Elliptically fibered Calabi-Yau threefolds: mirror symmetry and Jacobi forms

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SK, A. Klemm, and M. Huang arXiv:1501.04891 and work in progress

Geometry of the Main Example

- $\pi : X \to \mathbf{P}^2$ elliptically fibered Calabi-Yau threefold
- $X = \widetilde{X_{18}} \to X_{18} \subset \mathbf{P}(1,1,1,6,9)$, blowup $x_1 = x_2 = x_3 = 0$, Candelas, Font, K, Morrison hep-th/9403187
- $\pi(x_1, \ldots, x_5) = (x_1, x_2, x_3)$
- $\ell \subset \mathbf{P}^2$ line, $L = \pi^*\ell$, $E \subset X$ exc. div., a section of π
- Kähler cone generated by $\{H = E + 3L, L\}$
- Dual generators of Mori cone $\{f, \ell\}$, where $\ell \subset E \simeq \mathbf{P}^2$ and f elliptic fiber

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The Mirror Geometry

- $X^\circ \subset \mathbf{P}(1,1,1,6,9)/G$,

$$x_1^{18} + x_2^{18} + x_3^{18} - 18\psi x_1 x_2 x_3 x_4 x_5 - 3\phi x_1^6 x_2^6 x_3^6 = 0$$

- Additional symmetries:
- $(\psi, \phi) \mapsto (\zeta\psi, \zeta^6\phi), \ \zeta = \exp(2\pi i/18)$
- $I : (\rho, \phi) \mapsto (i\rho, \phi + \rho^6), \ \rho := (2 \cdot 3^4)^{1/3}\psi, \quad I^*(\Omega) = -\Omega$
- Invariant coordinates at large complex structure (maximal unipotent monodromy) $(s_1, s_2) = (\phi\psi^{-6}, \phi^{-3})$
- Become $(q, \tilde{Q}) = (\exp(2\pi i\tau), \exp(2\pi i\tilde{t}))$ in flat coordinates after mirror map

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Monodromy

- From periods observe an $SL(2, \mathbf{Z})$ action on a divisor with equation $Q = q^{3/2}\tilde{Q} = 0, \; S(Q) = -Q, \; T(Q) = -Q$
- So we shift coordinates to see modularity
- For later use, note that Q satisfies the same multiplier system as η^{12}

$$\eta^{12}(\tau + 1) = \eta^{12}\left(-\frac{1}{\tau}\right) = -\eta^{12}(\tau)$$

Shifted Coordinates

- $T := H + (3/2)L$, work in basis $\{T, L\}$
- $\omega = \tau T + tL = \tau H + (t + (3/2)\tau)L$
- Will explain this shift directly using homological mirror symmetry
- Topological string partition function $Z = Z(t, \tau, \lambda)$ (GW, PT,DT)
- $q = \exp(2\pi i\tau), \; Q = \exp(2\pi it), \lambda = 2\pi z$

$$Z = Z_0(\tau, z)\left(1 + \sum_{d=1}^{\infty} Z_d(\tau, z)Q^d\right),$$

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The Main Claim

- $Z = Z_0(\tau, z)\left(1 + \sum_{d=1}^{\infty} Z_d(\tau, z) Q^d\right)$
- $Z_d(\tau, z)$ is a weak Jacobi form of weight zero and index $d(d-3)/2$ with mulltipliers
- $$Z_d = \frac{\phi_d(\tau, z)}{\eta(\tau)^{36d} \prod_{k=1}^{d} \phi_{-2,1}(\tau, kz)},$$
  where $\phi_d(\tau, z)$ is a weak Jacobi form of weight $16d$ and index $(1/3)d(d-1)(d+4)$

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Motivation and Consistency Checks

