix A:

$$\Sigma(s) = A(s)E(s)A^T(s)$$

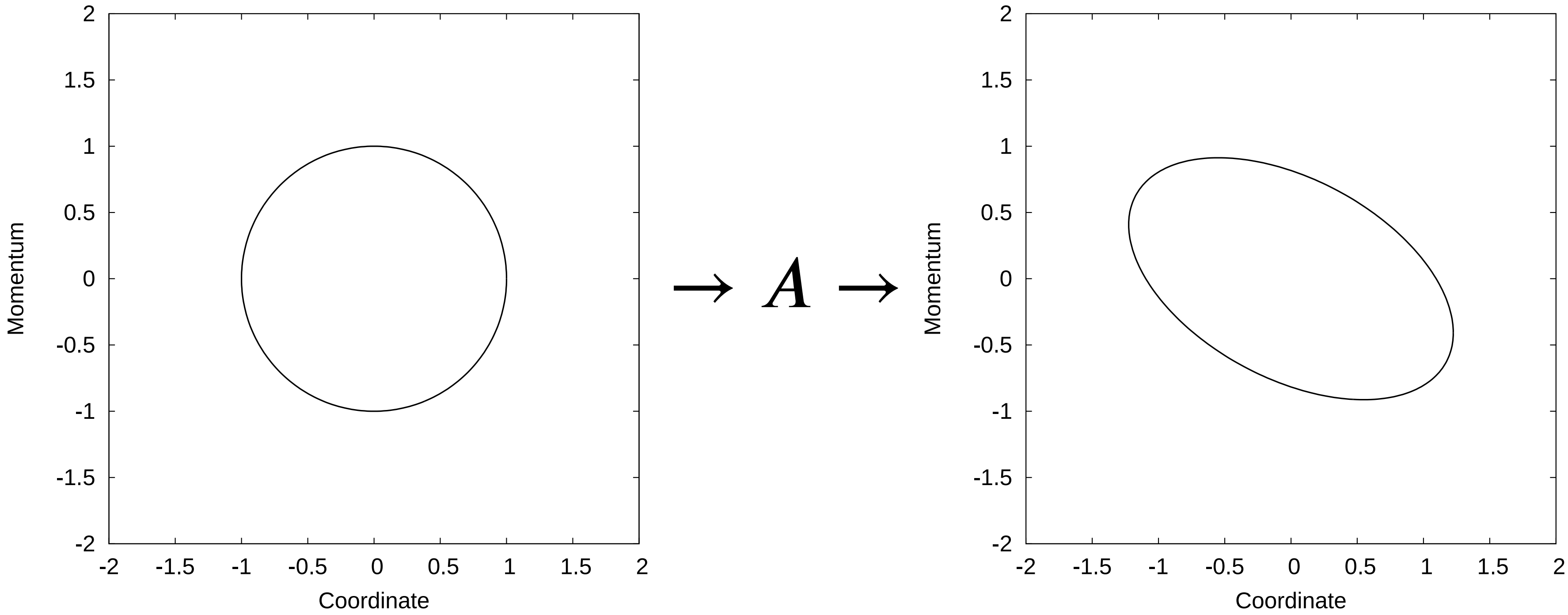
- E is diagonal, with 2×2 blocks of the form

$$\begin{bmatrix} \epsilon_i & 0 \\ 0 & \epsilon_i \end{bmatrix}$$

  - ϵ_i are "RMS" emittances
  - If use scaled momenta, these are un-normalized emittances, in m
  - If momenta not scaled, these are normalized. Units are eV s, scale by mc to get in m

Normalization of Σ

- A gives a *linear* canonical transform from a normalized phase space to the real phase space
  - Note this is without reference to any underlying Hamiltonian system



Normalization of Σ

- Columns A_{2i-1} and A_{2i}, scaled by a factor $\sqrt{2J_i}$, are the semi-axes of an ellipse
  - If particles evolve according to a Hamiltonian, the Poincaré integral invariant J_i for this ellipse remains constant
  - For partcle k, compute the scaling factors $\sqrt{2J_{i,k}}$. ϵ_i is the average of $J_{i,k}$ over k
    - This does *not* mean that ϵ_i is constant: only one particle on the curve; ellipse does not remain an ellipse

Normalization of Σ

- Decoupled system:

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xp} \\ \sigma_{xp} & \sigma_{pp} \end{bmatrix} \qquad \epsilon^2 = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2$$

$$A = \begin{bmatrix} \sqrt{\sigma_{xx}/\epsilon} & 0 \\ -\sigma_{xp}/\sqrt{\epsilon\sigma_{xx}} & \sqrt{\epsilon/\sigma_{xx}} \end{bmatrix} = \begin{bmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{bmatrix}$$

  - β, α are properties of the *beam*; earlier we wrote down similar quantities which were properties of a *periodic lattice*

Linearized Equations

- $\vec{g}(\vec{z}, s) = \vec{g}_0(s) + JH(s)\vec{z}$; H symmetric for a Hamiltonian system

$$\frac{d\Sigma}{ds} = JH(s)\Sigma(s) - \Sigma(s)H^T(s)J + \int \vec{x}\vec{x}^T \rho(\vec{x}, \vec{z}, s)\psi(\vec{z}, s)\, d\vec{x}\, d\vec{z}$$

- Equations for evolution of emittances, lattice functions

$$B(s) = A^{-1}(s)\frac{d\Sigma}{ds}{A^{-1}}^T(s)$$

$$\frac{d\epsilon_i}{ds} = \frac{1}{2}\operatorname{Tr} B_{ii} \qquad \frac{dA}{ds} = -AJC(B, \vec{\epsilon}, \vec{\xi})$$

Linearized Equations

- If Hamiltonian with no stochastics, emittances are constant
  - Need to use canonical momenta in computing second moment matrix
- Nonlinearities in Hamiltonian:
  - Other moments in distribution feed into second moments
  - Lead to apparent emittance growth, or even reduction, when viewed in terms of second moments

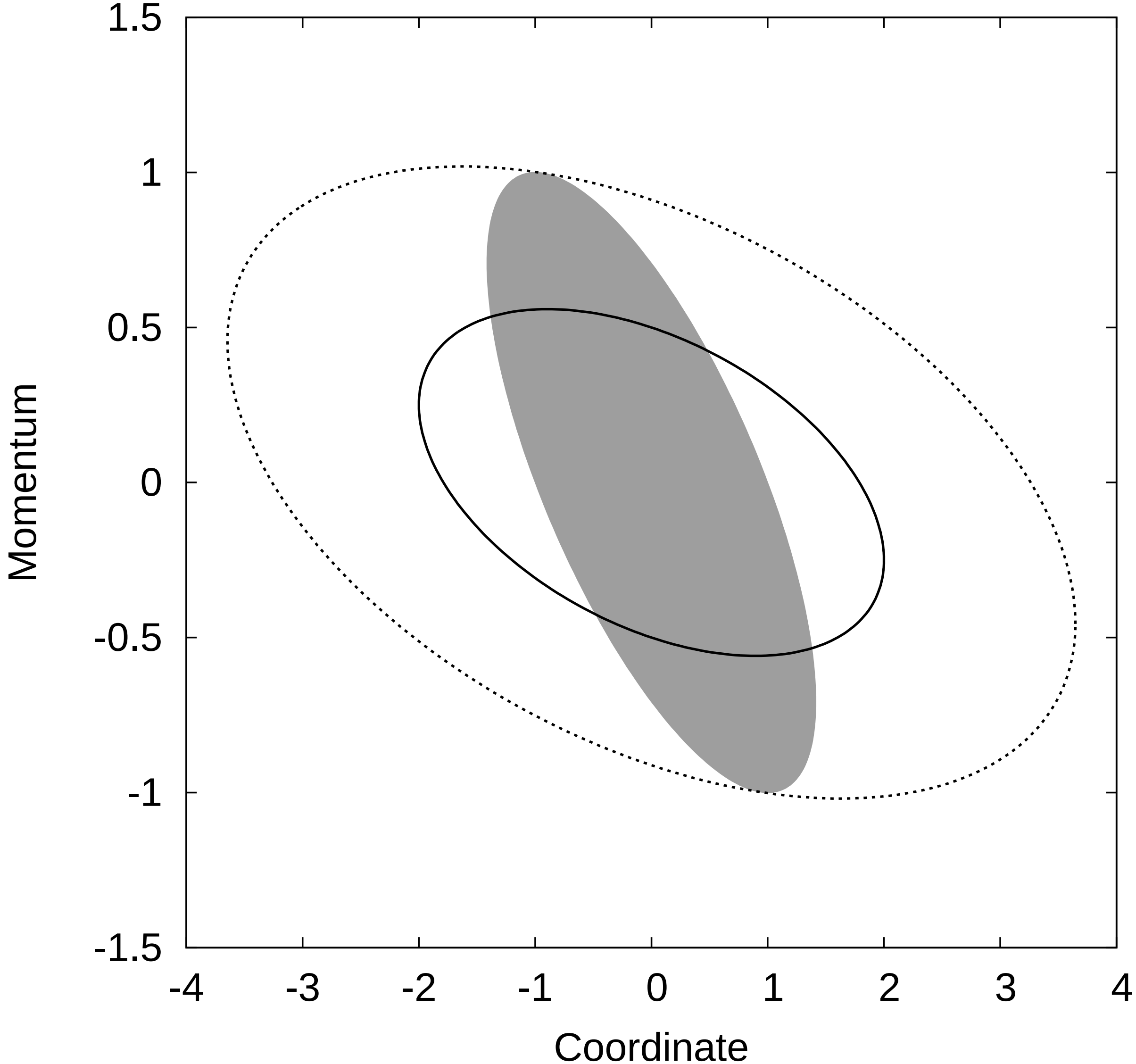
Matching

- Discussed two normalizations
  - Periodic lattice
  - Distribution
- If these normalizations are identical, then emittance would be invariant due to the Poincaré integral invariant
- If they are not identical, the beam is mismatched. This "effectively" increases emittances for two broad reasons
  - Effective beam size increase
  - Nonlinear distortion and filamentation

