Meromorphic connections and quivers

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String-Math 2016 conference, Paris

Plan of the talk

1. Crawley-Boevey's result on residue manifolds and quiver varieties
2. Polar-parts manifolds
3. Boalch's conjecture
4. Reflection transformations of isomonodromy equations

Residue manifolds

Let $O_1, O_2, \ldots, O_m \subset \mathfrak{gl}_n(\mathbb{C})$ be $\mathrm{GL}_n(\mathbb{C})$-adjoint orbits.
Consider the space

$$\mathcal{M}_n(O_1, \ldots, O_m) = \left\{ (A_i) \in \prod O_i \,\middle|\, \sum A_i = 0 \right\} / \, \mathrm{GL}_n(\mathbb{C}).$$

Take distinct $t_1, t_2, \ldots, t_m \in \mathbb{C}$.
The space parameterizes the isoclasses of logarithmic connections

$$d - \sum_{i=1}^{m} \frac{A_i}{x - t_i} \, dx \quad \text{on} \quad \mathcal{O}_{\mathbb{P}^1}^n$$

with the residue at each t_i lying in O_i and no other poles.
($\sum A_i = 0$ is the holomorphy condition at $x = \infty$.)

Residue manifolds

Identify $\mathfrak{gl}_n(\mathbb{C})^*$ with $\mathfrak{gl}_n(\mathbb{C})$ using the trace.
Then each O_i has the Kirillov–Kostant–Souriau (complex) symplectic structure and

$$\mu\colon \prod O_i \to \mathfrak{gl}_n(\mathbb{C}); \quad (A_i) \mapsto \sum A_i$$

is a $\mathrm{GL}_n(\mathbb{C})$-moment map. Therefore

$$\mathcal{M}_n(O_1, \ldots, O_m) = \mu^{-1}(0)/\operatorname{GL}_n(\mathbb{C}) = (O_1 \times \cdots \times O_m) /\!\!/ \operatorname{GL}_n(\mathbb{C})$$

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is a complex Hamiltonian reduction.

It may be singular but the open subset $\mathcal{M}_n^s(O_1, \dots, O_m)$ consisting of $[(A_i)]$ with no common invariant subspace of $\mathbb{C}^n$ (except $\{0\}, \mathbb{C}^n$) is a smooth complex symplectic manifold if it is non-empty.

Residue manifolds and quiver varieties

Theorem [Crawley-Boevey 2003]

$\mathcal{M}^s_n(O_1, \ldots, O_m)$ is isomorphic to a Nakajima quiver variety $\mathfrak{M}^s_Q(\boldsymbol{n}, \boldsymbol{\zeta})$.

The quiver variety $\mathfrak{M}^s_Q(\boldsymbol{n}, \boldsymbol{\zeta})$ is the complex symplectic manifold associated to

- a quiver Q (a directed graph consisting of vertices Q_0 and arrows Q_1)

  e.g. 

- $\boldsymbol{n} = (n_i) \in \mathbb{Z}^{Q_0}_{\geq 0}$, $\boldsymbol{\zeta} = (\zeta_i) \in \mathbb{C}^{Q_0}$

defined by

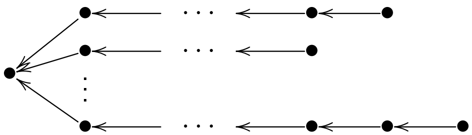
$$\left\{ \left(\mathbb{C}^{n_{\mathsf{s}(a)}} \underset{Y_a}{\overset{X_a}{\rightleftarrows}} \mathbb{C}^{n_{\mathsf{t}(a)}} \right)_{a \in Q_1} \;\middle|\; \sum_{\mathsf{t}(a)=i} X_a Y_a - \sum_{\mathsf{s}(a)=i} Y_a X_a = -\zeta_i I_{n_i} \right\}^{\text{irred}} \Big/ \prod \mathrm{GL}_{n_i}(\mathbb{C})$$

Residue manifolds and quiver varieties

Theorem [Crawley-Boevey 2003]

$\mathcal{M}_n^s(O_1, \ldots, O_m)$ is isomorphic to a Nakajima quiver variety $\mathfrak{M}_Q^s(\boldsymbol{n}, \boldsymbol{\zeta})$.

