Fredholm determinant and Nekrasov type representations for isomonodromic tau functions

Oleg Lisovyy

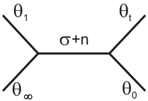
LMPT Tours

Paris, 27/06/2016

in collaboration with P. Gavrylenko

[Gamayun, Iorgov, OL, 1207.0787]

Painlevé VI tau function is a Fourier transform of $c = 1$ conformal block:

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\, \mathcal{B}\left(\vec{\theta}, \sigma + n, t\right) = \sum_{n\in\mathbb{Z}} e^{in\eta}$$  (t)

- 4 parameteres $\vec{\theta} \Longleftrightarrow$ external momenta
- 2 integration constants $(\sigma, \eta) \Longleftrightarrow$ internal momentum + Fourier conjugate variable
- isomonodromic deformation of rank $N = 2$ Fuchsian systems with 4 singular points on $\mathbb{P}^1$

[Iorgov, OL, Teschner, 1401.6104]

- understood in the framework of Liouville CFT and generalized to an arbitrary number of punctures (Garnier system)

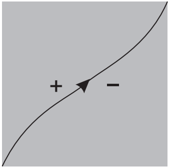
AGT correspondence

$$\mathcal{B}(t) = \mathcal{Z}_{\mathrm{inst}}(t) = \begin{array}{c}\text{combinatorial sum}\\ \text{over tuples of partitions}\end{array}$$ [Nekrasov, '04]

Questions :

- understand this combinatorial structure within the theory of monodromy preserving deformations
- generalize to higher rank/genus/beyond linear quivers
- conformal blocks of W_N beyond semi-degenerate case? T_N theory?

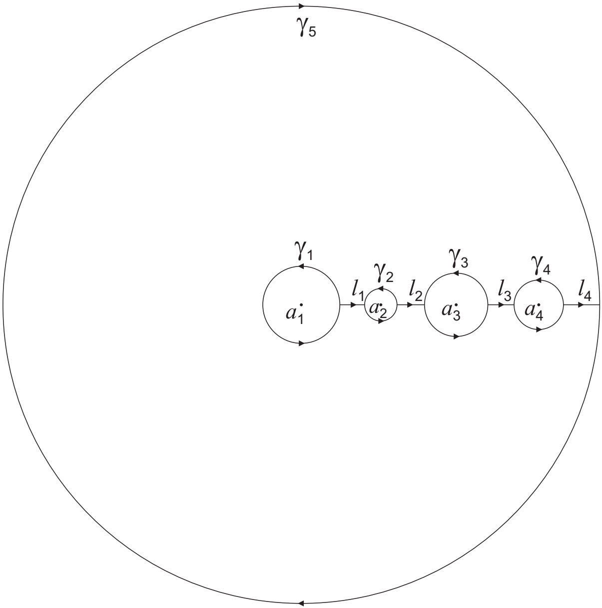
Riemann-Hilbert setup



- contour Γ on a Riemann surface Σ
- jump matrix $J : \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$

RHP defined by (Γ, J) is to find analytic invertible matrix function $\Psi : \Sigma \backslash \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$ whose boundary values satisfy

$$\Psi_+ = J \Psi_-$$



Contour Γ for $n = 5$

Jump data

- local exponents: n diagonal non-resonant $N \times N$ matrices $\Theta_k = \operatorname{diag}\{\theta_{k,1}, \ldots, \theta_{k,N}\}$ $(k = 1, \ldots, n)$ satisfying a consistency relation $\sum_{k=1}^{n} \operatorname{Tr} \Theta_k = 0$
- $2n$ connection matrices $C_{k,\pm} \in \mathrm{GL}(N, \mathbb{C})$ satisfying the constraints

$$\begin{aligned} M_{1\to k} &:= C_{k,-} e^{2\pi i \Theta_k} C_{k,+}^{-1} = C_{k+1,-} C_{k+1,+}^{-1}, \qquad k = 1, \ldots n-2, \\ M_{1\to n-1} &:= C_{n-1,-} e^{2\pi i \Theta_{n-1}} C_{n-1,+}^{-1} = C_{n,-} e^{-2\pi i \Theta_n} C_{n,+}^{-1}, \\ M_{1\to n} &:= \mathbf{1} = C_{n,-} C_{n,+}^{-1} = C_{1,-} C_{1,+}^{-1}, \end{aligned}$$