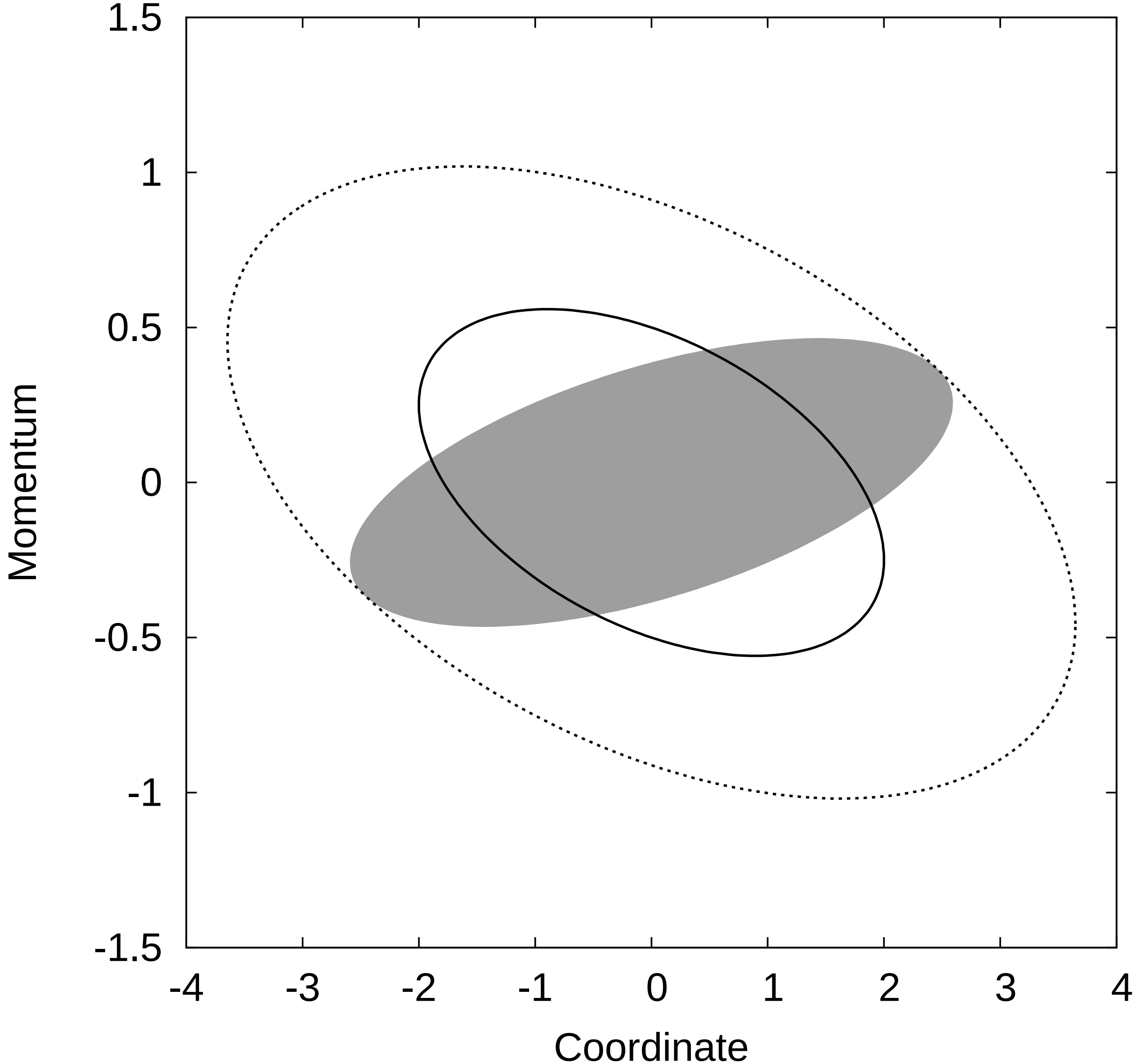
Linear Mismatch

- Assume beam distribution is ellipsoid
- Track in periodic lattice
- Matched: beam ellipse has same shape as ellipse particles follow; ellipse boundary invariant
- Mismatch: beam ellipse different from ellipse particles follow
  - Traces out ellipse of larger area, but same shape that particles follow
  - Beam acts like it has a large area (emittance)
- Poor-man's movie
- All together, filling in large ellipse

Linear Mismatch



Linear Mismatch



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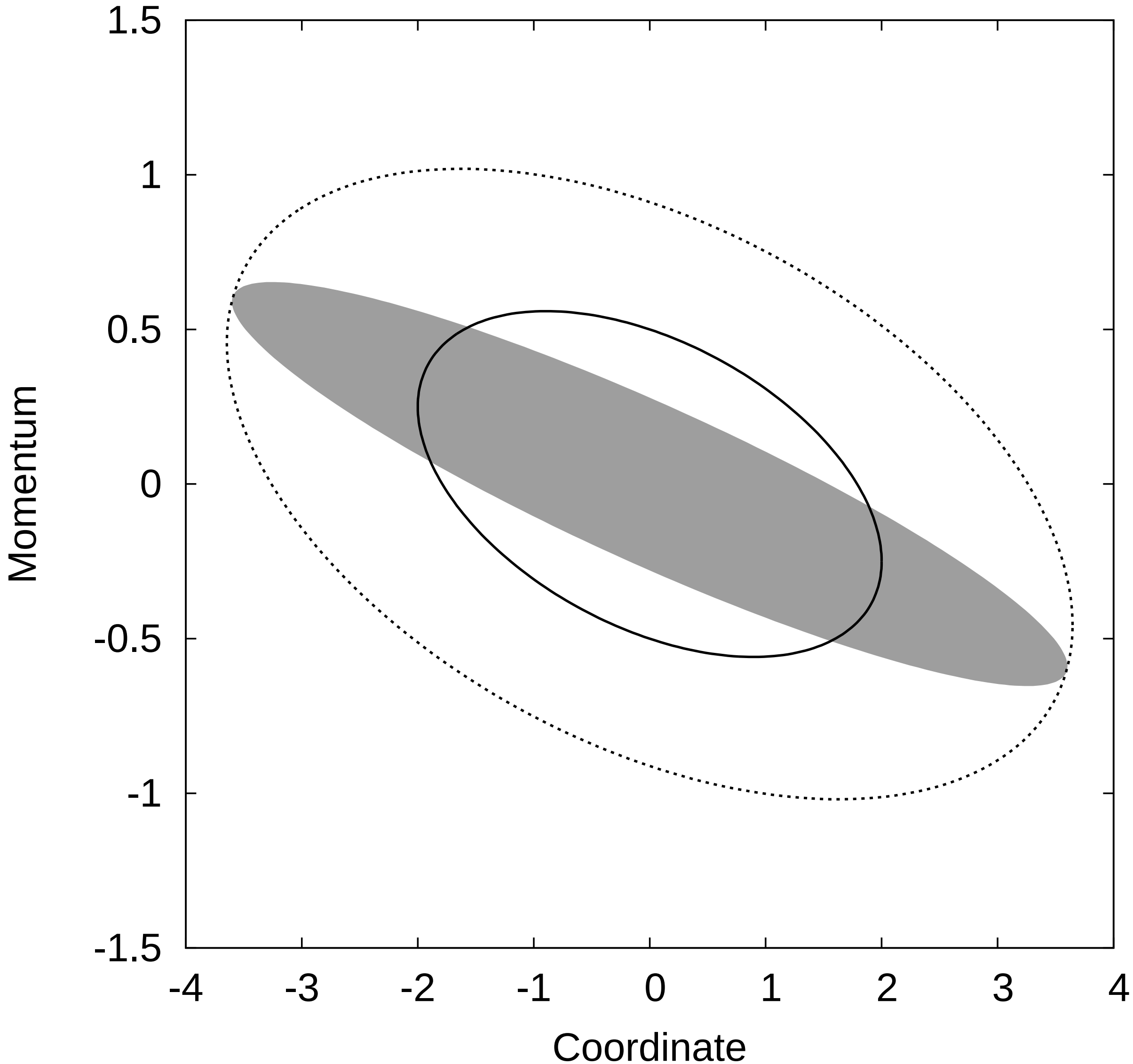

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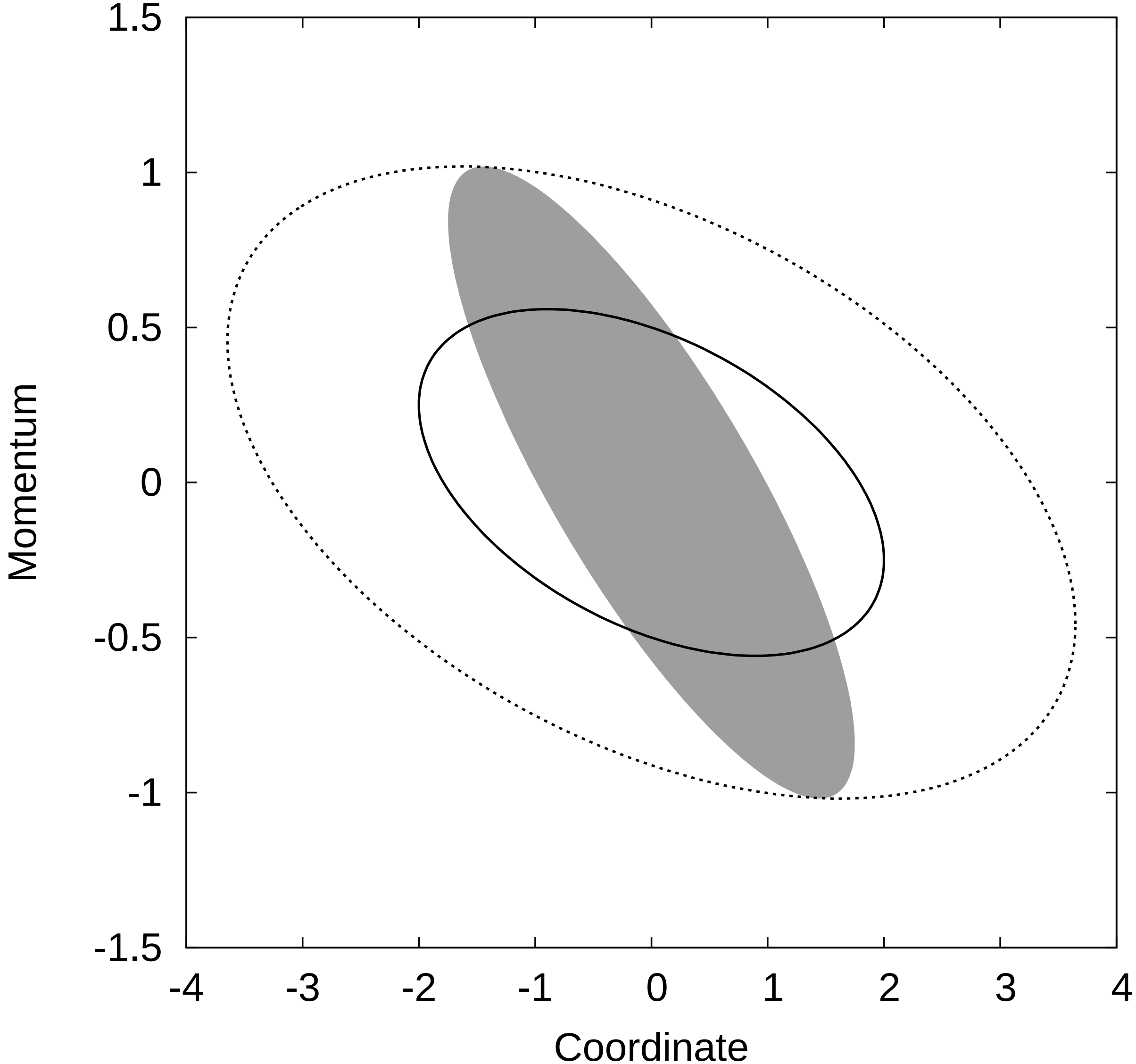

