SUSY field theory and geometric Langlands
The other side of the coin

Jörg Teschner

String-Math 2016
June 27, 2016

Based on work in progress with A. Balasubramanian, I. Coman-Lohi
and earlier work with E. Frenkel, S. Gukov

The geometric Langlands correspondence:

... has gained many faces. One of the earliest:

$$\boxed{{}^{\mathrm{L}}\mathfrak{g}\text{-opers on } C} \longrightarrow \boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G(C)}$$

$\mathfrak{sl}_N$-opers: DO's of the form $\partial_y^N - w_2(y)\partial_y + \cdots + w_N(y)$ on C.

$\mathcal{D}$-mod on $\mathrm{Bun}_G \sim$ Eqns. $\mathsf{H}_r\psi = E_r\psi$, H_r: q-Hitchin Hamiltonians

– Precursor of a spectral decomposition of $\mathcal{D}$-modules –

CFT-approach:

- Affine algebra $\hat{\mathfrak{g}}_k$ has large center at critical level $k = -h^\vee$.
- $\hat{\mathfrak{g}}_k$-reps parameterised by restrictions of opers to formal discs.
- Assign to resulting collection of $\hat{\mathfrak{g}}_k$-reps a space of conf. blocks.
- Space of conf. blocks: $\mathcal{D}$-moldule by $\hat{\mathfrak{g}}_k$-Ward identities.

$\exists$ many extensions: Local systems, categorical ....

Gauge theory: (Kapustin-Witten, Gukov-Witten)

Starting point: $N = 4$ SYM on $\mathcal{M}^4 = \mathbb{R} \times \mathbb{I} \times C$.

Topologically twisted theory effectively represented by sigma-model on $\mathbb{R} \times \mathbb{I}$, target $\mathcal{M}_{\mathrm{H}}(C, G)$.

- Integrability: $\mathcal{M}_{\mathrm{H}}(C, G) \sim$ tori $\mathfrak{F}_p$ fibered over points p of a base $\mathcal{B}$
- S-duality $\rightsquigarrow$ SYZ mirror symmetry.
  Fibration of (T-) dual tori $\simeq \mathcal{M}_{\mathrm{H}}(C, \check{G})$, where
  $\check{G}$: Langlands dual of group G.

Allows to understand (see below):

+ Hecke functors from $(4d)_{\mathrm{N}=4}$ Wilson- and 't Hooft loops
+ $\mathcal{D}$-module structure

Not so clear:

- Relation to CFT

2d TQFT-framework I

2d topological sigma model on strip $\mathbb{R} \times \mathbb{I}$, $\mathbb{I} = [0, \pi]$.

- $\mathcal{C}$: category of boundary conditions ("branes") at ends of $\mathbb{I}$.
- $\mathcal{Z}(\mathbb{I}_{\mathfrak{B}_1\mathfrak{B}_2}) \equiv \mathrm{Hom}(\mathfrak{B}_1, \mathfrak{B}_2)$.
- For distinguished brane $\mathfrak{B}_{\mathrm{cc}}$:
  - $\mathcal{A} := \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ has natural algebra structure,
  - $\mathcal{M}^{\mathrm{l}}_{\mathfrak{B}} := \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B})$ has natural (left) module structure,

  each of which defined by "joining strips".

Branes for Geometric Langlands

- (B, A, A)-brane $\mathfrak{F}_p$ supported on torus fiber over point $p \in \mathcal{B}$
- S-duality/mirror symmetry $\rightsquigarrow$ skyscraper sheaf on $\mathrm{Loc}_{\check{G}}(C)$ (moduli space of flat $\check{G}$-connections on C)
- $\rightsquigarrow$ Hecke eigenvalue property.
- $\mathcal{D}$-mod structure: From brane $\mathfrak{B}_{\mathrm{cc}}$: alg. $\mathcal{A} \equiv \mathcal{D}_{-h^\vee}(\mathrm{Bun}_G)$, deformation of $\mathrm{Fun}(\mathcal{M}_{\mathrm{H}}(C)) \simeq \mathrm{Fun}(T^*\mathrm{Bun}_G)$.

Similar: AGT via 2d topological Sigma models (Nekrasov-Witten)

Starting point: $N = 2$ SUSY field theory of class $\mathcal{S}$ on $\mathcal{M}^4_{\epsilon_1\epsilon_2}$, where $\mathcal{M}^4_{\epsilon_1\epsilon_2}$: $S^1_{\epsilon_1} \times S^1_{\epsilon_2}$-circle fibration over $\mathbb{R} \times \mathbb{I}$ with Ω-deform..