Jump matrix J

$$\begin{aligned} J(z)\Big|_{\ell_k} &= M_{1\to k}^{-1}, \qquad k = 1, \ldots, n-1, \\ J(z)\Big|_{\gamma_k} &= (a_k - z)^{-\Theta_k} C_{k,\pm}^{-1}, \qquad \Im z \gtrless 0, \quad k = 1, \ldots, n-1, \\ J(z)\Big|_{\gamma_n} &= (-z)^{\Theta_n} C_{n,\pm}^{-1}, \qquad \Im z \gtrless 0. \end{aligned}$$

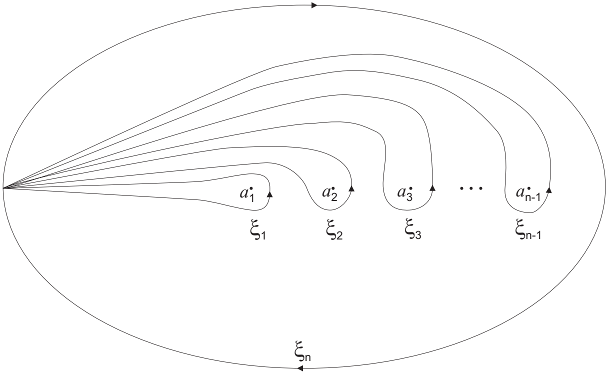
Fundamental matrix solution

$$\Phi(z) = \begin{cases} \Psi(z), & z \text{ outside } \gamma_{1\ldots n}, \\ C_k (a_k - z)^{\Theta_k} \Psi(z), & z \text{ inside } \gamma_k, \quad k = 1, \ldots, n-1, \\ C_n (-z)^{-\Theta_n} \Psi(z), & z \text{ inside } \gamma_n. \end{cases}$$

- only piecewise constant jumps on $\mathbb{R}_{>0}$
- matrix $\Phi^{-1}\partial_z\Phi$ meromorphic on $\mathbb{P}^1$ with poles only possible at $a_1, \ldots, a_n$
- local analysis shows that

$$\partial_z\Phi = \Phi A(z), \qquad A(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}$$

with $A_k = \Psi(a_k)^{-1}\Theta_k\Psi(a_k)$



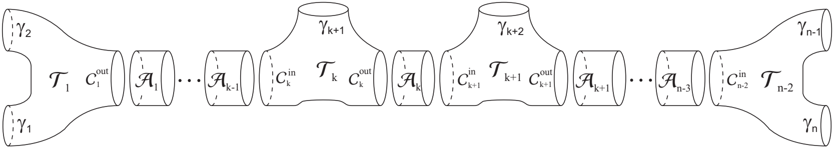
Monodromy representation $\rho : \pi_1 \left(\mathbb{P}^1 \backslash a\right) \to \mathrm{GL}\left(N, \mathbb{C}\right)$ generated by

$$M_k = \rho\left(\xi_k\right) = M_{1\to k-1}{}^{-1} M_{1\to k}$$

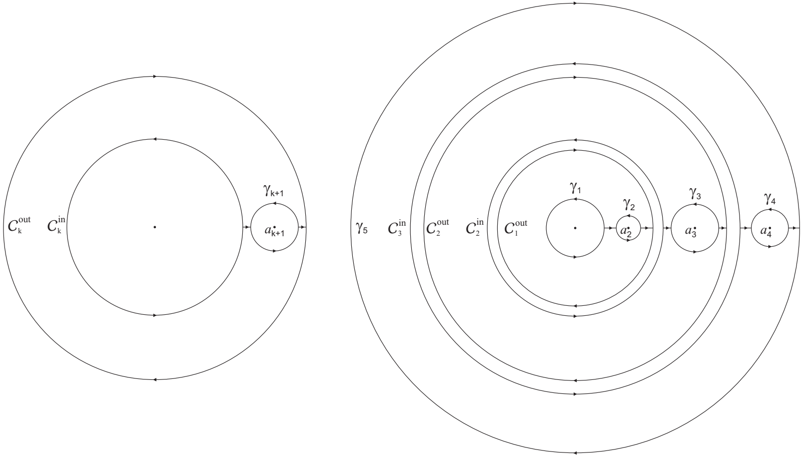
Assume that all $M_{1\to k} = M_1 \dots M_k$ are diagonalizable,

$$M_{1\to k} = S_k e^{2\pi i \mathfrak{S}_k} S_k^{-1}, \qquad \mathfrak{S}_k = \mathrm{diag}\left\{\sigma_{k,1}, \dots, \sigma_{k,N}\right\}.$$

