Derivation of modular anomaly equation in compact elliptic Calabi-Yau spaces

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Based on MH, S. Katz and A. Klemm, arXiv:1501.04891; and work in progress.

Outline

- Introduction and previous works

- The case of compact elliptic Calabi-Yau three-folds (with a main example: elliptic fibration over $\mathbb{CP}^2$)

  1. Genus zero anomaly from differential equations

  2. Higher genus modular anomaly from BCOV holomorphic anomaly (with refinement)

- Summary and Conclusion

Introduction and previous works

- Topological strings: A $N = (2, 2)$ supersymmetric non-linear sigma model from world sheet Σ to target space X.

  $$\Phi_i : \Sigma \to X$$

  Topological string theory is the most interesting and free of world sheet anomaly, when the target space X is a Calabi-Yau 3-fold.

- There are two types of topological twisting: A-model and B-model. We are interested in the topological string partition function

  $$Z = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}(t_i)\right)$$

  where t_i are Kahler moduli in the case of A-model, and complex structure moduli in the case of B-model. The A-model topological string free energy counts holomorphic curves, has rigorous mathematical definition as Gromov-Witten invariants.

- Mirror symmetry relates topological A-model on manifold M to topological B-model on its mirror manifold W. Some very difficult mathematical problems of enumerative geometry can be easily solved by topological B-model methods.

A-model on M

B-model on W

- A long standing problem: How to solve topological strings on compact Calabi-Yau three-folds? The non-compact Calabi-Yau three-folds are basically described by a Riemann surface, so the geometric structure is simpler than the compact case. Many methods work only on the non-compact case.

- The recent works with S. Katz and A. Klemm propose that the topological string partition functions on compact elliptic Calabi-Yau three-folds can be written in terms of Weak Jacobi Forms. This provides topological invariants for fixed base degree, for all fiber degrees and all genera.

- Combined with the B-model method of holomorphic anomaly equation and boundary conditions, we can solve topological strings to very high base degree (for all fiber degrees and all genera), or very high genus (for all base and fiber degrees). MH, S. Katz and A. Klemm, arXiv:1501.04891, [HKK15]. See S. Katz's talk.

S.T. Yau

"... which has culminated in the first all-genus results for the Gromov-Witten theory of compact versions of these manifolds [HKK15]..."
Yau et al, arXiv:1511.01310

- In this talk I will focus on a particular aspect of the works, the derivation of modular anomaly equation.

A quick introduction to modular forms

- (Holomorphic) modular forms $f(\tau), \mathrm{Im}(\tau) > 0$ of weight k satisfies

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau), \qquad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})$$

- Since $f(\tau + 1) = f(\tau)$, we have the Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i\tau}$$

- Eisenstein series $\mathbb{G}_k(\tau) = -\frac{B_k}{2k} E_k(\tau)$,

$$\mathbb{G}_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{m,n} \frac{1}{(m\tau + n)^k} = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1}$$

  where the first sum is over all integer pairs $(m, n) \neq (0, 0)$, B_k is the kth Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$ is the sum of kth power of positive divisors of n. It is a modular form of weight k for even $k > 2$.

- Theorem: Modular forms are homogenous polynomials of the Eisenstein series E_4 and E_6.

- We immediately have some nice identities, e.g.

$$E_4(\tau)^2 = E_8(\tau), \quad E_4(\tau)E_6(\tau) = E_{10}(\tau), \quad \cdots . \tag{2}$$

- The second Eisenstein series $E_2(\tau)$ transforms with a shift under S-duality, and can be made modular by adding an anti-holomorphic piece

$$E_2(-\frac{1}{\tau}) = \tau^2(E_2(\tau) + \frac{12}{2\pi i\tau}), \quad \hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{6i}{\pi(\tau-\bar{\tau})}$$

- The homogenous polynomials of E_2, E_4 and E_6 are known as quasi-modular forms (Zagier). The modular anomaly refers to the partial derivative with respect to E_2.

