Exact WKB analysis, cluster algebras and Painlevé equations

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Contents

WKB solutions and Voros symbols

- Schrödinger equation on a compact Riemann surface C:

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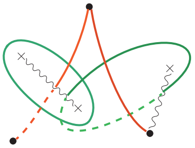

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$\beta_i \in H_1(\Sigma, P, \mathbb{Z})$

$\Sigma \subset T^*C$ is the **spectral curve**:

(locally $\Sigma = \{(x,\xi) \mid \xi^2 = Q_0(x)\}$)

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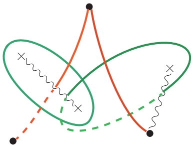

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- **Exact WKB analysis** = WKB + Borel resummation: [Voros 83], [Ecalle 81],...

Borel summation and Stokes graph

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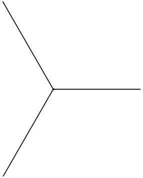
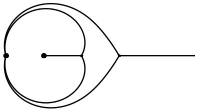
- **Stokes graph** (= **spectral network**) consists of **trajectories**:

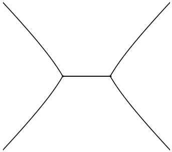
$$\mathrm{Im}\left(\int_v^x \sqrt{Q_0(x')}\, dx' \right) = 0 \qquad (v : \text{a zero (or simple-pole) of } Q_0(x)\, dx^2)$$



$Q_0(x) = x.$



$Q_0(x) = \dfrac{(x-2)(x-3)}{x^2(x-1)^2}.$



$Q_0(x) = 1 - x^2.$

Recall: $\psi(x,\hbar) = \exp\left(\hbar^{-1} \cdot \int_\alpha \lambda(x,\hbar)\, dx\right), \quad \lambda(x,\hbar) = \sqrt{Q_0(x)} + O(\hbar)$.

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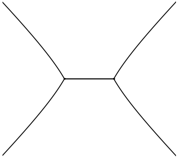

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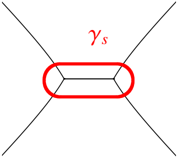

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  - Borel transform has singularities at
    $\omega_m = m \cdot \oint_{\gamma_s} \sqrt{Q_0}\, dx$ ($m \in \mathbb{Z}_{\neq 0}$).
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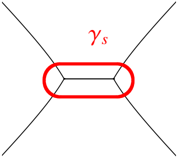

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    Here $\gamma_s \in H_1(\Sigma, \mathbb{Z})$ is the cycle around the saddle ("saddle class").
  - The corresponding Stokes jump is:

$$\mathcal{S}_-[\psi_\alpha] = \mathcal{S}_+\left[\psi_\alpha \cdot \left(1 + e^{V_{\gamma_s}}\right)^{\langle \alpha, \gamma_s \rangle}\right]$$

where

$$e^{V_\gamma} = \exp\left(\oint_{\gamma_s} \lambda(x,\hbar)\, dx\right) \ : \ \text{Voros symbol}$$

(c.f., [Delabaere-Dillinger-Pham 93], [Aoki-Kawai-Takei 08].)

Cluster mutation = Stokes phenomenon

- For each rectangular Stokes region D_i, set

$$e^{V_{\gamma_i}} = \exp\left(\hbar^{-1} \cdot \oint_{\gamma_i} \lambda(x,\hbar)\, dx\right)$$

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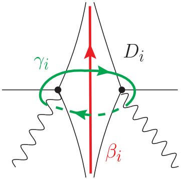

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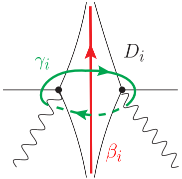


Theorem [Gaiotto-Moore-Neitzke 09, I-Nakanishi 14]

Stokes jump for these Voros symbols = cluster mutations:

- Borel sum of Voros symbol $e^{W_{\beta_i}}$ = cluster x-variable.
- Borel sum of Voros symbol $e^{V_{\gamma_i}}$ = cluster y-variable.