Auxiliary 3-point RHPs



- we are going to associate to the n-point RHP $n - 2$ 3-point RHPs assigned to different trinions



Contour $\Gamma^{[k]}$ (left) and $\hat{\Gamma}$ for $n = 5$ (right)

- jumps on the boundary circles $\mathcal{C}_{k-1}^{\text{out}}$, $\mathcal{C}_k^{\text{in}}$ mimic regular singularities characterized by counterclockwise monodromies $M_{1 \to k}$

Cauchy-Plemelj operators

- associate to every trinion $\mathcal{T}_k$ with $k = 2, \ldots, n-3$ the spaces of vector-valued functions

$$\mathcal{H}^{[k]} = \bigoplus_{\epsilon=\mathrm{in,out}} \left(\mathcal{H}^{[k]}_{\epsilon,+} \oplus \mathcal{H}^{[k]}_{\epsilon,-} \right), \qquad \mathcal{H}^{[k]}_{\epsilon,\pm} = \mathbb{C}^N \otimes \mathcal{V}_{\pm}\left(\mathcal{C}^{\epsilon}_k\right).$$

- elements $f^{[k]} \in \mathcal{H}^{[k]}$ will be written as

$$f^{[k]} = \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix}.$$

- define an operator $\mathcal{P}^{[k]} : \mathcal{H}^{[k]} \to \mathcal{H}^{[k]}$ by

$$\mathcal{P}^{[k]} f^{[k]}(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k \cup \mathcal{C}^{\mathrm{out}}_k} \frac{\Psi^{[k]}_{+}(z)\, \Psi^{[k]}_{+}(z')^{-1} f^{[k]}(z')\, dz'}{z - z'}$$

Lemma. We have $\left(\mathcal{P}^{[k]}\right)^2 = \mathcal{P}^{[k]}$ and $\ker \mathcal{P}^{[k]} = \mathcal{H}^{[k]}_{\mathrm{in},+} \oplus \mathcal{H}^{[k]}_{\mathrm{out},-}$. Moreover, $\mathcal{P}^{[k]}$ can be explicitly written as

$$\mathcal{P}^{[k]} : \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix} \mapsto \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a}^{[k]} & \mathsf{b}^{[k]} \\ \mathsf{c}^{[k]} & \mathsf{d}^{[k]} \end{pmatrix} \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix},$$

where the operators $\mathsf{a}^{[k]}$, $\mathsf{b}^{[k]}$, $\mathsf{c}^{[k]}$, $\mathsf{d}^{[k]}$ are defined by

$$\left(\mathsf{a}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{b}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{c}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k,$$

$$\left(\mathsf{d}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k.$$

- introduce the total space

$$\mathcal{H} := \bigoplus_{k=1}^{n-2} \mathcal{H}^{[k]}.$$

- there is a splitting

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

$$\mathcal{H}_\pm := \mathcal{H}^{[1]}_{\mathrm{out},\pm} \oplus \left(\mathcal{H}^{[2]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[2]}_{\mathrm{out},\pm}\right) \oplus \ldots \oplus \left(\mathcal{H}^{[n-3]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[n-3]}_{\mathrm{out},\pm}\right) \oplus \mathcal{H}^{[n-2]}_{\mathrm{in},\mp}.$$

- combine the 3-point projections $\mathcal{P}^{[k]}$ into an operator $\mathcal{P}_\oplus : \mathcal{H} \to \mathcal{H}$ given by the direct sum

$$\mathcal{P}_\oplus = \mathcal{P}^{[1]} \oplus \ldots \oplus \mathcal{P}^{[n-2]}.$$

- similarly, define another projection $\mathcal{P}_\Sigma : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{P}_\Sigma f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\Sigma} \frac{\hat{\Psi}_+(z)\, \hat{\Psi}_+(z')^{-1} f(z')\, dz'}{z - z'}, \quad \mathcal{C}_\Sigma := \bigcup_{k=1}^{n-3} \mathcal{C}_k^{\mathrm{out}} \cup \mathcal{C}_{k+1}^{\mathrm{in}}.$$