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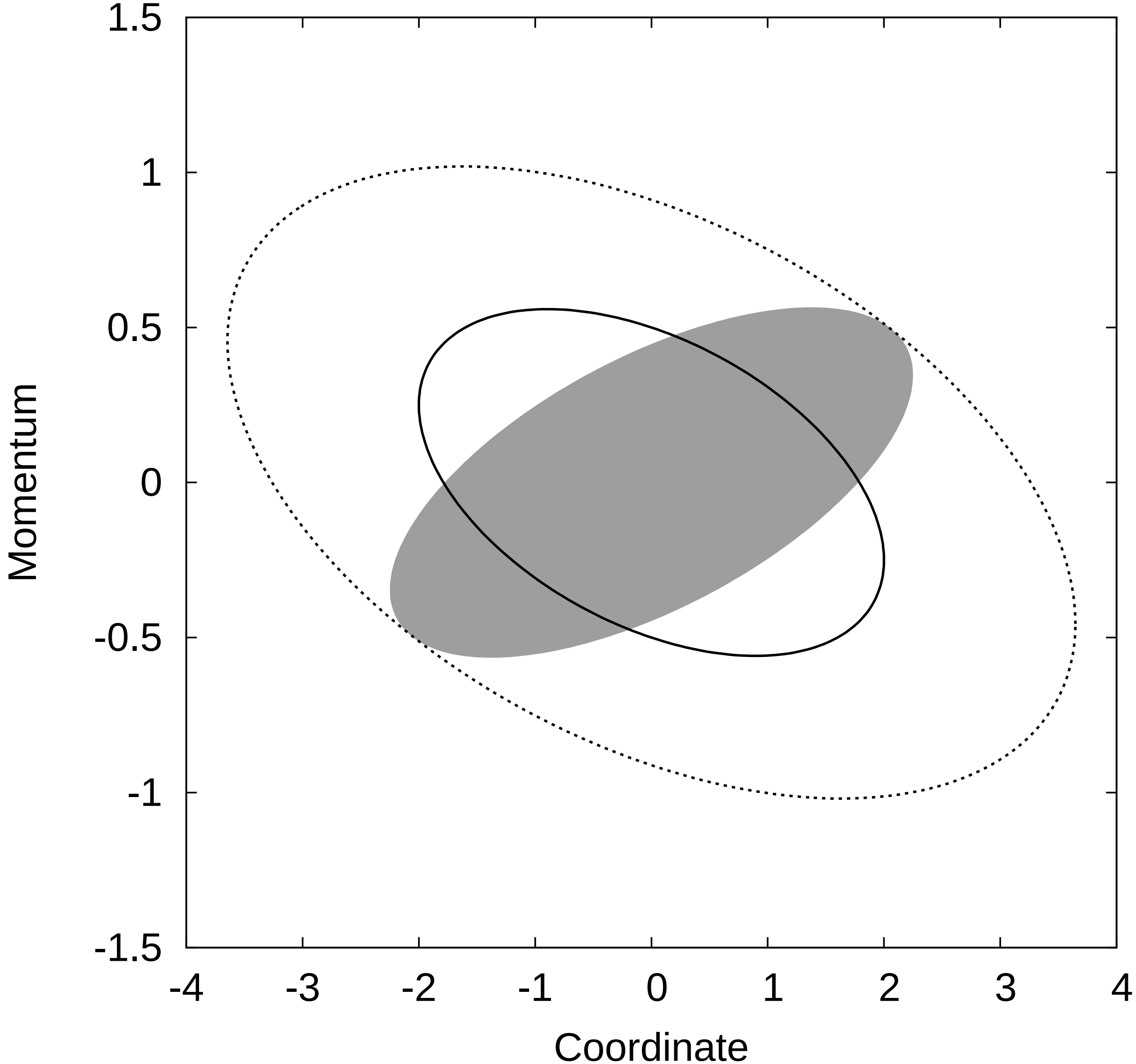

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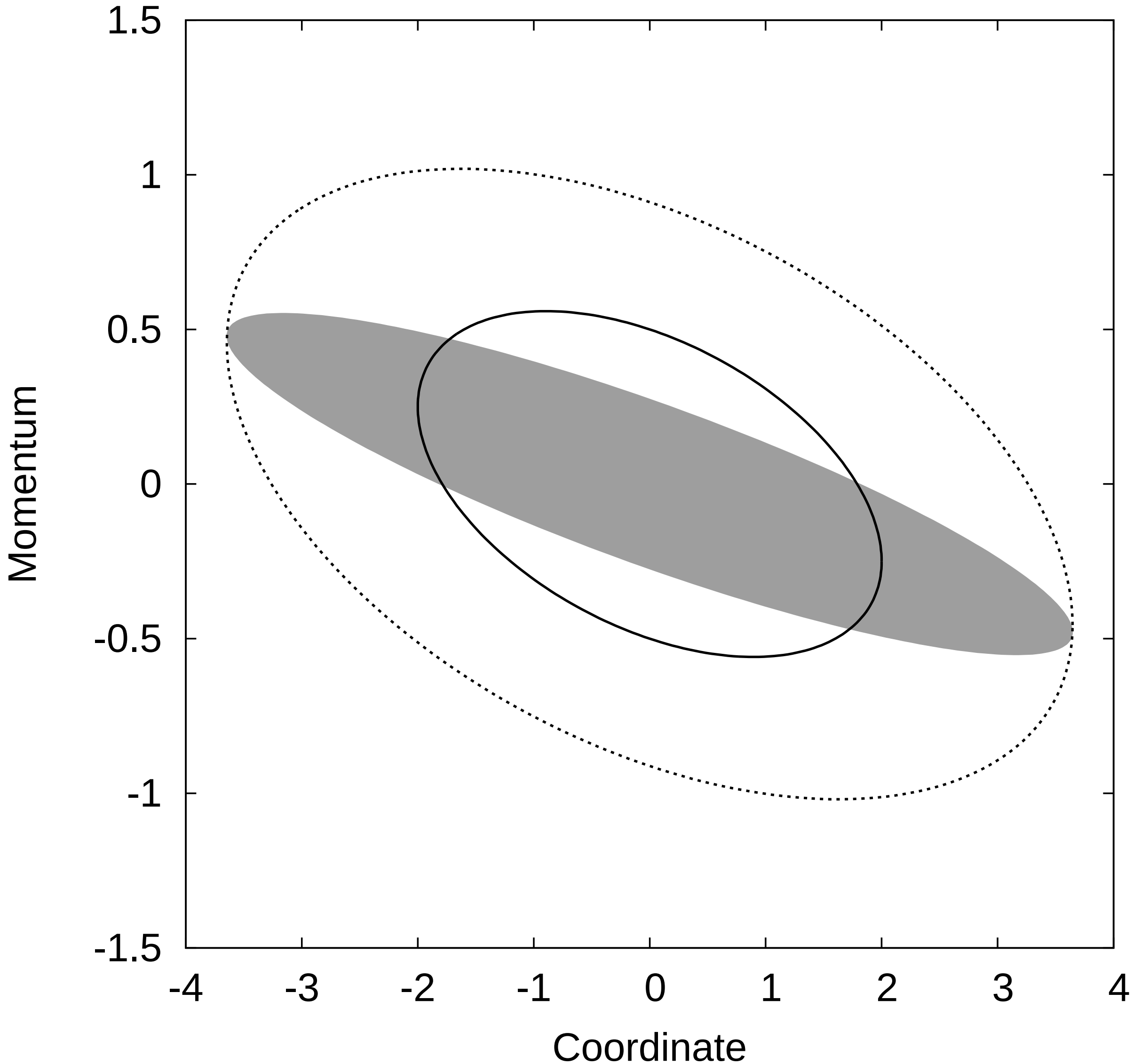

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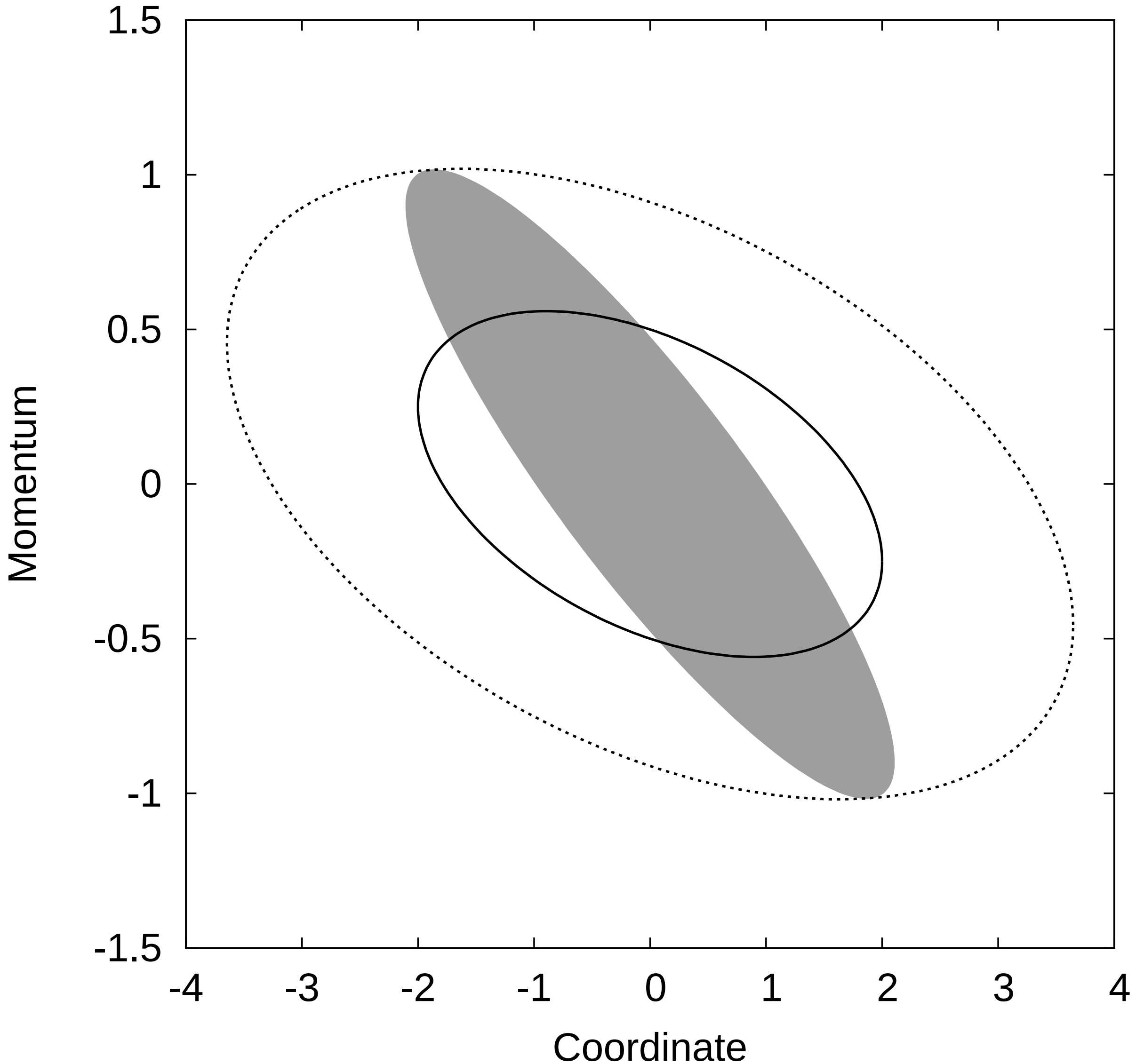