Effectively represented by reduction on $S^1_{\epsilon_1} \times S^1_{\epsilon_2}$

- $\rightsquigarrow$ $2d$ sigma model, target $\mathcal{M}_{\mathrm{H}}(C)$,
- $\rightsquigarrow$ boundary conditions represented by (A, B, A)-brane $\mathfrak{B}_{\mathrm{cc}}(G) \simeq \mathfrak{B}_{\mathrm{op}}(\check{G})$

$\mathfrak{B}_{\mathrm{op}}(\check{G})$: The Lagrangian submfd. in $\mathcal{M}_{\mathrm{char}}(C) \simeq \mathrm{Hom}(\pi_1(C), G)/G$, represented by monodromies of DO's $\partial_y^N - w_2(y)\partial_y + \cdots + w_N(y)$ on C.

Relation to Virasoro conformal blocks:

Nekrasov and Witten argue that

- Algebra $\mathcal{A} \simeq \mathcal{A}^{\mathfrak{g}}_{\epsilon_1/\epsilon_2} \simeq$ quant. alg. of functions on $\mathcal{M}_{\mathrm{char}}(C)$
- $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{op}}) \simeq \mathrm{Hom}(\check{\mathfrak{B}}_{\mathrm{op}}, \check{\mathfrak{B}}_{\mathrm{cc}})$ naturally bimodule

$$\mathcal{A}^{\mathfrak{g}}_{\epsilon_1/\epsilon_2} \triangleright \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{op}}) \simeq \mathrm{Hom}(\check{\mathfrak{B}}_{\mathrm{op}}, \check{\mathfrak{B}}_{\mathrm{cc}}) \triangleleft \mathcal{A}^{\check{\mathfrak{g}}}_{\epsilon_2/\epsilon_1}$$

$\Rightarrow$ $\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}) \simeq \mathrm{CB}(C, \mathfrak{Vir})$ (modular duality $\rightarrow$ later).

Nature of modular duality

- Two **commuting** actions of algebras $\mathcal{A}_\hbar$ and $\check{\mathcal{A}}_{1/\hbar}$ realised on *same* **maximal** domain $\mathcal{S}$

  Example: Let

$$\begin{aligned}\mathsf{E}f(k) &= \left[\tfrac{Q}{2} + is - ik\right]_b f(k+ib) \\ \mathsf{F}f(k) &= \left[\tfrac{Q}{2} + is + ik\right]_b f(k+ib)\end{aligned} \qquad \mathsf{K}f(k) = e^{-\pi bk} f(k)\,,$$

  where $[x]_b = \frac{\sin \pi bx}{\sin \pi b^2}$ and $Q = b + b^{-1}$.

  Claim: Operators E, F, K and $\tilde{\mathsf{E}}$, $\tilde{\mathsf{F}}$, $\tilde{\mathsf{K}}$, obtained by $b \to b^{-1}$, generate **commuting** actions of $\mathcal{U}_q(\mathfrak{sl}_2)$, $q = e^{\pi i b^2}$, and $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}_2)$, $q = e^{\pi i b^{-2}}$ on common **maximal** domain $\mathcal{S} \subset L^2(\mathbb{R})$.

$\Rightarrow$ $\exists$ Canonical topology on $\mathcal{S}$ defined by $\mathcal{A}_\hbar$-module structure

$\Rightarrow$ $\exists$ Canonical topological dual $\mathcal{S}'$.

Similar story for $\mathcal{A} \simeq \mathcal{A}^{\mathfrak{g}}_{\epsilon_1/\epsilon_2}$.

A unified picture?

Start with $6d$ theory $\mathfrak{X}$ on mfd. of the form $\mathcal{M}^4_{\epsilon_1\epsilon_2} \times C$, where $\mathcal{M}^4_{\epsilon_1\epsilon_2}$: circle fibration over $\mathbb{R} \times \mathbb{I}$.

NW Top. twist on $\mathcal{M}^4_{\epsilon_1\epsilon_2}$, reduction on $C \rightsquigarrow$ AGT

KW Top. twist on $\mathbb{R} \times \mathbb{I} \times C$, reduction on $S^1_{\epsilon_1} \times S^1_{\epsilon_2} \rightsquigarrow$ Geometric Langlands a là Kapustin-Witten.

Further reduction to 2d: The **same** TQFT !?!?

Results do not quite seem to fit:

- Reduction **KW** should give sthg. **topological** on C, but description of $\mathfrak{B}_{\rm cc}$ used to see $\mathcal{D}$-modules uses **complex structure** of C.
- Reduction **NW** should give sthg. **conformal** on C, but description of $\mathfrak{B}_{\rm cc}$ used to see $\mathcal{A}_q$-modules uses description as $\mathcal{M}_{\rm flat}$ which is **topological**.