- it is easy to show that $\mathcal{P}_\Sigma \mathcal{P}_\oplus = \mathcal{P}_\oplus$ and $\mathcal{P}_\oplus \mathcal{P}_\Sigma = \mathcal{P}_\Sigma$
- the space
$$\mathcal{H}_\mathcal{T} := \operatorname{im} \mathcal{P}_\oplus = \operatorname{im} \mathcal{P}_\Sigma .$$
can be thought of as the subspace of functions on the union of boundary circles $\mathcal{C}_k^{\mathrm{in}}$, $\mathcal{C}_k^{\mathrm{out}}$ that can be continued inside $\bigcup_{k=1}^{n-2} \mathcal{T}_k$ with monodromy and singular behavior of the n-point fundamental matrix solution $\Phi(z)$
- varying the positions of singular points, one obtains a trajectory of $\mathcal{H}_\mathcal{T}$ in the infinite-dimensional Grassmannian $\mathrm{Gr}(\mathcal{H})$ defined with respect to the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$
- each of the subspaces $\mathcal{H}_\pm$ may be identified with $N(n-3)$ copies of the space $L^2(S^1)$ of functions on a circle; the factor $n-3$ corresponds to the number of annuli and N is the rank of the appropriate RHP

- introduce operators $\mathcal{P}_{\oplus,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ and $\mathcal{P}_{\Sigma,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ given by restrictions of $\mathcal{P}_\oplus$ and $\mathcal{P}_\Sigma$ to $\mathcal{H}_+$
- define $L \in \mathrm{End}\,(\mathcal{H}_+)$ defined by

$$L := {\mathcal{P}_{\oplus,+}}^{-1}\mathcal{P}_{\Sigma,+}$$

- there exists a basis in which $L^{-1} = \mathbf{1} - K$, with

$$K = \begin{pmatrix} U_1 & V_1 & 0 & . & 0 \\ W_1 & U_2 & V_2 & . & 0 \\ 0 & W_2 & U_3 & . & . \\ . & . & . & . & V_{n-4} \\ 0 & 0 & . & W_{n-4} & U_{n-3} \end{pmatrix}, \; \vec{g} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{n-3} \end{pmatrix}, \; \tilde{g}_k = \begin{pmatrix} g^{[k]}_{\mathrm{out},+} \\ g^{[k+1]}_{\mathrm{in},-} \end{pmatrix},$$

$$U_k = \begin{pmatrix} 0 & \mathrm{a}^{[k+1]} \\ \mathrm{d}^{[k]} & 0 \end{pmatrix}, \; V_k = \begin{pmatrix} \mathrm{b}^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \; W_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{c}^{[k+1]} \end{pmatrix}$$

Definition

The tau function associated to the Riemann-Hilbert problem for Ψ is defined as

$$\tau(a) := \det\left(L^{-1}\right)$$

Theorem

We have

$$\tau(a) = \Upsilon(a)^{-1}\tau_{\mathrm{JMU}}(a),$$

where $\tau_{\mathrm{JMU}}(a)$ is defined up to a prefactor independent of a by

$$d_a \ln \tau_{\mathrm{JMU}} = \sum_{1\le k<l\le n-1} \operatorname{Tr} A_k A_l \, d\ln(a_k - a_l),$$

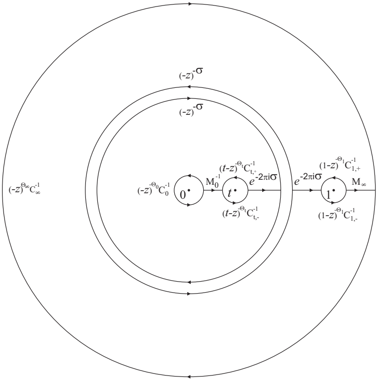
and $\Upsilon(a) = \prod_{k=2}^{n-2} a_k^{\bar{\Delta}_k - \bar{\Delta}_{k-1} - \Delta_k}$, with $\Delta_k = \frac{1}{2}\operatorname{Tr}\Theta_k^2$, $\bar{\Delta}_k = \frac{1}{2}\operatorname{Tr}\mathfrak{S}_k^2$

Example $(n = 4)$

$$\tau_{\mathrm{JMU}}(t) = t^{\frac{1}{2}\,\mathrm{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right)} \det\left(\mathbf{1} - U\right),$$

with

$$U = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \mathrm{End}\left(\mathcal{H}_{\mathcal{C}}\right)$$



where the operators $\mathsf{a} \equiv \mathsf{a}^{[2]} : \Pi_- \mathcal{H}_{\mathcal{C}} \to \Pi_+ \mathcal{H}_{\mathcal{C}}$ and $\mathsf{d} \equiv \mathsf{d}^{[1]} : \Pi_+ \mathcal{H}_{\mathcal{C}} \to \Pi_- \mathcal{H}_{\mathcal{C}}$ are given by