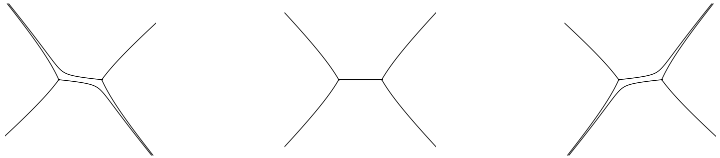
$$x'_i = \begin{cases} x_k^{-1}\left(\prod_{j=1}^{n} x_j^{[-b_{jk}]_+}\right)(1+y_k) & i = k \\ x_i & i \neq k. \end{cases} \qquad y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i y_k^{[b_{ki}]_+}(1+y_k)^{-b_{ki}} & i \neq k. \end{cases}$$

$b_{ij} = \langle \gamma_i, \gamma_j \rangle$, $[a]_+ = \max(a,0)$.

$$y_i = c_i \cdot \prod_{j=1}^{n} (x_j)^{b_{ji}} \qquad c_i = \exp\left(\hbar^{-1} \oint_{\gamma_i} \sqrt{Q_0(x)}\, dx\right) \quad : \text{“coefficient”}$$

Mutation for path / cycles

- Saddle connection causes **“mutation”** of Stokes graph:



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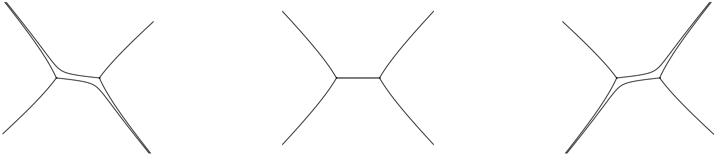

- Mutation formula for path/cycles:

$$\beta_i{}' = \begin{cases} -\beta_k + \sum_{j=1}^{n} [-\langle \gamma_j, \gamma_k \rangle]_+ \cdot \beta_j & i = k \\ \beta_i & i \neq k. \end{cases} \qquad \gamma_i{}' = \begin{cases} -\gamma_k & i = k \\ \gamma_i + [\langle \gamma_k, \gamma_i \rangle]_+ \cdot \gamma_k & i \neq k. \end{cases}$$

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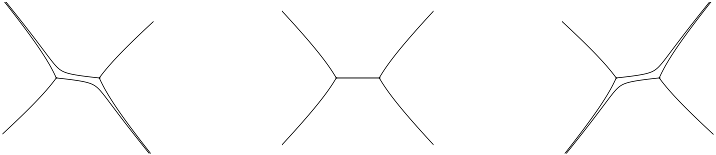

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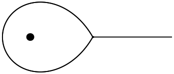
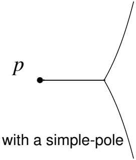
- Delabaere-Dillinger-Pham's formula:

$$\begin{aligned} \mathcal{S}_-[e^{V_\gamma}] &= \mathcal{S}_+\left[e^{V_\gamma} \cdot \left(1 + e^{V_{\gamma_s}}\right)^{\langle \gamma, \gamma_s \rangle}\right] \\ \mathcal{S}_-[e^{W_\beta}] &= \mathcal{S}_+\left[e^{W_\beta} \cdot \left(1 + e^{V_{\gamma_s}}\right)^{\langle \beta, \gamma_s \rangle}\right] \end{aligned}$$

More on Voros symbols and cluster algebras



(i) loop-type saddle



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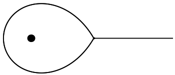

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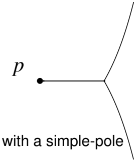

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Theorem [I-Nakanishi 14, Aoki-I-Takahashi 16]

(i) For the loop-type saddle around a double-pole:

$$\begin{aligned} \mathcal{S}_-[e^{V_\gamma}] &= \mathcal{S}_+[e^{V_\gamma}] \\ \mathcal{S}_-[e^{W_\beta}] &= \mathcal{S}_+\left[e^{W_\beta} \cdot \left(1 - e^{V_{\gamma_s}}\right)^{-\langle \beta, \gamma_s \rangle}\right] \end{aligned}$$

"**Local rescaling**" of cluster x-variable.