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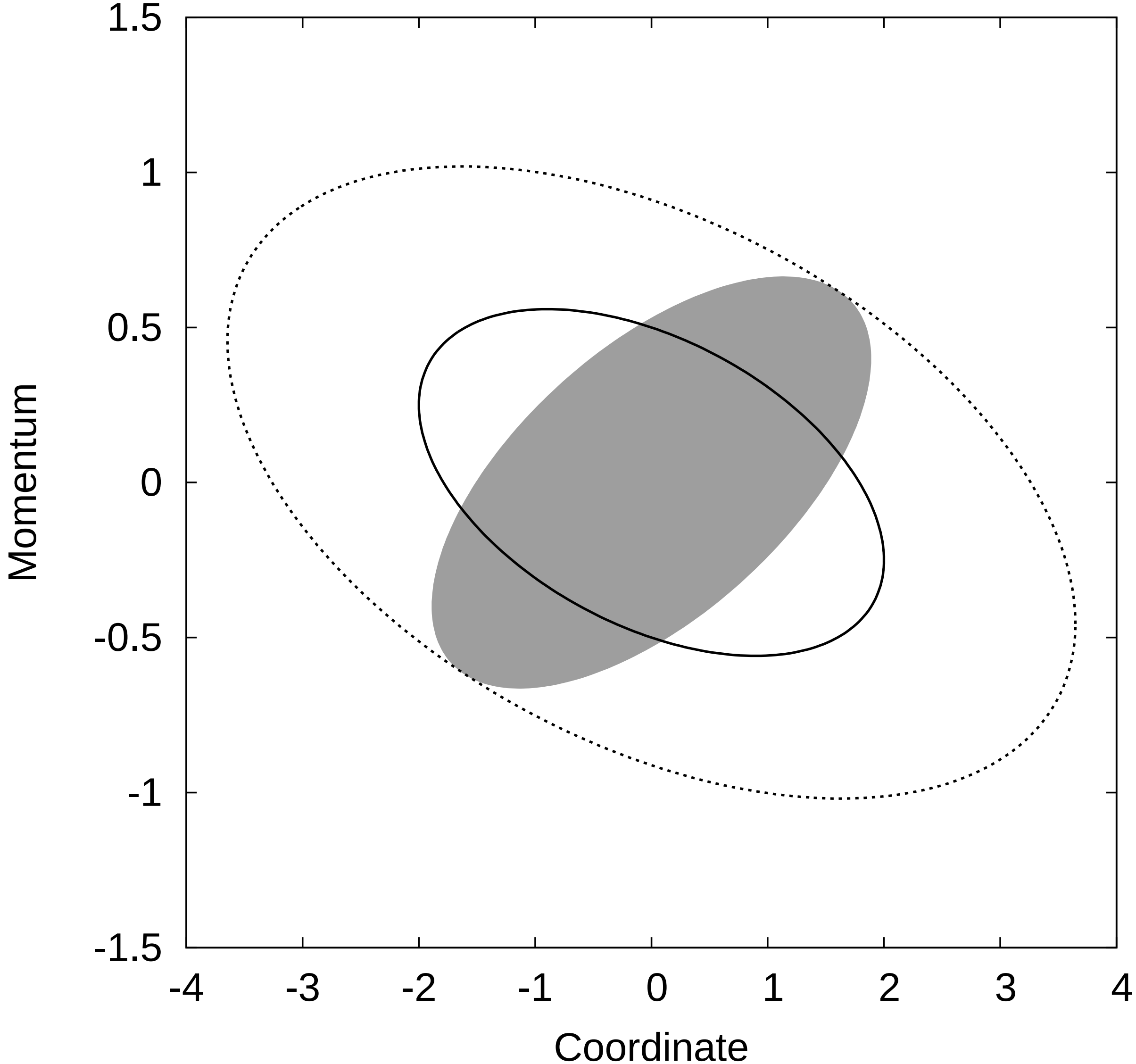

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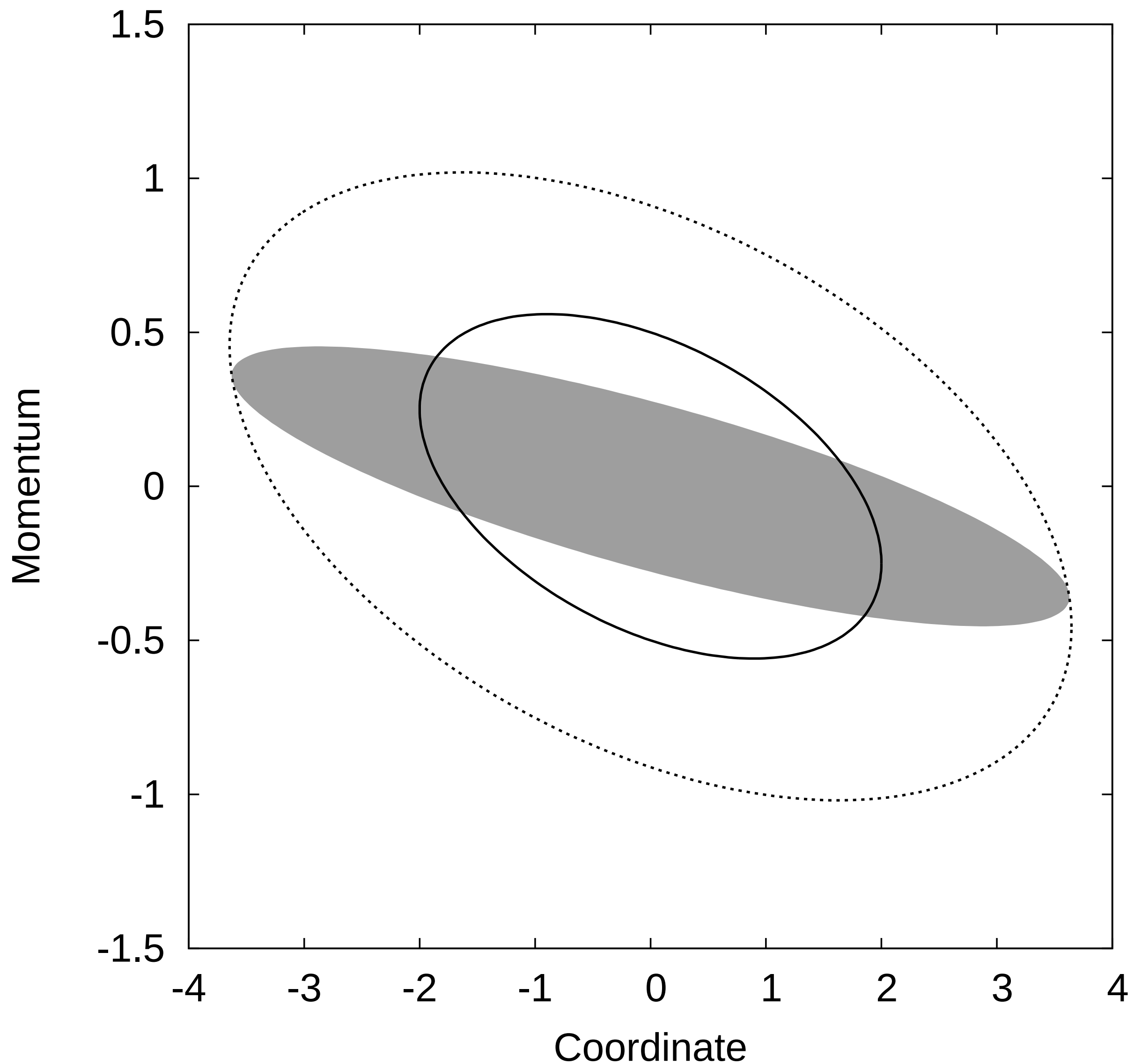

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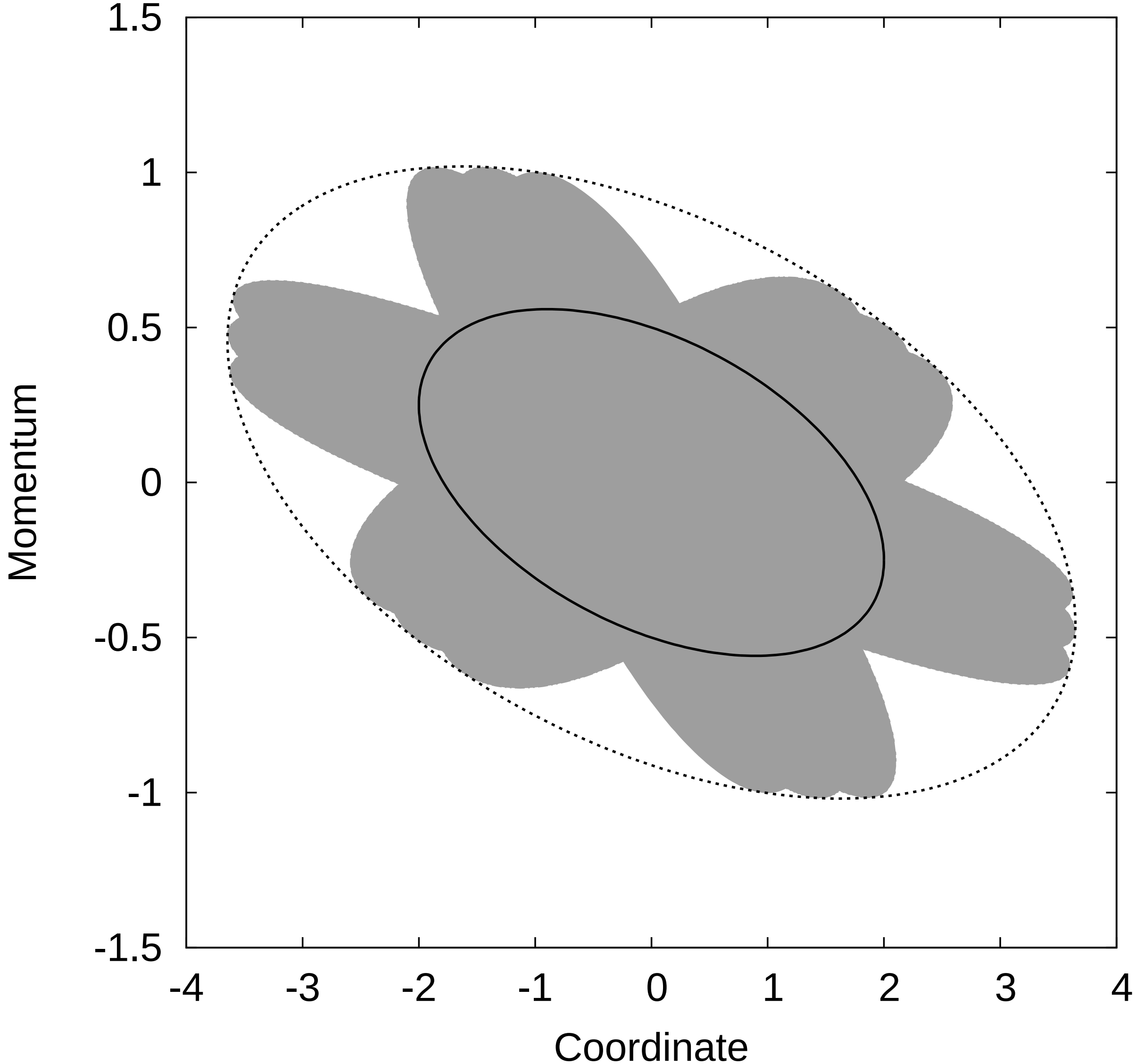
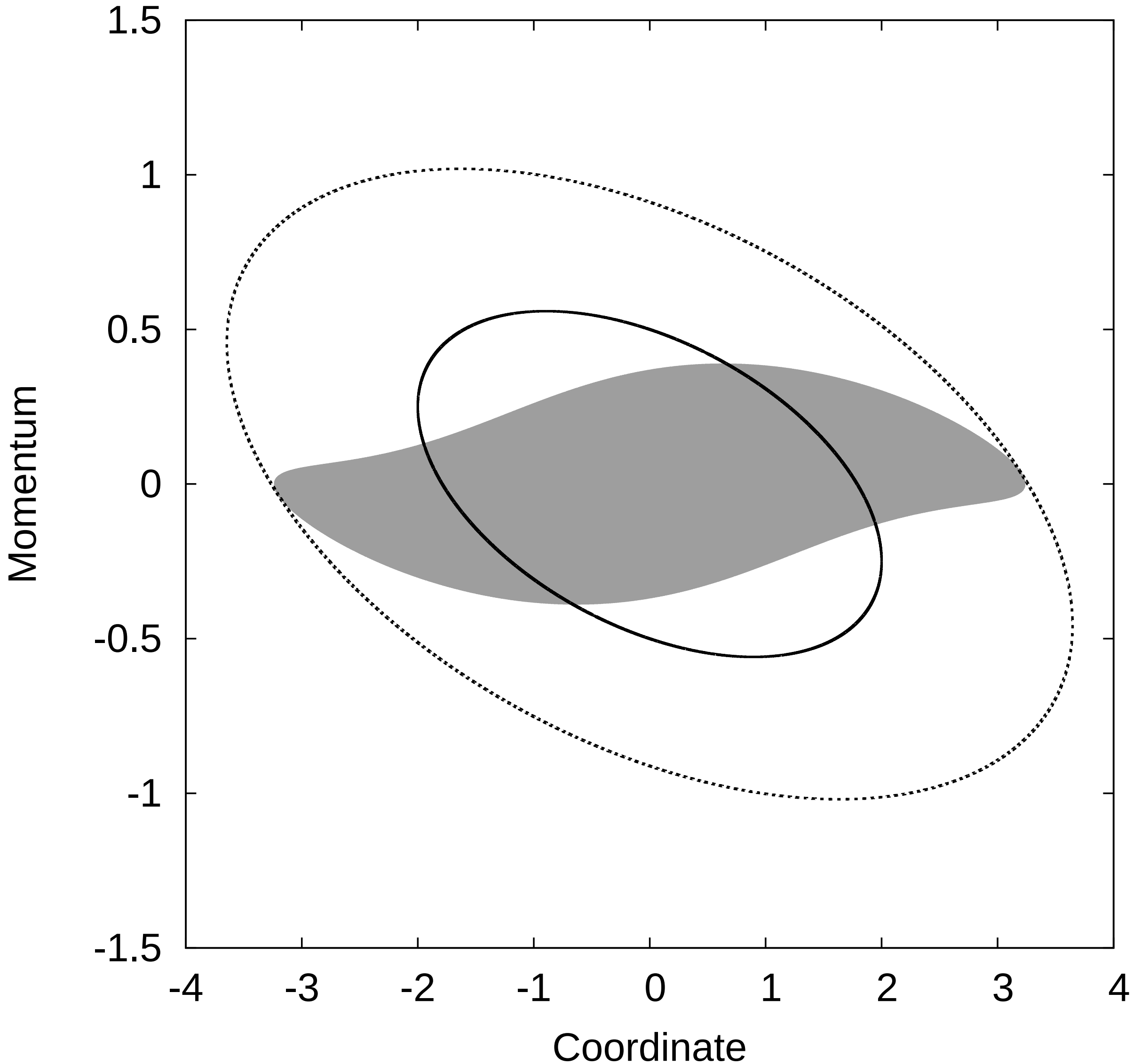