Let d_i be the degree of the minimal polynomial of elements of O_i.
The quiver Q appearing in Theorem is the "star-shaped" quiver



with m legs of length $d_i - 1$, $i = 1, \ldots, m$. Roughly speaking,

- $\boldsymbol{\zeta}$ depends on the eigenvalues of $O_1, \ldots, O_m$
- $\boldsymbol{n}$ depends on the multiplicities of eigenvalues

As an application he solved the additive Deligne–Simpson problem, i.e., gave a criterion for the emptiness of $\mathcal{M}_n^s(O_1, \ldots, O_m)$.

Reflection transformations of Schlesinger equations

Each loop-free vertex $i \in Q_0$ defines a reflection $s_i \colon \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$:

$$s_i(\boldsymbol{n}) = \boldsymbol{n}', \quad n'_j = \begin{cases} -n_i + \sum_{j \neq i} n_j \#\{\text{arrows joining } i \text{ and } j\} & (j = i) \\ n_j & (j \neq i) \end{cases}$$

Reflection functors [Nakajima, Rump, Lusztig, ...]

$$\mathfrak{M}^s_Q(\boldsymbol{n}, \boldsymbol{\zeta}) \simeq \mathfrak{M}^s_Q(s_i(\boldsymbol{n}), s_i^T(\boldsymbol{\zeta})) \quad (\zeta_i \neq 0)$$

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Through Crawley-Boevey's isomorphisms,

- the reflection at the central vertex relates to the additive analogue of Katz's middle convolution in the sense of Dettweiler–Reiter [Boalch]
- the other reflections come from scalar shifts of residues.

They induce transformations of Schlesinger equations (isomonodromy equations for non-resonant logarithmic connections) [Haraoka–Filipuk].

Polar-part orbits

As a generalization of $\mathrm{GL}_n(\mathbb{C})$-(co)adjoint orbits, consider orbits of the $\mathrm{GL}_n(\mathbb{C}[\![z]\!])$-action on $\mathfrak{gl}_n(\mathbb{C}[z^{-1}])\frac{dz}{z}$ defined by

$$g \cdot A = \text{the principal part of } gAg^{-1} \in \mathfrak{gl}_n(\mathbb{C}(\!(z)\!))dz.$$

- This action preserves the order $\mathrm{ord}(A) \in \mathbb{Z}_{<0}$.
- If $\mathrm{ord}(g(z) - I_n) + \mathrm{ord}(A) \geq 0$ then $g \cdot A = A$.

Hence every orbit is finite-dimensional, and in fact may be viewed as a coadjoint orbit of $\mathrm{GL}_n(\mathbb{C}[z]/(z^k)) \simeq \mathrm{GL}_n(\mathbb{C}[\![z]\!])/\{I_n + O(z^k)\}$ for sufficiently large k via the pairing

$$\mathfrak{gl}_n(\mathbb{C}[\![z]\!]) \times \mathfrak{gl}_n(\mathbb{C}[z^{-1}])\frac{dz}{z} \to \mathbb{C}; \quad (X, B) \mapsto \operatorname*{res}_{z=0} \mathrm{tr}(XB).$$

Every orbit of order ≥ -1 may be identified with a $\mathrm{GL}_n(\mathbb{C})$-(co)adjoint orbit.

Polar-parts manifolds

Let $O_1, O_2, \ldots, O_m \subset \mathfrak{gl}_n(\mathbb{C}[z^{-1}])\frac{dz}{z}$ be $\mathrm{GL}_n(\mathbb{C}[\![z]\!])$-orbits.
Consider the space

$$\mathcal{M}_n(O_1, \ldots, O_m) = \left\{ (A_i) \in \prod O_i \,\middle|\, \sum \operatorname*{res}_{z=0} A_i = 0 \right\} / \, \mathrm{GL}_n(\mathbb{C}).$$

Take $t_1, t_2, \ldots, t_m \in \mathbb{C}$ (distinct).
The space parameterizes the isoclasses of meromorphic connections $d - A$ on $\mathcal{O}_{\mathbb{P}^1}^n$ with poles at $t_1, \ldots, t_m$ (and nowhere else) such that

$$(\text{the principal part of } A \text{ at } x = t_i) \in O_i \quad (\text{with } z = x - t_i).$$

The open subset $\mathcal{M}_n^s(O_1, \ldots, O_m)$ consisting of $[(A_i)]$ with no nonzero subspace $V \subsetneq \mathbb{C}^n$ satisfying

$$A_i(V) \subset V \otimes \mathbb{C}[z^{-1}]\frac{dz}{z} \quad (i = 1, 2, \ldots, m)$$

is a smooth complex symplectic manifold if it is non-empty.