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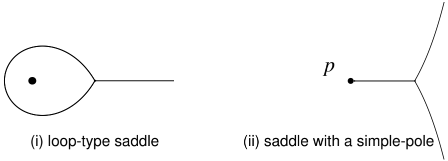

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t_p : characteristic exponent at p

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"**Local rescaling**" of cluster x-variable.

(ii) For a saddle connecting a simple-zero and a simple-pole p:

$$\begin{aligned} \mathcal{S}_-[e^{V_\gamma}] &= \mathcal{S}_+\left[e^{V_\gamma} \cdot \left(1 + (t_p + t_p^{-1}) \cdot e^{V_{\gamma_s}} + e^{2V_{\gamma_s}}\right)^{\langle \gamma, \gamma_s \rangle}\right] \\ \mathcal{S}_-[e^{W_\beta}] &= \mathcal{S}_+\left[e^{W_\beta} \cdot \left(1 + (t_p + t_p^{-1}) \cdot e^{V_{\gamma_s}} + e^{2V_{\gamma_s}}\right)^{\langle \beta, \gamma_s \rangle}\right] \end{aligned}$$

"**Generalized cluster transform**" (c.f., [Chekhov-Shapiro 11]).

([Koike 00] : Exact WKB analysis near a simple-pole.)

Contents

Painlevé equations with a small parameter

- **Painlevé equations** are discovered by Painlevé and Gambier:

$$(P_{\rm I}) \quad : \quad \hbar^2 \frac{d^2q}{dt^2} = 6q^2 + t$$

$$(P_{\rm II}) \quad : \quad \hbar^2 \frac{d^2q}{dt^2} = 2q^3 + tq + \alpha$$

$$(P_{\rm VI}) \quad : \quad \hbar^2 \frac{d^2q}{dt^2} = \frac{\hbar^2}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \hbar^2\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt}$$
$$+ \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{\alpha_\infty^2}{4} - \frac{\alpha_0^2}{4}\frac{t}{q^2} + \frac{\alpha_1^2}{4}\frac{t-1}{(q-1)^2} - \frac{\alpha_t^2}{4}\frac{t(t-1)}{(q-t)^2}\right)$$

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&\qquad + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{\alpha_\infty^2}{4} - \frac{\alpha_0^2}{4}\frac{t}{q^2} + \frac{\alpha_1^2}{4}\frac{t-1}{(q-1)^2} - \frac{\alpha_t^2}{4}\frac{t(t-1)}{(q-t)^2}\right)
\end{aligned}
$$

- **Many nice properties**:
  Painlevé property (movable singularity must be a pole), isomonodromy deformation, Hamiltonian description, affine-Weyl symmetry, space of initial conditions (= quiver variety), non-linear Stokes phenomenon, conformal block expansion of solutions, ... (See [Fokas-Its-Kapaev-Novokshenov 06].)

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&\qquad + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{\alpha_\infty^2}{4} - \frac{\alpha_0^2}{4}\frac{t}{q^2} + \frac{\alpha_1^2}{4}\frac{t-1}{(q-1)^2} - \frac{\alpha_t^2}{4}\frac{t(t-1)}{(q-t)^2}\right)
\end{aligned}
$$

- **Many nice properties**:
  Painlevé property (movable singularity must be a pole), isomonodromy deformation, Hamiltonian description, affine-Weyl symmetry, space of initial conditions (= quiver variety), non-linear Stokes phenomenon, conformal block expansion of solutions, ... (See [Fokas-Its-Kapaev-Novokshenov 06].)

P-Voros symbol of $(P_{\mathrm{II}}) : \hbar^2 \frac{d^2 q}{dt^2} = 2q^3 + tq + \alpha$

- Formal power series solution

$$q(t,\hbar) = \sum_{n \geq 0} \hbar^n q_n(t) = q_0(t) + \hbar q_1(t) + \hbar^2 q_2(t) + \cdots$$

  - Top term satisfies $2q_0^3 + tq_0 + \alpha = 0$.
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$$q_{\rm trans}(t,\hbar;A) = \sum_{k\geq 0} A^k \cdot q^{(k)}(t,\hbar) \cdot e^{k\phi(t)/\hbar}, \quad \phi(t) = \int^t \sqrt{6q_0(s)^2 + s}\, ds$$