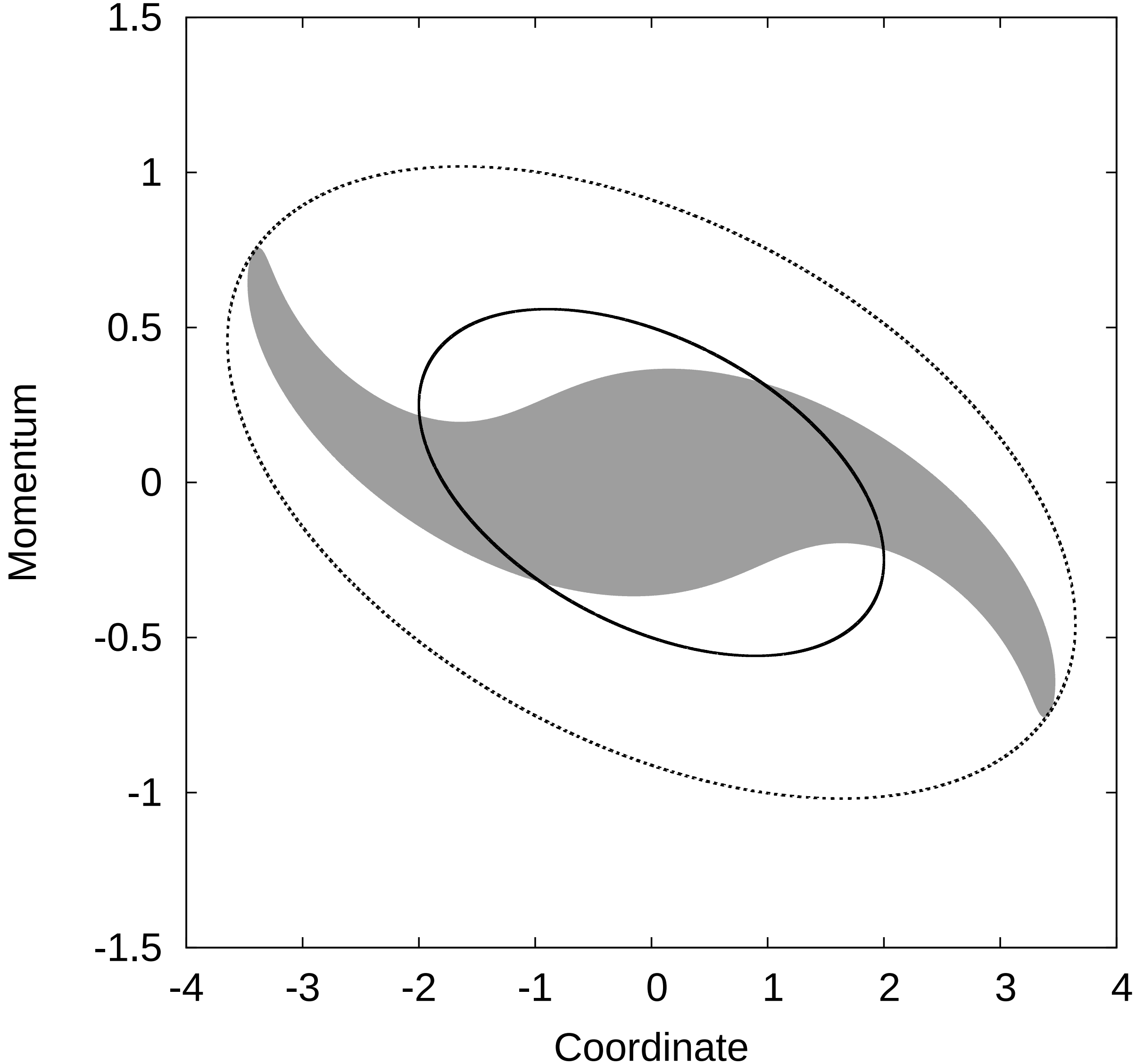
Mismatch with Nonlinearity

- With linear motion, ellipse remains elliptical, emittance same
- Add nonlinearity: tune shift with amplitude
- Distribution fills in larger area
  - Cannot easily recover from this: effective emittance growth
- Emittance from second order moment matrix immediately increases, even though area does not