- Appearence of Jacobi forms in related situations
  - Elliptic genus of E-strings Haghighat, Lockhart, Vafa 1406.0850; Kim, Kim, Lee, Park, Vafa 1411.2324; Cai, Huang, Sun 1411.2801
  - 6D SCFT Haghighat, Klemm, Lockhart, Vafa, 1412.3152
- $E_2 \mapsto \widehat{E_2} = E_2 - 3/(\pi\tau_2)$ sends $Z \mapsto \mathfrak{Z}$. Jacobi form ansatz implies $\mathfrak{Z}$ automatically satisfies modular anomaly equation from Alim, Scheidegger, Yau, Zhou 1205.1784, 1306.0002
- We can now *derive* the modular anomaly equation directly from the wave function version of the holomorphic anomaly equation Bershadsky, Cecotti, Ooguri, Vafa hep-th/9302103, Witten hep-th/9306122

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- $$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau_\gamma = \frac{a\tau + b}{c\tau + d}, \; z_\gamma = \frac{z}{c\tau + d}$$

- **Definition** A *Jacobi form of weight k and index m* is a function $\phi(\tau, z)$ satisfying

$$\phi\left(\tau_\gamma, z_\gamma\right) = (c\tau + d)^k \, e^{\frac{2\pi i m c z^2}{c\tau + d}} \, \phi(\tau, z)$$

$$\phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\phi(\tau, z)$$

- A modular form of weight k is a weak Jacobi form of weight k and index 0 and conversely
- ϕ has Fourier expansion $\phi = \sum_{n,r} c(n,r) q^n y^r, \; y = \exp(2\pi i z)$
- A *weak Jacobi form* satisfies $c(n,r) = 0$ unless $n \geq 0$, i.e. holomorphic in q and z (but not necessarily in y)

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Structure of Weak Jacobi Forms

- Have a weak Jacobi form of weight -2 and index 1

$$\phi_{-2,1} = \left(y - 2 + y^{-1}\right) \prod_{n=1}^{\infty} \frac{(1 - yq^n)^2 \left(1 - y^{-1}q^n\right)^2}{(1 - q^n)^4}$$

- $\phi_{0,1} = (1/2)$(elliptic genus of K3) is a weak Jacobi form of weight 0 and index 1
- **Proposition.** The weak Jacobi forms form a **C**-algebra, bigraded by weight and index. This algebra is freely generated by $E_4, E_6, \phi_{0,1}$, and $\phi_{-2,1}$

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$d = 1$

- $\phi_1(\tau, z)$ has weight $16d = 16$ and index $(1/3)d(d-1)(d+4) = 0$
- So ϕ_1 is an ordinary modular form of weight 16
- $\phi_1 = \alpha E_4^4 + \beta E_4 E_6^2$ for some α, β
- ϕ_1, hence Z_1 is completely determined by 2 BPS (or PT) invariants!
- Curves in class $\beta = \ell$ are parametrized by lines in $\mathbf{P}^2$, a $\mathbf{P}^2$, so $n_\ell^0 = \chi(\mathbf{P}^2) = 3$
- Curves in class $\beta = \ell + f$ are the union of a line and a fiber, parametrized by a $\mathbf{P}^1$-bundle over $\mathbf{P}^2$, $n_{\ell+f}^1 = -\chi = -6$
- $\phi_1 = (1/48)(31E_4^4 + 113E_4E_6^2)$

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Table of $d = 1$ BPS Numbers

$g \backslash e$	0	1	2	3	4	5
0	3	-1080	143370	204071184	21772947555	1076518252152
1	0	-6	2142	-280284	-408993990	-44771454090
2	0	0	9	-3192	412965	614459160
3	0	0	0	-12	4230	-541440
4	0	0	0	0	15	-5256
5	0	0	0	0	0	-18
6	0	0	0	0	0	0

Table: Some BPS invariants for base degree $d = 1$. Infinitely many BPS numbers have been verified geometrically