  - $k = 1$ part $\varphi(t,\hbar;A) = A \cdot q^{(1)}(t,\hbar) \cdot e^{\phi(t)/\hbar}$ satisfies

$$\hbar^2 \frac{d^2\varphi}{dt^2} = \left(6q^{(0)}(t,\hbar)^2 + t\right) \cdot \varphi \quad \cdots \quad (*)$$

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$$V_\gamma = \hbar^{-1} \cdot \oint_\gamma \lambda(t,\hbar)\, dt, \qquad W_\beta = \hbar^{-1} \cdot \int_\beta \left(\lambda(t,\hbar) - \lambda_0(t)\right) dt$$

P-Voros symbol of $(P_{\mathrm{II}}) : \hbar^2 \frac{d^2 q}{dt^2} = 2q^3 + tq + \alpha$

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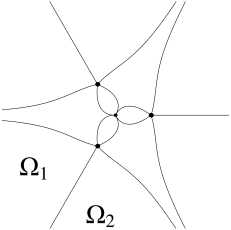
Theorem [I 14]

$$V_\gamma = 2\pi i \alpha \cdot \hbar^{-1}, \quad W_\beta = \sum_{g \geq 0} \frac{(1 - 2^{1-2g}) \cdot B_{2g}}{2g(2g-1)} \cdot \left(\frac{\hbar}{\alpha}\right)^{2g-1}$$

Non-linear Stokes phenomenon for $(P_{\mathrm{II}}) : \hbar^2 \dfrac{d^2 q}{dt^2} = 2q^3 + tq + \alpha$



(P_{II}) with $\alpha = 1$ (on q_0-plane)

- P**-Stokes graph**: $\mathrm{Im} \int_r^t \sqrt{6q_0(s)^2 + s}\, ds = 0.$ ([Kawai-Takei 96])

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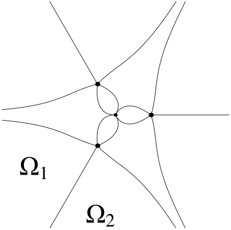

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  $\rightsquigarrow q_{\mathrm{trans}}(t, \hbar; A)$ is Borel summable on the region $\mathrm{Re}\, \phi(t) < 0$.

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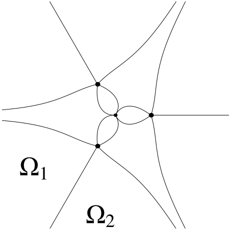

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Non-linear Stokes phenomenon (c.f., [Kapaev 04])

$$\mathcal{S}_{\Omega_1}[q(t,\hbar)] = \mathcal{S}_{\Omega_2}[q_{\rm trans}(t,\hbar; A)]$$

The constant A is chosen as $A = -\dfrac{i}{2\sqrt{\pi}} \cdot \dfrac{\sqrt{2\pi} \cdot e^{-\alpha/\hbar} \cdot (\alpha/\hbar)^{\alpha/\hbar}}{\Gamma(\frac{\alpha}{\hbar} + \frac{1}{2})}$.

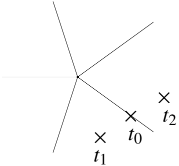
A = (non-linear Stokes multiplier of $(P_{\rm I})$) $\times$ (Borel sum of the Voros symbol e^{W_β}.)

Stokes graph of isomonodromy system and mutation

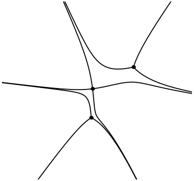
Isomonodromic deformation and exact WKB ([Kawai-Takei 96]):

$$\hbar^2 \frac{\partial^2 \psi}{\partial x^2} = Q_{\rm II}(x, t, \hbar) \cdot \psi,$$

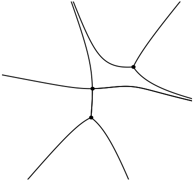
$$Q_{\rm II} = x^4 + tx^2 + 2\alpha x + 2H_{\rm II} - \hbar \frac{p}{x-q} + \hbar^2 \frac{3}{4(x-q)^2}$$



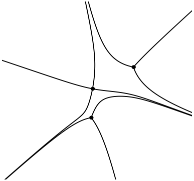
: P-Stokes graph (on t-plane)



at $t = t_1$.