To see the other side of the coin: Brane $\mathfrak{B}_x^{\mathrm{WZ}}$

- In $(2d)_{\mathrm{TFT}}$: Supported on fiber of $\mathcal{M}_{\mathrm{H}}(C) \sim T^*\mathrm{Bun}_G(C)$ over point $x \in \mathrm{Bun}_G(C)$
- In $(4d)_{\mathrm{AGT}}$: **Surface operator** defined by singular behaviour of *all* gauge fields
- In $(4d)_{N=4}$: **Zero** (!) Nahm pole.
- In $(6d)$: **Co-dim 2 defect** wrapping $\mathcal{M}^2 \times C$.

Features:

- Valid SUSY boundary condition in $N = 4$ (Gaiotto-Witten)
- Has global G symmetry $\rightsquigarrow$ gauge symmetry G on C.

Emergence of current algebra:

Claim:

$$\boxed{\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_x^{\mathrm{WZ}}) \simeq \mathrm{CB}(C, \hat{\mathfrak{g}}_k)}$$

as $\mathcal{A} \simeq \mathcal{D}_k(\mathrm{Bun}_G)$-modules, where

- $\mathrm{CB}(C, \hat{\mathfrak{g}}_k)$ space of conformal blocks for $\hat{\mathfrak{g}}_k$,
- $k + h^\vee = -\epsilon_2/\epsilon_1$.

Available support

- In $(2d)_{\mathrm{TFT}}$: Arguments à la KW, GW, NW.
- In $(4d)_{\mathrm{AGT}}$: Instanton calculus (Braverman, Alday-Tachikawa, Negut, Nawata, Nekrasov)
- In $(4d)_{N=4}$: Arguments à la Gaiotto-Witten, Cordova-Jafferis.
- In $(11d)_{\mathrm{M}}$: Arguments of Frenkel-Gukov-J.T..

Relation to Separation of Variables

Proposal of Frenkel-Gukov-J.T. can be formulated as the statement that the co-dim 2 surface operator has an effective IR description in terms of a collection of co-dim 4 surface operators.

> *Supported on points on C. Need $h = 4g - 4 + n$ co-dim 4 surface ops. for $\mathfrak{g} = \mathfrak{sl}_2$. Case $\mathfrak{g} = \mathfrak{sl}_3$: Work in progress.*

Co-dim 4 surface operators modify conformal blocks $\mathrm{CB}(C, \mathfrak{Vir})$ in AGT-correspondence by insertions of degenerate representations. (Alday-Gaiotto-Gukov-Tachikawa-Verlinde)

$\Leftrightarrow$ Representation of $\mathrm{CB}(C, \hat{\mathfrak{sl}}_{2,k})$ in terms of $\mathrm{CB}(C, \mathfrak{Vir})$.

This representation can be see as a quantisation of the Separation of Variables method in integrable models.

Generalisation to $\hat{\mathfrak{sl}}_{3,k}$: In progress.

One road to quantum geometric Langlands - I

Mathematical translation:

$$\begin{array}{ccc} \mathcal{D}_k\text{-mod on } \mathrm{Bun}_{\check{G}} & \overset{\text{q-}\mathfrak{GL}}{\Longleftrightarrow} & \mathcal{D}_k\text{-mod on } \mathrm{Bun}_G \\ \mathfrak{SOV}\Big\Updownarrow & & \Big\Updownarrow\mathfrak{SOV} \\ \mathcal{W}_k(\mathfrak{g})\text{-conformal blocks} & \overset{\mathfrak{id}}{\Longleftrightarrow} & \mathcal{W}_{\check{k}}(\check{\mathfrak{g}})\text{-conformal blocks} \end{array}$$

- q-$\mathfrak{GL}$: Quantum Geometric Langlands
- $\mathfrak{SOV}$: Separation of variables
  Math: Passage to Whittaker model ("coefficient functor")
- $\mathfrak{id}$: Duality of $\mathcal{W}$-algebras of Feigin and Frenkel

Comparison Beilinson-Drinfeld with Kapustin-Witten:

$\mathcal{D}$-modules Δ_k^{wz} at $k = -h^\vee$ corresponding to sheaves of conformal blocks in CFT can be described as **double fibration**:

- Fixing bundle: $\Delta_{-h^\vee}^{\mathrm{wz}}|_x \simeq \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_x^{\mathrm{wz}})$
- Fixing oper: $\Delta_{-h^\vee}^{\mathrm{wz}}|_p \simeq \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{F}_p)$

  Left: Beilinson-Drinfeld, right: Kapustin-Witten.