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}(z, z')\, g(z')\, dz', \quad \mathsf{a}(z, z') = \frac{\Psi^{[R]}(z)\, \Psi^{[R]}(z')^{-1} - \mathbf{1}}{z - z'},$$

$$(\mathsf{d}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{d}(z, z')\, g(z')\, dz', \quad \mathsf{d}(z, z') = \frac{\mathbf{1} - \Psi^{[L]}(z)\, \Psi^{[L]}(z')^{-1}}{z - z'}.$$

Fourier basis

Let us represent the elements of $\mathcal{H}_{\mathcal{C}}$ by their Laurent series inside $\mathcal{A}$,

$$f(z) = \sum_{p \in \mathbb{Z}'} f^p z^{-\frac{1}{2}+p}, \qquad f^p \in \mathbb{C}^N,$$

and write integral kernels of 3-point projection operators $\mathrm{a}^{[k]}$, $\mathrm{b}^{[k]}$, $\mathrm{c}^{[k]}$, $\mathrm{d}^{[k]}$ as

$$\mathrm{a}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1} - \mathbf{1}}{z-z'} = \sum_{p,q \in \mathbb{Z}'_+} \mathrm{a}^{[k]}{}_{-q}^{\ p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}, \qquad z,z' \in \mathcal{C}_k^{\mathrm{in}},$$

$$\mathrm{b}^{[k]}(z,z') := -\frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q \in \mathbb{Z}'_+} \mathrm{b}^{[k]}{}_{q}^{p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}-q}, \quad z \in \mathcal{C}_k^{\mathrm{in}}, z' \in \mathcal{C}_k^{\mathrm{out}}$$

$$\mathrm{c}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q \in \mathbb{Z}'_+} \mathrm{c}^{[k]}{}_{-q}^{-p}\, z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}+q}, \ z \in \mathcal{C}_k^{\mathrm{out}}, z' \in \mathcal{C}_k^{\mathrm{in}}$$

$$\mathrm{d}^{[k]}(z,z') := \frac{\mathbf{1} - \Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q \in \mathbb{Z}'_+} \mathrm{d}^{[k]}{}^{-p}_{\ q} z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}-q}, \quad z,z' \in \mathcal{C}_k^{\mathrm{out}}.$$

Von Koch's formula

Let $A \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$ be a matrix indexed by a discrete and possibly infinite set $\mathfrak{X}$. The basic tool for expanding $\tau(a)$ is the formula

$$\det(\mathbf{1}+A) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det A_{\mathfrak{Y}},$$

where $\det A_{\mathfrak{Y}}$ denotes the $|\mathfrak{Y}|\times|\mathfrak{Y}|$ principal minor obtained by restriction of A to a subset $\mathfrak{Y}\subseteq\mathfrak{X}$.

In our case : A is K in the Fourier basis. Elements of $\mathfrak{X}$ are multi-indices which encode the following data:

- positions of the blocks $\mathsf{a}^{[k]}$, $\mathsf{b}^{[k]}$, $\mathsf{c}^{[k]}$, $\mathsf{d}^{[k]}$ in K
- a half-integer Fourier index of the appropriate block;
- a color index in $\{1,\ldots,N\}$.

Combine Fourier and color indices into one multi-index

$$\imath = (p,\alpha) \in \mathfrak{N} := \mathbb{Z}' \times \{1,\ldots,N\}$$

Unordered sets $\{\imath_1,\ldots,\imath_m\}\in 2^{\mathfrak{N}}$ of such multi-indices are denoted by I or J. Given $M\in\mathbb{C}^{\mathfrak{N}\times\mathfrak{N}}$, we denote by M_I^J its restriction to rows I and columns J.

Principal minor

$$
\begin{pmatrix}
0 & \left(a^{[2]}\right)^{I_1}_{J_1} & \left(b^{[2]}\right)^{I_1}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
\left(d^{[1]}\right)^{J_1}_{I_1} & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & \left(a^{[3]}\right)^{I_2}_{J_2} & \left(b^{[3]}\right)^{I_2}_{I_3} & \cdot & \cdot & 0 & 0 \\
0 & \left(c^{[2]}\right)^{J_2}_{J_1} & \left(d^{[2]}\right)^{J_2}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \left(c^{[3]}\right)^{J_3}_{J_2} & \left(d^{[3]}\right)^{J_3}_{I_3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \left(b^{[n-3]}\right)^{I_{n-2}}_{I_{n-3}} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot & 0 & 0 & \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & \left(a^{[n-2]}\right)^{I_{n-3}}_{J_{n-3}} \\
0 & 0 & 0 & 0 & \cdot & \cdot & \left(c^{[n-3]}\right)^{J_{n-3}}_{J_{n-4}} & \left(d^{[n-3]}\right)^{J_{n-3}}_{I_{n-3}} & 0
\end{pmatrix}
$$