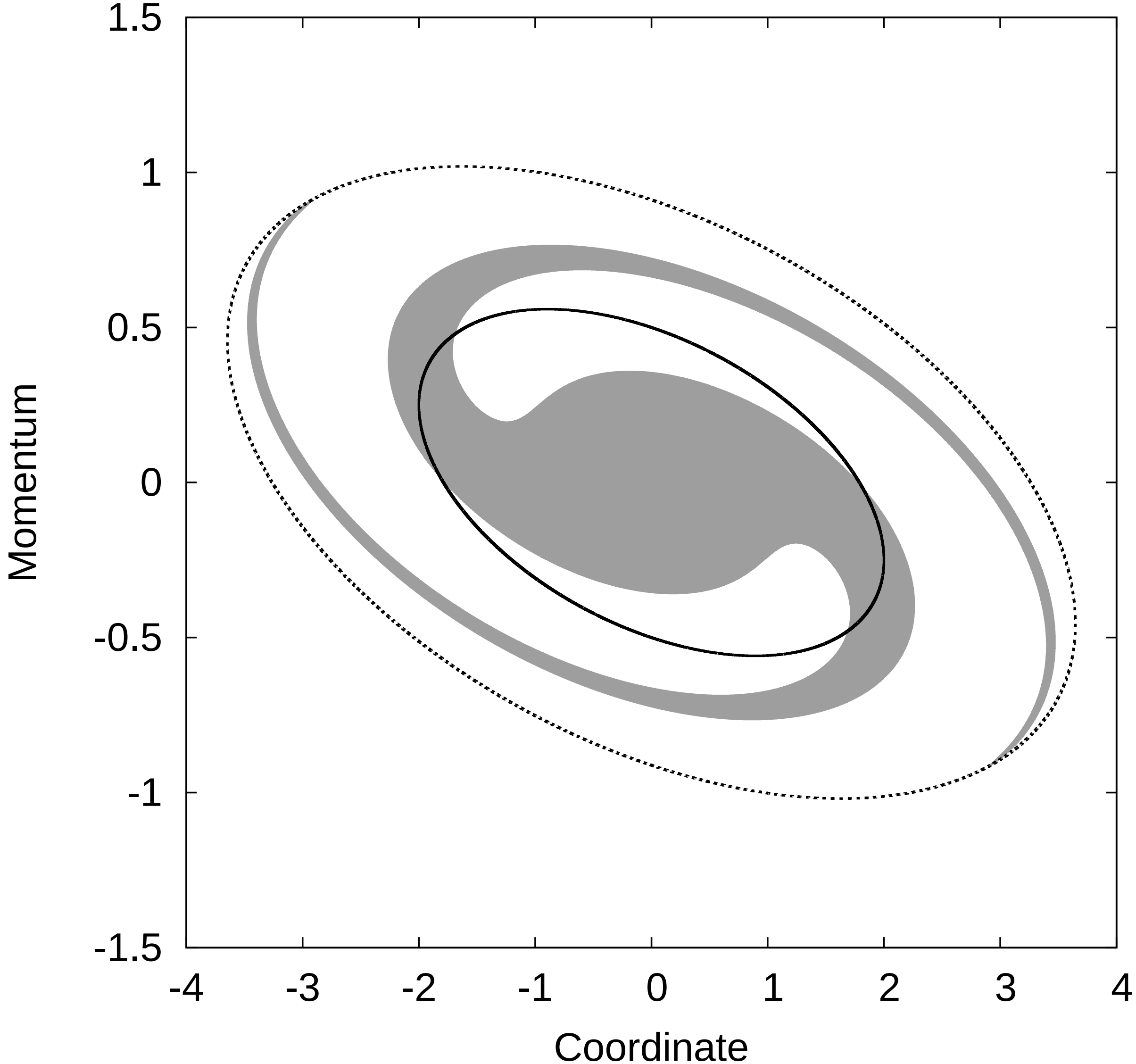
Mismatch with Nonlinearity



Mismatch with Nonlinearity



Mismatch with Nonlinearity



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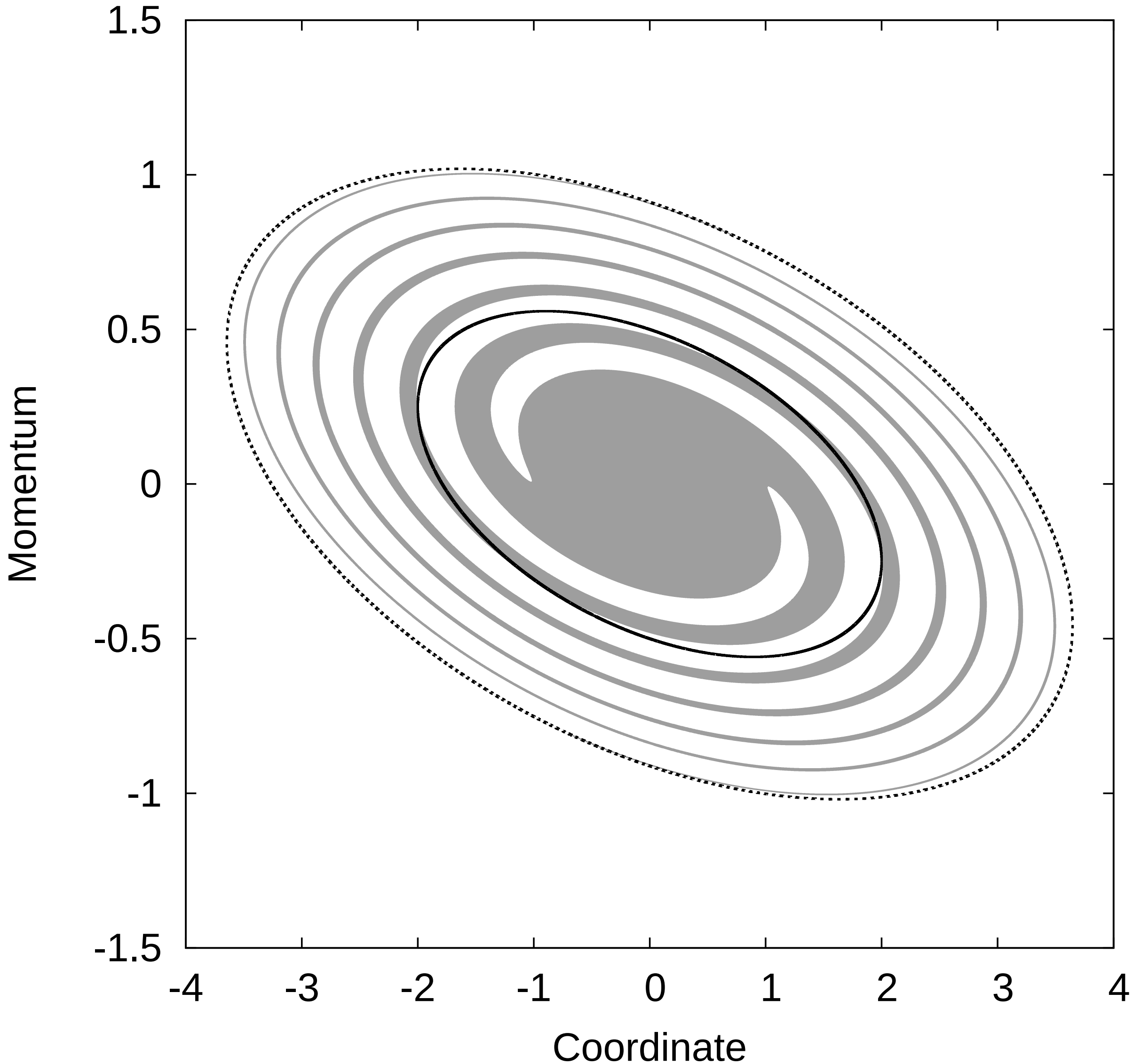
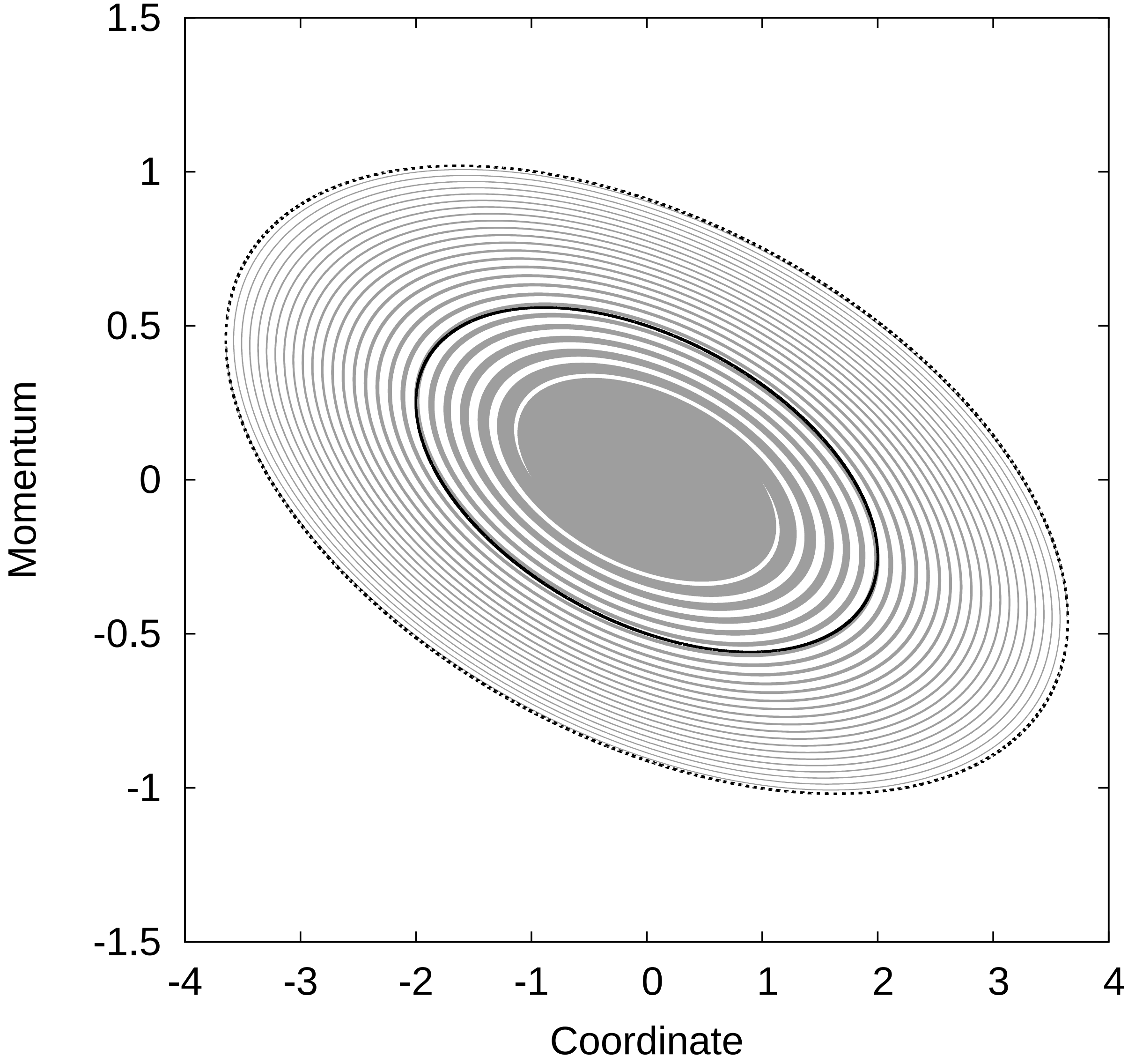


Mismatch with Nonlinearity



Rotationally Invariant Distribution

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xp} & 0 & L/2 \\ \sigma_{xp} & \sigma_{pp} & -L/2 & 0 \\ 0 & -L/2 & \sigma_{xx} & \sigma_{xp} \\ L/2 & 0 & \sigma_{xp} & \sigma_{pp} \end{bmatrix}$$

$$A = \begin{bmatrix} \sqrt{\sigma_{xx}/(2\epsilon)} & 0 & 0 & \sqrt{\sigma_{xx}/(2\epsilon)} \\ \sigma_{xp}/\sqrt{2\epsilon\sigma_{xx}} & \sqrt{\epsilon/(2\sigma_{xx})} & -\sqrt{\epsilon/(2\sigma_{xx})} & \sigma_{xp}/\sqrt{2\epsilon\sigma_{xx}} \\ 0 & \sqrt{\sigma_{xx}/(2\epsilon)} & \sqrt{\sigma_{xx}/(2\epsilon)} & 0 \\ -\sqrt{\epsilon/(2\sigma_{xx})} & \sigma_{xp}/\sqrt{2\epsilon\sigma_{xx}} & \sigma_{xp}/\sqrt{2\epsilon\sigma_{xx}} & \sqrt{\epsilon/(2\sigma_{xx})} \end{bmatrix}$$

$$\epsilon^2 = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2$$

Emittances: $\epsilon \pm L/2$

Rotationally Invariant Distribution

- Same form using canonical or kinetic momenta
- σ_{pp} and L change when transforming canonical to kinetic momenta
- Emittances invariant for linear, Hamiltonian system only in canonical coordinates
- Eigenmodes are two helicities
- A doesn't care about angular momentum L
- $\sqrt{\sigma_{xx}/\epsilon}$ behaves like the beta function; note ϵ is not the emittance
- Product of the emittances, $\epsilon^2 - L^2/4$, is identical with kinetic or canonical momenta

Rotationally Invariant Distribution

- Emittance evolution

$$\frac{d(\epsilon \pm L/2)}{ds} = -\frac{m_{10}}{\beta c p}\left(1 \mp \frac{zeB_s}{2}\frac{\sigma_{xx}}{\epsilon}\right)(\epsilon \pm L/2) + \frac{S_{\mathrm{MS}}}{2}\frac{\sigma_{xx}}{\epsilon}$$

- If B_s nonzero in absorber
  - One equilibrium emittance worse (even nonexistent)
  - Angular momentum can be generated
  - σ_{xx}/ϵ appears
- Beta function change at absorber depends on beam emittance at absorber
  - To get “matched” beamline, need to choose an emittance

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Rotationally Invariant Distribution

Muon Accelerator Program

- Alternatively, look at product of emittances: $\epsilon_4^2 = \epsilon^2 - L^2/4$

$$\frac{d\epsilon_4^2}{ds} = -2\frac{m_{10}}{\beta c p}\epsilon_4^2 + S_{\mathrm{MS}}\sigma_{xx}$$

- For evolution of ϵ_4 instead of ϵ_4^2, now a different version of the beta function appears:

$$\frac{d\epsilon_4}{ds} = -\frac{m_{10}}{\beta c p}\epsilon_4 + \frac{S_{\mathrm{MS}}}{2}\frac{\sigma_{xx}}{\epsilon_4}$$