$d = 2$

- $k = 32, \ m = 4$
- Have a 17-dimensional space of weak Jacobi forms of weight 32 and index 4
- Constraints: enumerative geometry (which suffices in this case), holomorphic anomaly, conifold gap condition, and invariance under I. Many more than 17 constraints, yet have a unique solution
- Following Zagier, put $A = \phi_{-2,1}, \ B = \phi_{0,1}, Q = E_4, R = E_6$
- $\phi_2 = (1/23887872)B^4Q^2\left(31Q^3 + 113R^2\right)^2 + (1/1146617856)[2507892B^3AQ^7R + 9070872B^3AQ^4R^3 + 2355828B^3AQR^5 + 36469B^2A^2Q^9 + 764613B^2A^2Q^6R^2 - 823017B^2A^2Q^3R^4 + 21935B^2A^2R^6 - 9004644BA^3Q^8R - 30250296BA^3Q^5R^3 - 6530148BA^3Q^2R^5 + 31A^4Q^{10} + 5986623A^4Q^7R^2 + 19960101A^4Q^4R^4 + 4908413A^4QR^6]$

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Table of $d = 2$ BPS Numbers

$g \backslash e$	0	1	2	3	4	5
0	-6	2700	-574560	74810520	-49933059660	7772494870800
1	0	15	-8574	2126358	521856996	1122213103092
2	0	0	-36	20826	-5904756	-47646003780
3	0	0	0	66	-45729	627574428
4	0	0	0	0	-132	-453960
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Further Results

- Have found ϕ_d for $d \leq 7$
- We expect our methods to work in principle to determine ϕ_d for $d \leq 20$
- This gives all BPS invariants for $d \leq 20$ and all g and e
- We expect the ϕ_d for $d \leq 20$ to determine the F_g for $g \leq 189$ by solving the holomorphic anomaly equation
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Computational Methods

- Gopakumar-Vafa invariants n^g_β vanish for g above *Castelnuovo bound* $p_a(d,e)$: for an irreducible curve C of class $\beta = d\ell + ef$ with $e > 0$,

$$p_a \le p_a(d,e) = de - \frac{1}{2}\left(3d^2 - d - 2\right)$$

- Regularity of $P^g_d = \eta^{36d} F^g_d$ in B-model
- Invariance of F^g under I (studied previously in 1306.0002)
- Conifold gap condition: $x = 0$ conifold locus in B-model

$$F^g = \frac{c}{x^{2g-2}} + O(x^0)$$

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Generalizations

- $\pi : X \to B$ elliptic fibration with a unique section E, fibers irreducible in codimension 1
- Curve class $\beta + df,\ \beta \in H_2(E, \mathbf{Z}) = H_2(B, \mathbf{Z})$
- Shift coordinates by $c_1(B)/2$
- $Z = Z_0(\tau, z)\left(1 + \sum_\beta Z_\beta(\tau, z) Q^\beta\right)$
- $Z_\beta(\tau, z)$ is a weak Jacobi form of weight zero and index $\beta \cdot (\beta - c_1(B))/2$ with multipliers
- $$Z_\beta = \frac{\phi_d(\tau, z)}{\eta(\tau)^{12 c_1(B)\cdot\beta} \prod_{j=1}^{b_2(B)} \prod_{k=1}^{\beta_j} \phi_{-2,1}(\tau, kz)},$$

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T-duality

- Can see the $\mathrm{SL}(2, \mathbf{Z})$ acting directly on X without mirror symmetry, following Kontsevich
- Autoequivalence of $D^b(X)$ (actually twisted derived category) following Bridgeland alg-geom/9705002; Andreas, Curio, Hernández Ruipérez, Yau math/0012196
- X is itself moduli of sheaves on X supported on fibers

$$p \mapsto \mathcal{O}_{f_p}(p - p_0) =: \mathcal{P}_p$$

where $f_p = \pi^{-1}(\pi(p))$ is the fiber over p and $p_0 = f_p \cap E$
- Globalize to Poincaré sheaf on $X \times_B X$

$$\mathcal{P} = \mathcal{I}^* \otimes \mathcal{O}(-E \times_B X - X \times_B E)$$