- The graded ring of quasi-modular forms is closed under derivative with respect to τ, due to the Ramanujan identities

$$\begin{aligned} \frac{1}{2\pi i}\frac{d}{d\tau}E_2 &= \frac{1}{12}(E_2^2 - E_4), \\ \frac{1}{2\pi i}\frac{d}{d\tau}E_4 &= \frac{1}{3}(E_2E_4 - E_6), \\ \frac{1}{2\pi i}\frac{d}{d\tau}E_6 &= \frac{1}{2}(E_2E_6 - E_4^2). \end{aligned} \tag{3}$$

- Equivalently, we can defined the Maass derivative $D_\tau = \partial_\tau + \frac{k}{\tau - \bar{\tau}}$, where k the modular weight of the object acted on by D_τ. Then the usual Leibniz Rule is valid $D_\tau(f_1f_2) = f_1D_\tau f_2 + f_2D_\tau f_1$, and the Ramanujan identities are invariant by replacing

$$\partial_\tau \to D_\tau, \quad E_2(\tau) \to \widehat{E}_2(\tau, \bar{\tau}) \tag{4}$$

A useful formula

- Suppose P_k is a rational function of quasi-modular forms, with modular weight k, then we have the formula

$$\partial_{E_2}\partial_t P_k = \partial_t \partial_{E_2} P_k + \frac{k}{12} P_k, \tag{5}$$

where we use the notation $t = \log(q) = 2\pi i \tau$.

- The derivation uses the Maass derivative and replace E_2 with $\widehat{E}_2$. The derivative with respect to E_2 can be related to the anti-holomorphic derivative

$$\bar{\partial}_{\bar{\tau}} = (\bar{\partial}_{\bar{\tau}} \widehat{E}_2)\partial_{\widehat{E}_2} = \frac{6}{\pi i(\tau - \bar{\tau})^2}\partial_{\widehat{E}_2} \tag{6}$$

We can compute the derivative

$$\bar{\partial}_{\bar{\tau}} D_\tau P_k = (\partial_\tau + \frac{k}{(\tau - \bar{\tau})})\bar{\partial}_{\bar{\tau}} P_k + (\bar{\partial}_{\bar{\tau}} \frac{k}{(\tau - \bar{\tau})}) P_k \tag{7}$$

Taking the holomorphic limit and cancel out the infinitesimal factor $(\tau - \bar{\tau})^{-2}$ we arrive at the formula (5).

Modular anomaly: genus zero case

- A simpler non-compact case: half K3 model. The geometry of the half K3 surface can be constructed by blowing up nine points on $\mathbb{P}^2$. For simplicity we consider the "massless limit" with only the a base and a fiber class.

- By string dualities, the topological string amplitudes on the half K3 Calabi-Yau threefold are equivalent to the partition function of the six-dimensional non-critical E-string compactified on a circle. The winding and momentum numbers of the E-string on the compactified circle correspond to the wrapping numbers n_b and n_e on the base and elliptic fiber in the second homology classes in the half K3 surface.

- The modular anomaly equation is first discovered in J. A. Minahan, D. Nemeschansky and N. P. Warner, hep-th/9707149; J. A. Minahan, D. Nemeschansky, C. Vafa and N. P. Warner, hep- th/9802168.

- The genus zero amplitude $F^{(0)}$ can be computed by Picard-Fuchs differential equations $\mathcal{L}_1 f(z_e, z_b) = \mathcal{L}_2 f(z_e, z_b) = 0$ with

$$\begin{aligned} \mathcal{L}_1 &= \theta_e(\theta_e - \theta_b) - 12 z_e (6\theta_e + 5)(6\theta_e + 1), \\ \mathcal{L}_2 &= [\theta_b - z_b(\theta_b - \theta_e)]\theta_b, \end{aligned} \tag{8}$$

where z_b and z_e are complex structure moduli of the base and the elliptic fiber, and $\theta_i := z_i \frac{\partial}{\partial z_i}$. We follow the approach in A. Klemm, J. Manschot and T. Wotschke, arXiv:1205.1795.

- We solve the equations around the large volume point correspond to $(z_e, z_b) = (0, 0)$. The power series and single logarithmic solutions are

$$\begin{aligned} w_0(z_e) &= 1 + 60 z_e + 13860 z_e^2 + \mathcal{O}(z_e^3), \\ w_e(z_e) &= w_0 \log(z_e) + 312 z_e + 77652 z_e^2 + \mathcal{O}(z_e^3), \\ w_b(z_e, z_b) &= w_0 \log(z_b) + 60 z_e + 20790 z_e^2 - 60 z_e z_b + \mathcal{O}(z^3) \end{aligned} \tag{9}$$

The mirror maps are defined as $t_i = \frac{w_i}{w_0} \sim \log(z_i)$.