(Second iso uses choice of reference oper/uniformisation)

This is how $\mathcal{B}_x^{\mathrm{wz}}$ represents the "other side of the coin".

Generalisation **opers** $\rightarrow$ **local systems**: Opers with singularities.

Purely topological branes - I

There exist many branes $\mathfrak{B}_{\mathcal{L}}$ w/o dependence on cplx. structure of C, e.g. corresponding to Lagrangians $\mathcal{L}$ in character variety.

- Algebra $\mathcal{A} \simeq \mathcal{A}^{\mathfrak{g}}_{\epsilon_1/\epsilon_2} \simeq$ quant. alg. of functions on $\mathcal{M}_{\rm char}(C)$
- $\mathrm{Hom}(\mathfrak{B}_{\rm cc}, \mathfrak{B}) \simeq \mathrm{Hom}(\check{\mathfrak{B}}, \check{\mathfrak{B}}_{\rm cc})$ naturally bimodule

How to relate the topological to the conformal story

(Or: Betti versus De Rham quantum geometric Langlands corr.)

- Proposal (claim for $\mathfrak{g} = \mathfrak{sl}_2$)

$$
\begin{aligned}
\mathrm{Hom}(\mathfrak{B}_{\rm cc}, \mathfrak{B}^{\rm WZ}_x) &\simeq \mathrm{CB}_{\mathcal{E}_x}(C, \hat{\mathfrak{g}}_k), \\
\mathrm{Hom}(\mathfrak{B}_{\rm cc}, \mathfrak{B}_{\mathcal{L}_a}) &\simeq \mathrm{CB}_{\mathcal{L}_a}(C, \hat{\mathfrak{g}}_k),
\end{aligned}
$$

  as modules of $\mathcal{A}^{\mathfrak{g}}_q$ (CB: via Verlinde loop operators)
- Mathematically: A conjecture of Kazhdan-Lusztig type
- For $\mathfrak{g} = \mathfrak{sl}_2$: Follows from results of J.T.-Vartanov + SOV.

2d TQFT-framework II

2d topological sigma model on strip $\mathbb{R} \times \mathbb{I}$, $\mathbb{I} = [0, \pi]$.

Recall: $\mathcal{C}$: category of branes, $\mathcal{Z}(\mathbb{I}_{\mathfrak{B}_1\mathfrak{B}_2}) \equiv \mathrm{Hom}(\mathfrak{B}_1, \mathfrak{B}_2)$,
$\mathcal{A} := \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$ has natural algebra structure, and
$\mathcal{A}^{\mathfrak{g}}_{\epsilon_1/\epsilon_2} \triangleright \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}) \simeq \mathrm{Hom}(\mathfrak{B}, \mathfrak{B}_{cc}) \triangleleft \mathcal{A}^{\check{\mathfrak{g}}}_{\epsilon_2/\epsilon_1}$.

- Let $\mathbb{V}_{\mathfrak{B}_1\mathfrak{B}_2} \simeq \mathbb{R}_{<0} \times \mathbb{I}_{\mathfrak{B}_1\mathfrak{B}_2} \cup \{\infty\}$ be a triangle with "upper" side removed. Define **state** $\Psi_{\mathfrak{B}_1\mathfrak{B}_2} \equiv \mathcal{Z}(\mathbb{V}_{\mathfrak{B}_1\mathfrak{B}_2}) \in \mathcal{H}_{\mathfrak{B}_1\mathfrak{B}_2}$.
- Let $\mathbb{T}^{\mathfrak{B}_3}_{\mathfrak{B}_1\mathfrak{B}_2} \simeq \mathbb{R}_{\leq 0} \times \mathbb{I}_{\mathfrak{B}_1\mathfrak{B}_2} \cup \{\infty\}$ be a triangle with "upper" side decorated with $\mathfrak{B}_3$. Define **wave-function** of $\Psi_{\mathfrak{B}_1\mathfrak{B}_2}$:

$$\Psi_{\mathfrak{B}_1\mathfrak{B}_2}(\mathfrak{B}_3) \equiv \mathcal{Z}(\mathbb{T}^{\mathfrak{B}_3}_{\mathfrak{B}_1\mathfrak{B}_2}) \in \mathbb{C}\,.$$

- $(4d)_{\mathrm{AGT}}$-description: Wave-fct. defd. by path-integral over half-ellipsoid $\mathbb{E}^{4,-}_{\epsilon_1,\epsilon_2} \simeq \mathbb{R}_- \times \mathbb{I} \times S^1_{\epsilon_1} \times S^1_{\epsilon_2}$; b.c.: fixing scalar zero modes a on $\partial\mathbb{E}^{4,-}_{\epsilon_1,\epsilon_2}$. Resulting brane: $\mathfrak{B}_{\mathcal{L}_a}$.
- Output: Chiral partition function $\Psi_{\mathfrak{B}_{cc}\mathfrak{B}^{\mathrm{WZ}}_x}(\mathfrak{B}_{\mathcal{L}_a}) \equiv \mathcal{Z}_C(a, x)$. Also known as instanton partition function in $4d$.