Rotationally Invariant Distribution

- For a solenoid lattice in the Larmor frame, equations of motion act like a linear focusing lattice with scaled strength $[qB_s/(2p)]^2$
- Using this, compute beta functions for a periodic lattice
- For a matched beam, that beta function should be $\sqrt{\sigma_{xx}/\epsilon}$

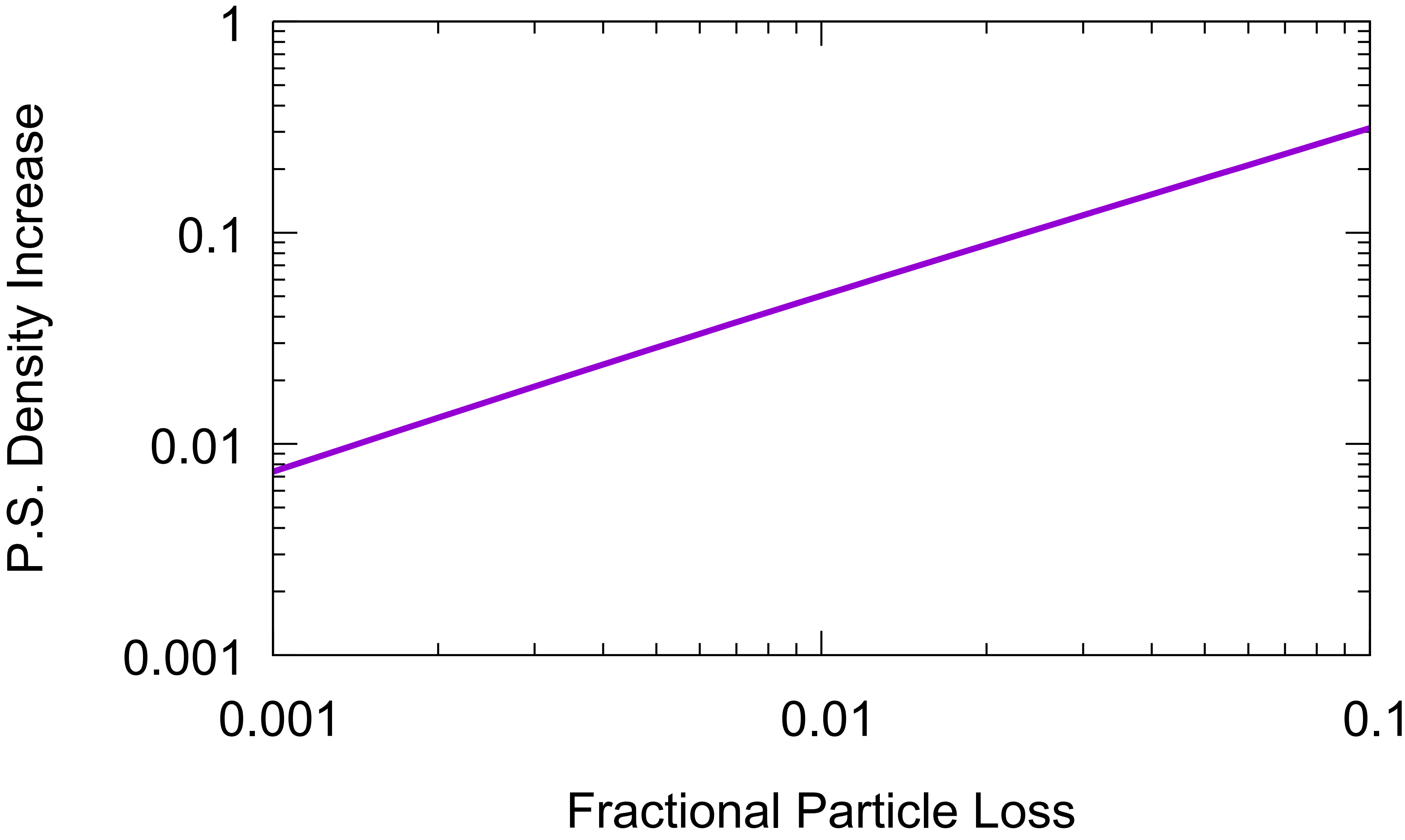
Rotational Invariance: Matching

- In 4-D phase space, a second order matrix has 10 independent parameters
- In a roataionally symmetric lattice, these correspond to 10 modes:
  - Two rotationally symmetric matched helical modes
  - Two rotationally symmetric beta beating modes
  - 6 modes that lack rotational symmetry
- Ideally, initial particle selection should make a distribution that is rotationally symmetric and reasonably well-matched
  - Matching assuming the matched distribution in the tracker region is that for a constant solenoid

Caution in Using Second Moments

- Simple example: start with Gaussian, then truncate
- Use N/ϵ^2 as measure of transverse phase space density
- Vary how deep I cut into the Gaussian
- Plot fractional particle loss and increase in N/ϵ^2
- Can get apparent phase space density increase, with no real cooling
- Truncation of a mere 1% will lead to apparent MICE-level cooling performance
- Message: you need to rule out this sort of effect

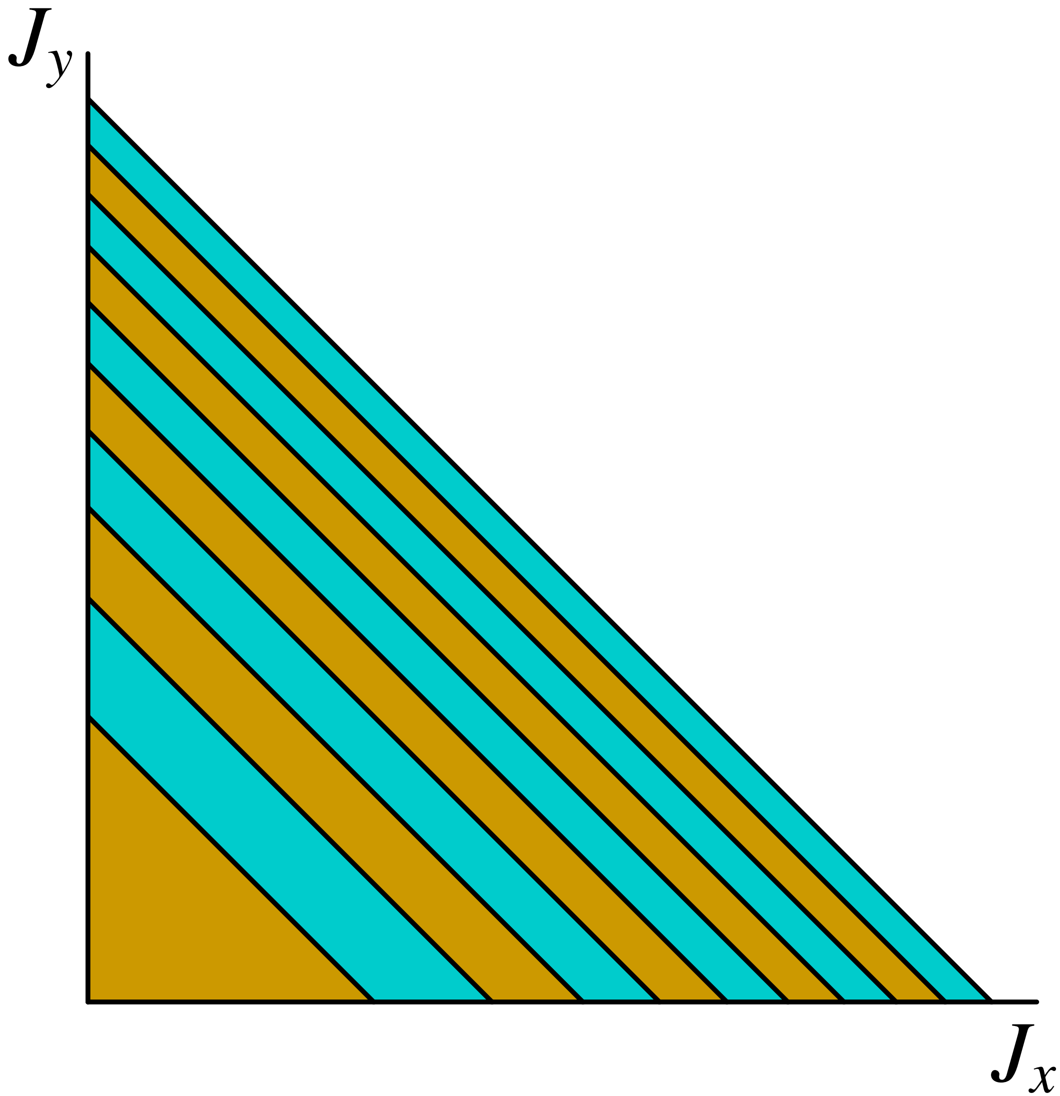
Caution in Using Second Moments



Alternative: Histogram Density

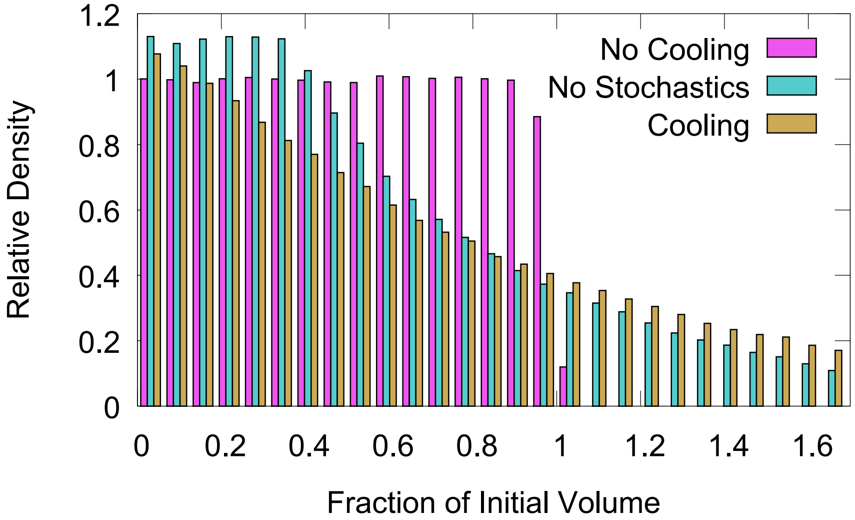
- Begin with uniform distribution
- At the end, make a transform to action-angle variables
  - Based on constant solenoid field
- Create uniform sized bins in action space
- Plot histogram vs. amplitude
- If more particles in bin than would be there for original uniform distribution, you've increased phase space density