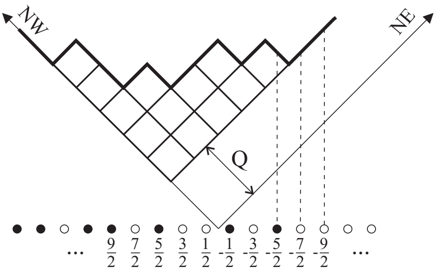
- vanishes unless balance condition $|I_k| = |J_k|$ is satisfied
- factorization into a product of elementary determinants

$$
Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right) := (-1)^{|I_k|} \det \begin{pmatrix} \left(a^{[k]}\right)^{I_{k-1}}_{J_{k-1}} & \left(b^{[k]}\right)^{I_{k-1}}_{I_k} \\ \left(c^{[k]}\right)^{J_k}_{J_{k-1}} & \left(d^{[k]}\right)^{J_k}_{I_k} \end{pmatrix}
$$

Corollary: Fredholm determinant $\tau(a)$ is given by

$$\tau(a) = \sum_{(\vec{I},\vec{J})\in\mathbf{Conf}_+} \prod_{k=1}^{n-2} Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right)$$

- The set Conf_+ of proper balanced configurations $\left(\vec{I},\vec{J}\right)$ may be described in terms of Maya diagrams and charged partitions
- A Maya diagram is a map $\mathrm{m}: \mathbb{Z}' \to \{-1,1\}$ subject to the condition $\mathrm{m}(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- $\mathrm{charge}(\mathrm{m}) = \sharp(\mathrm{particles}) - \sharp(\mathrm{holes})$
- balanced configurations (I_k, J_k) are in one-to-one correspondence with N-tuples of Maya diagrams of zero total charge



- here the charge $Q(\mathrm{m}) = 2$ and the positions of particles and holes are given by $\mathrm{p}(\mathrm{m}) = \left(\frac{13}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}\right)$ and $\mathrm{h}(\mathrm{m}) = \left(-\frac{5}{2}, -\frac{1}{2}\right)$
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q^{(\alpha)} = 0 \right\}.$$

Alternative combinatorial notation :

$$Z_{\vec{Y}_k, \vec{Q}_k}^{\vec{Y}_{k-1}, \vec{Q}_{k-1}} \left(\mathcal{T}^{[k]}\right) := Z_{I_k, J_k}^{I_{k-1}, J_{k-1}} \left(\mathcal{T}^{[k]}\right),$$

Theorem

Fredholm determinant $\tau(a)$ can be written as a combinatorial series

$$\tau(a) = \sum_{\vec{Q}_1,\ldots\vec{Q}_{n-3}\in\mathfrak{Q}_N} \sum_{\vec{Y}_1,\ldots\vec{Y}_{n-3}\in\mathbb{Y}^N} \prod_{k=1}^{n-2} Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}\left(\mathcal{T}^{[k]}\right)$$

- elementary determinants $Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}$ are constructed from matrix elements of 3-point Plemelj operators in Fourier basis
- in rank $N=2$, they are given by Cauchy matrices conjugated by diagonal factors $\Rightarrow$ explicitly computable !!!
- the result coincides with dual Nekrasov partition function for $U(2)$ linear quiver gauge theory with $\epsilon_1+\epsilon_2=0$
- rank $N \Rightarrow$ a sum of $N-1$ Cauchy matrices (unless additional spectral conditions are imposed)

Example

Complete expansion of Painlevé VI tau function at $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta} \mathcal{Z}(\vec{\theta}, \sigma + n; t),$$

The function $\mathcal{Z}(\vec{\theta}, \sigma; t)$ is explicitly given by

$$\mathcal{Z}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{Z}_{\lambda,\mu}\left(\vec{\theta}, \sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{Z}_{\lambda,\mu}\left(\vec{\theta}, \sigma\right) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$
$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$
$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$

Conclusions

1. Isomonodromic tau functions of Fuchsian systems can be written as block Fredholm determinants whose kernels are built of fundamental solutions of 3-point Fuchsian systems
2. Expanding these determinants in Fourier basis leads to combinatorial series over tuples over tuples of partitions
3. The coefficients of the series can be computed explicitly when 3-point solutions have hypergeometric representations (in particular for $N = 2$)