Boalch's conjecture

Assume

- O_1 contains an element of the form

$$\begin{pmatrix} \lambda_1(z^{-1})\, I_{n_1} + L_1 & & & \\ & \lambda_2(z^{-1})\, I_{n_2} + L_2 & & \\ & & \ddots & \\ & & & \lambda_\ell(z^{-1})\, I_{n_\ell} + L_\ell \end{pmatrix} \frac{dz}{z}$$

  with $\lambda_j(t) \in t\mathbb{C}[t]$ and $L_j \in \mathfrak{gl}_{n_j}(\mathbb{C})$.
- $O_2, \ldots, O_m$ are logarithmic.

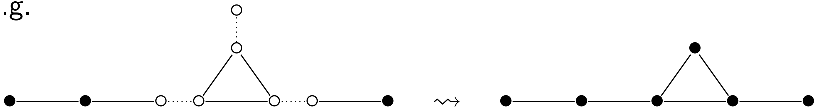
Theorem (Boalch's conjecture) [Boalch ($\mathrm{ord}(O_1) \geq -3$), Hiroe–Y]
$\mathcal{M}_n^s(O_1, \ldots, O_m)$ is isomorphic to a quiver variety.

Construction of the quiver

The underlying graph of the quiver appearing in Theorem is constructed as follows (for simplicity assume that $O_2, \ldots, O_m$ are not scalar)

1. Define a finite graph Γ_0 by
   - the vertices $= \{\lambda_1, \ldots, \lambda_\ell\} \subset t\mathbb{C}[t]$
   - joining each $\lambda_j \neq \lambda_k$ by $(\deg(\lambda_j - \lambda_k) - 1)$ edges
2. Extend Γ_0 by
   - adding $(m-1)$ vertices $2, 3, \ldots, m$
   - joining each i and λ_j by one edge
3. Glue a leg (a graph of type A) to each vertex
   - the length of i-th leg $= \deg(\text{the minimal poly. of } O_i) - 2$
   - the length of λ_j-th leg $= \deg(\text{the minimal poly. of } L_j) - 1$

e.g.



Reflection transformations

In this situation it is hard to check that the reflection functors induce transformations of isomonodromy equations.
Using an irregular singular analogue of the additive middle convolution (developed by Takemura and Y) instead, Hiroe constructed isomorphisms

$$\begin{array}{ccc} \mathcal{M}_n^s(O_1,\dots,O_m) & \xrightarrow{S_i'} & \mathcal{M}_n^s(O_1',\dots,O_m') \\ \simeq\downarrow & \circlearrowright & \downarrow\simeq \\ \mathfrak{M}_Q^s(\boldsymbol{n},\boldsymbol{\zeta}) & \xrightarrow[\simeq]{} & \mathfrak{M}_Q^s(s_i(\boldsymbol{n}), s_i^T(\boldsymbol{\zeta})) \end{array}$$

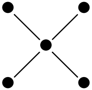
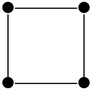
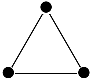
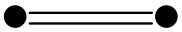
Theorem [Y]

Under some mild assumption S_i' induce transformations of isomonodromy equations.

In the case $\operatorname{ord}(O_1) \geq -3$ similar transformations were constructed earlier by Boalch.

Examples: Painlevé equations

When $n = 2$, $\dim \mathcal{M}_n^s(O_1, \ldots, O_m) = 2$ one gets Painlevé equations

$(m, -\operatorname{ord}(O_1))$	$(4,1)$	$(3,2)$	$(2,3)$	$(1,4)$
Isomonodromy eq.	P_{VI}	P_{V}	P_{IV}	P_{II}
Underlying graph				

In each case one recovers Okamoto's Weyl group symmetry of the Painlevé equation (except the action of the lattice part).