Alternative: Histogram Density

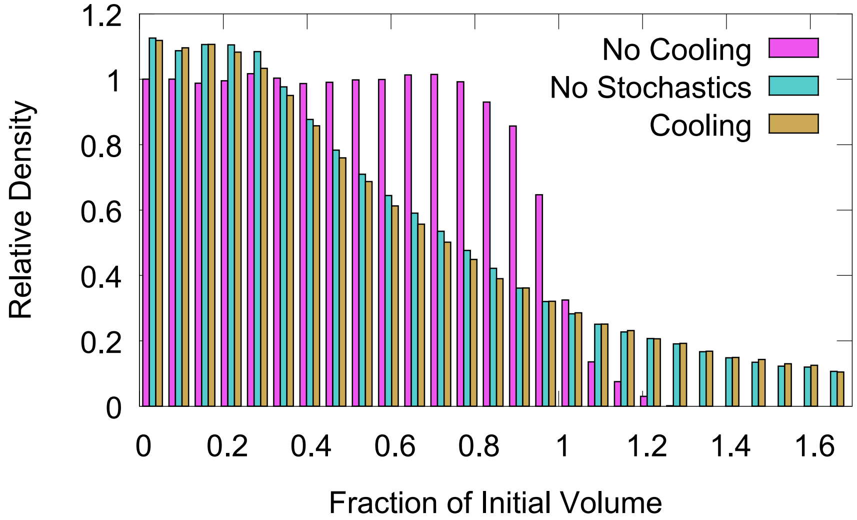


Phase Space Density

Small Emittance



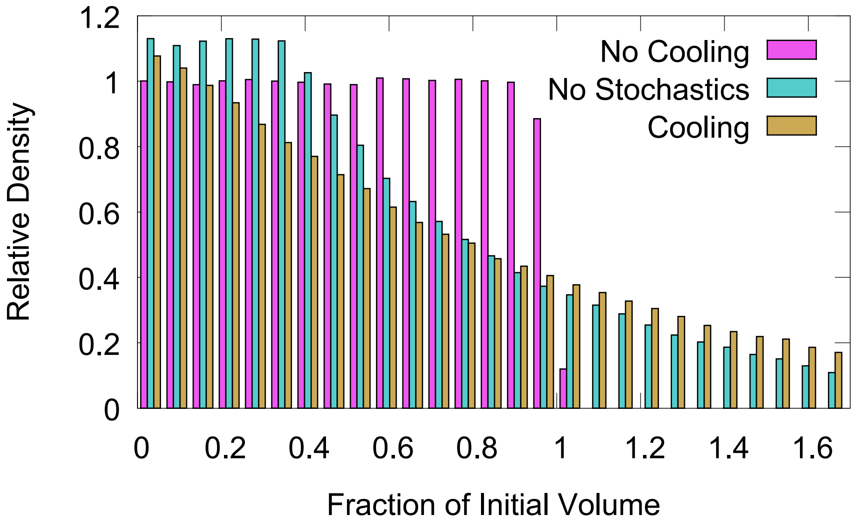
Large Emittance



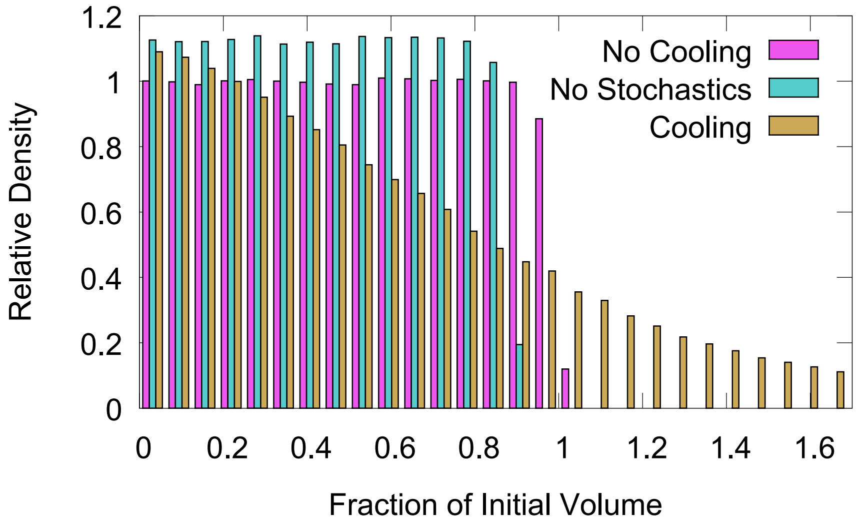
- Increase in phase space density at low amplitudes, even for small amplitude
- Large emittance has more phase space with density increase
- Note tails even without stochastics

Phase Space Density

Small, No Match



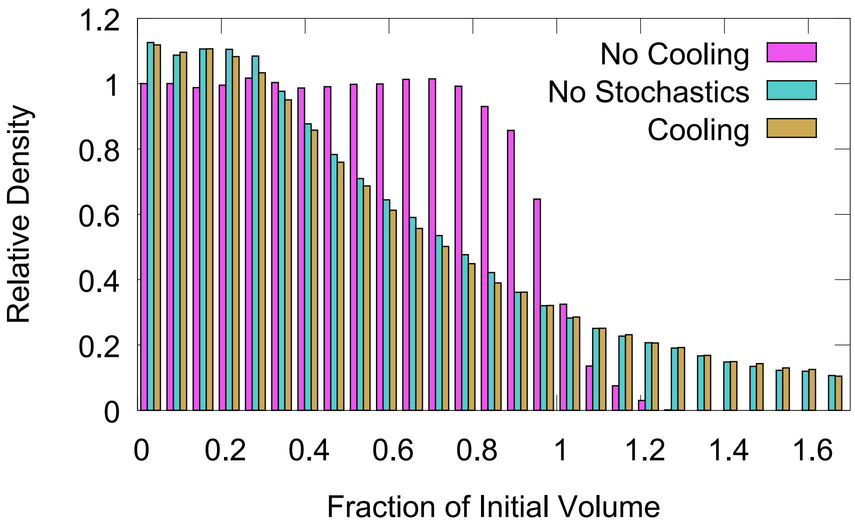
Small, Match



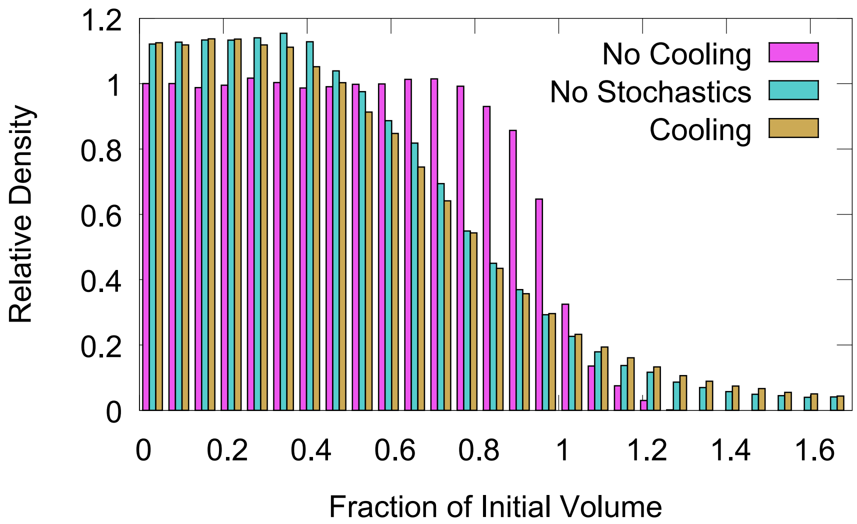
- Change the final linear transform to remove the tails without stochastics.
  - Not a match based on lattice: linear transform to improve cooling measure
- Shows better cooling even with stochastics

Phase Space Density

Large, No Match

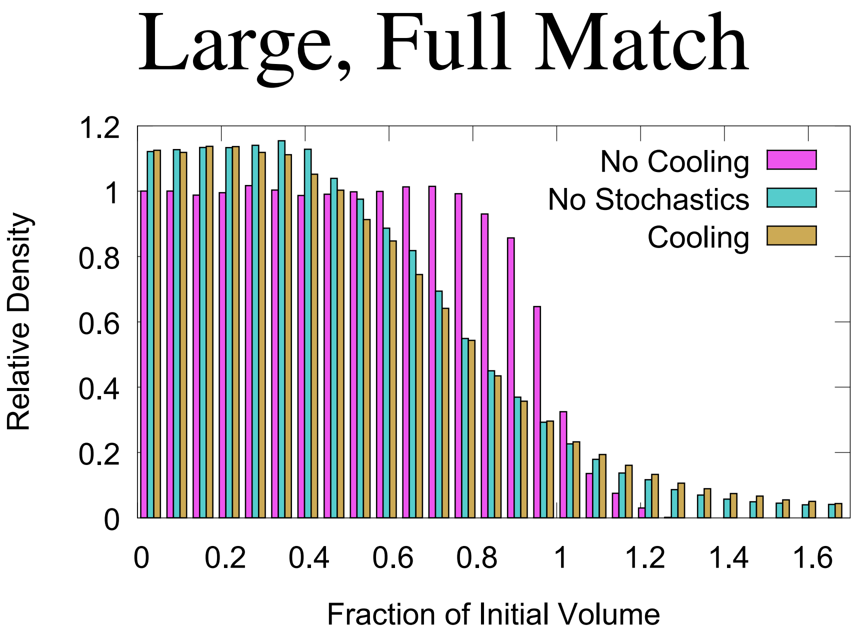


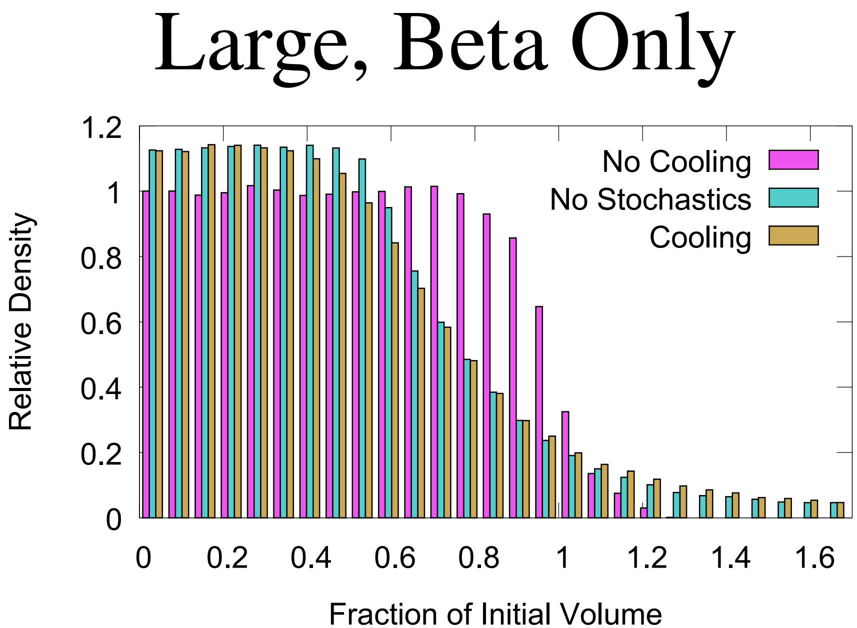
Large, Match



- The match also improves the situation for large emittance
- Tails still remain
- Impact of nonlinearity

Phase Space Density





- Curiously, a more restricted match appears to perform better
- Fault of my measure of matching
- Should try correct linear match based on linear map with absorber

Conclusions

- The Poincaré integral invariant, and its consequence Louiville's theorem, give us invariants for a Hamiltonian system that we will change with ionization cooling
  - Using the beam's second-order matrix, we can define emittances which characterize these quantities for a beam

Conclusions

- Courant-Snyder beta functions can be defined for a lattice or a beam.
  - The functions for a beam are defined in terms of the second moment matrix.
  - The lattice determines how those functions evolve for a beam.
  - The functions for a beam are well-defined even for non-Hamiltonian systems
  - Matching refers to the process of making these two definitions, lattice and beam, agree

Conclusions

- The second moment matrix is an imperfect characterization of the Poincaré integral invariant and the phase space density
  - It will change due to nonlinearity
  - It can change due to effects other than a change in phase space density
  - It may be more precise to look at phase space density directly