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$$n_0^0 = -\chi(X)/2,\ n_{kf}^0 = -\chi(X) = 60c_1^2,\ n_{kf}^1 = \chi(B),\ (k > 0)$$

Y. Toda math/1103.4229

- Determines all Z_0 and F_0^g. In particular

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$$F_0^g = \frac{15c_1^2 B_{2g} B_{2g-2}}{2g(g-1)(2g-2)!} E_{2g-2}(q), \qquad g \geq 2$$

- F_0^g is modular except for $g = 2$

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Degree 0

- Only nonzero BPS invariants in multiple of fiber class

$$n_0^0 = -\chi(X)/2, \; n_{kf}^0 = -\chi(X) = 60c_1^2, \; n_{kf}^1 = \chi(B), \; (k > 0)$$

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Holomorphic Anomaly

- $$\left(\frac{\partial}{\partial \bar{z}^{\bar{\alpha}}} + \frac{\lambda^2}{2} C_{\bar{\alpha}}^{\beta\gamma} D_{z^\beta} D_{z^\gamma}\right) \mathfrak{Z} = 0$$

  $C_{\bar{\alpha}}^{\beta\gamma} = e^{2K} \bar{C}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} G^{\bar{\beta}\beta} G^{\bar{\gamma}\gamma}$, G metric on moduli space of X°, C Yukawa coupling, D covariant derivative
- Calculate metric and Yukawa couplings at large complex structure to conclude from holomorphic anomaly in shifted coordinates

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial \bar{\tau}} - \frac{\lambda^2}{8\tau_2^2}\left(\beta - \frac{c_1(B)}{2}\right)\right)^2 \mathfrak{Z}_\beta = 0$$

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Almost holomorphic modular anomaly

- Rewrite in terms of almost holomorphic modular forms and use $\mathfrak{F}^2$ result

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Poles

- $$Z = \exp\left(\sum_{\beta,g,m} n_\beta^g \frac{1}{m}\left(2\sin\left(m\pi z\right)\right)^{2g-2} q^{m\beta}\right)$$

  Gopakumar-Vafa hep-th/9812127
- From $g = 0$ can have poles at torsion points of z
- Back to $\mathbf{P}^2$ to simpilify notation, Q^d has a zero of order $3d/2$ at $q = 0$ due to shift, introducing a pole of order $3d/2$ in Z^d to compensate
- $\phi_{-2,1}(\tau, z)$ has a zero at $z = 0$
- $$Z_d(\tau, z)\eta(\tau)^{36d} \prod_{k=1}^{d} \phi_{-2,1}\left(\tau, kz\right)$$

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End of the argument

- Letting

$$\phi_d(\tau, z) = Z_d(\tau, z)\eta(\tau)^{36d} \prod_{k=1}^{d} \phi_{-2,1}(\tau, kz)$$

we conclude that

$$Z_d = \frac{\phi_d(\tau, z)}{\eta(\tau)^{36d} \prod_{k=1}^{d} \phi_{-2,1}(\tau, kz)}$$

Work in Progress

- Refined $SL_2 \times SL_2$ BPS invariants
- More sections: relate to E-string calculation of BPS invariants of dP_n Huang-Klemm-Poretschkin 1308.0619
- Reducible elliptic fibers/enhanced gauge symmetries: richer monodromy structure, as Seidel-Thomas twists interact with $SL(2, \mathbf{Z})$

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Some refined invariants

$$(d,e)=(0,1): \qquad [\frac{1}{2},1]+546[0,0],$$

$$(d,e)=(1,1): \qquad [\frac{1}{2},\frac{3}{2}]+[\frac{1}{2},\frac{1}{2}]+546[0,\frac{1}{2}],$$

$$(d,e)=(2,1): \qquad [\frac{1}{2},3]+[\frac{1}{2},2]+[\frac{1}{2},1]+546[0,\ 2].$$

Conclusions

- The partition function of an elliptically fibered CY can be described by Jacobi forms
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