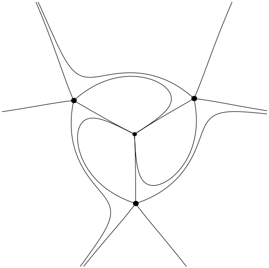
at $t = t_0$



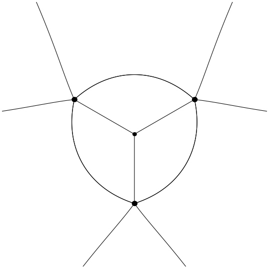
at $t = t_2$.

Stokes graphs of isomonodromy system (on x-plane).

Saddle connections in P-Stokes graph



$\alpha = i + \varepsilon$



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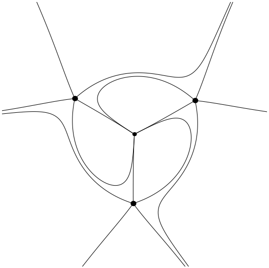
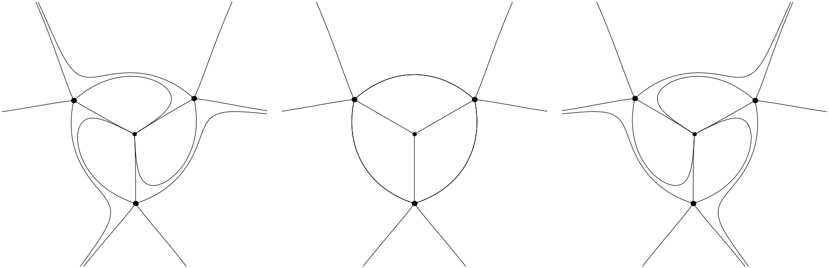


$\alpha = i - \varepsilon$

Saddle connections in P-Stokes graph



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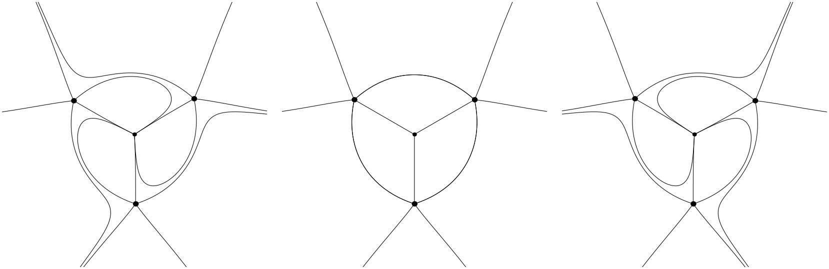
These saddle connections play the role of **wall of marginal stability**:

Theorem [I 13, 14]

- If t lies *outside* of the wall, the Borel transform has singular points at $y = 2m\pi i\alpha$ ($m \in \mathbb{Z}_{\neq 0}$) and

$$\mathcal{S}_{+}[q_{\rm trans}(t,\hbar;A)] = \mathcal{S}_{-}[q_{\rm trans}(t,\hbar;\tilde{A})] \text{ with } \tilde{A} = (1 + e^{2\pi i\alpha/\hbar}) \cdot A$$

Saddle connections in P-Stokes graph



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Problems

Exact WKB analysis of linear differential equations:

- Higher rank ODEs (Aoki-Kawai-Koike-Takei, Gaiotto-Moore-Neitzke, Katzarkov-Noll-Pandit-Simpson, ...).
  - Borel summablity.
  - Exact WKB analysis $\leftrightarrow$ cluster algebra dictionary.
- Relation / application to topological recursion.
- Exact qunatization condition (Marino, Grassi, Kashani-Poor, ...)

Exact WKB analysis of non-linear differential equations:

- 2-parameter formal solution of Painlevé equations (Aoki-Kawai-Takei).

$$q(t,\hbar;A,B) = \sum_{k_1,k_2 \ge 0} A^{k_1} B^{k_2} \cdot q^{(k_1,k_2)}(t,\hbar) \cdot e^{(k_1-k_2)\phi(t)/\hbar}$$

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Thank you for your attention !