Fixing Lagrangians

To fix a Lagrangian use Darboux-coordinates (a, t) for $\mathcal{M}_{\mathrm{char}}(C)$,

$$\Omega = \sum_r da_r \wedge dt_r \,.$$

(for $\mathfrak{sl}_2$: `Nekrasov-Rosly-Shatashvili`; "Good" coords. for $\mathfrak{g}$?)

Fixing $a \rightsquigarrow$ Lagrangian $\mathcal{L}_a$. We have:

$$\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathcal{L}_a}) \simeq \mathrm{CB}_a(C, \hat{\mathfrak{g}}_k)$$

for eigen-space $\mathrm{CB}_a(C, \hat{\mathfrak{g}}_k)$ with eigenvalue a of (gen.) Verlinde loop operators associated to $\mathcal{L}$ within $\mathrm{CB}(C, \hat{\mathfrak{g}}_k)$.

Relation to quantisation of character variety $\mathcal{M}_{\mathrm{char}}(C)$:

- Lagrangian $\mathcal{L}$ with coords. $a \rightsquigarrow$ comm. subalg. of $\mathcal{A}^{\mathfrak{g}}_{\epsilon_1/\epsilon_2}$,

$\rightsquigarrow$ $\exists$ Representations $\pi_{\mathcal{L}}$ for $\mathcal{S}$ as spaces of functions $f(a)$ on $\mathcal{L}$,

$\rightsquigarrow$ $\exists$ Distributions $\delta_a \in \mathcal{S}'$, $a \in \mathcal{L}$, $\exists$ modules $\pi_{\mathcal{L}}(\mathcal{A}^{\mathfrak{g}}_{\epsilon_1/\epsilon_2})\,\delta_a$.

How to recover ordinary geometric Langlands:

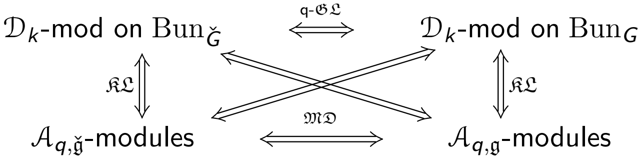
- Fixing half of Darboux coordinates $a \rightsquigarrow$ Lagrangian $\mathcal{L}_a$
- Riemann-Hilbert $\rightsquigarrow$ point ρ_a in $\mathrm{Op}_{\mathfrak{g}}(C)$
- WKB analysis: Brane $\mathfrak{B}_a \to \mathfrak{F}_{p_a}$ for $k + h^\vee \to 0$.

$\rightsquigarrow$ **Fixing a fiber** $\mathfrak{F}_p$.

Warning: Work needed here!

Another road towards quantum geometric Langlands:

(math translation)



- q-$\mathfrak{GL}$: Quantum Geometric Langlands
- $\mathfrak{KL}$: Kazhdan-Lusztig type equivalence
- $\mathfrak{MD}$: Modular duality

Features

- Non-algebraic
- But brings back honest **harmonic analysis**!

Summary

Outline of a picture containing stories of Beilinson-Drinfeld, Kapustin-Witten, Alday-Gaiotto-Tachikawa, and Nekrasov-Witten.

1) Brane $\mathcal{B}_x^{\mathrm{wz}}$ represents the other side of the GL-coin:

- Fixing bundle: $\Delta_k^{\mathrm{wz}}|_x \simeq \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_x^{\mathrm{wz}})$
- Fixing oper: $\Delta_k^{\mathrm{wz}}|_p \simeq \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{F}_p)$

2) Consideration of Lagrangian branes in $\mathcal{M}_{\mathrm{char}}(C)$

- can be used to describe spectral decomposition for $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_x^{\mathrm{wz}})$,
- gives chiral partition functions $\mathcal{Z}(a, x)$: analog of automorphic forms.

Outtakes

- Hecke action
- Opers to general local systems

Looking forward to seeing you next year....

at String-Math 2017 in Hamburg !

Thomas Wolf, www.foto-tw.de

de.wikipedia.org