- Define the modular parameter τ by the implicit relation

$$J(\tau) = \frac{1728 E_4^3}{E_4^3 - E_6^2} = \frac{1}{z_e(1-432 z_e)}. \tag{10}$$

Using the Ramanujan identities, we can show that $E_4(\tau)^{\frac{1}{4}}$ and $\tau E_4(\tau)^{\frac{1}{4}}$ are solutions to the differential equations. So the mirror map is simply $t_e = 2\pi i \tau$, and we have $w_0(z_e) = E_4(\tau)^{\frac{1}{4}}$.

- The genus zero amplitude $F^{(0)}(t_e, t_b)$ is defined by the fact that $\partial_{t_b} F^{(0)}$ is a double logarithmic solution of the differential equations. After some analysis, it can be shown that the instanton part can be written as quasi-modular forms

$$P^{(0)} \equiv F^{(0)}_{inst} = \sum_{n=1}^{\infty} \frac{e^{\frac{nt_e}{2}}}{\eta^{12n}} P_{6n-2}(E_2, E_4, E_6) e^{nt_b}, \tag{11}$$

and satisfy a recursive modular anomaly equation

$$\partial_{E_2} P_{6n-2} = -\frac{1}{24} \sum_{k=1}^{n-1} k(n-k) P_{6k-2} P_{6(n-k)-2} \tag{12}$$

I am very brief here and shall give more details for a compact case.

- The modular anomaly equation is also derived by Seiberg-Witten curve in J. A. Minahan, D. Nemeschansky and N. P. Warner, hep-th/9707149. However, the above approach by A. Klemm, J. Manschot and T. Wotschke, arXiv:1205.1795 can easily see the quasi-modularity (11).

- The modular anomaly equation is generalized to higher genus in S. Hosono, M. H. Saito and A. Takahashi, [arXiv:hep-th/9901151]. The generalization for general compact elliptic Calabi-Yau manifolds is proposed in A. Klemm, J. Manschot and T. Wotschke, arXiv:1205.1795; M. Alim and E. Scheidegger, arXiv:1205.1784. The refinement (for half K3 model) is proposed in M. -x. Huang, A. Klemm and M. Poretschkin, arXiv:1308.0619.

- The weak Jacobi Form satisfies the modular anomaly equation with the elliptic parameter identified with genus expansion parameter, and provide a completion of the "modular ambiguity".

- The vanishings of Gopakumar-Vafa invariants. For genus zero we have the expansion

$$P^{(0)} = \sum_{m=1}^{\infty} \sum_{n_e,n_b} \frac{n^{n_e,n_b}}{m^3} e^{-m(n_e t_e + n_b t_b)}, \tag{13}$$

where n^{n_e,n_b} are integer genus zero Gopakumar-Vafa invariants. Geometric arguments state that $n^{n_e,n_b} = 0$ for $n_b > n_e$ and $n_b > 1$. These vanishing conditions completely redundantly fix the modular ambiguity.

- How to derive the vanishing conditions from differential equation? This seems much more difficult than deriving the modular anomaly equation. C. Vafa proposed the problem to Don Zagier soon after their paper appeared in 1998.

Cumrun Vafa

How to solve this problem?

Don Zagier

- The compact case: our main example is elliptic fibration over $\mathbb{P}^2$. The Picard-Fuchs operators are

$$\begin{aligned} \mathcal{L}_1 &= \theta_e(\theta_e - 3\theta_b) - 12 z_e (6\theta_e + 1)(6\theta_e + 5), \\ \mathcal{L}_2 &= \theta_b^3 + z_b \prod_{i=0}^{2} (3\theta_b - \theta_e + i). \end{aligned} \tag{14}$$

- The main difference with the non-compact case in e.g. (8): There is no solution dependent only on fiber parameter z_e. So the mirror map for elliptic fiber depends on both z_e and z_b.

- To proceed, we define an auxiliary variable $\tilde{q} = e^{\tilde{t}} = e^{2\pi i \tilde{\tau}}$, which is related to z_e by

$$J(\tilde{\tau}) = \frac{1}{z_e(1 - 432 z_e)}, \tag{15}$$

- As this becomes very technical, It is difficult to explain all the details in a talk. Here I provide only the rough outline of the derivation. For more details see the paper with S. Katz and A. Klemm.

- We can write the ansatz for the z_b power series solution to the equations (14)
$$w = \sum_{n=0}^{\infty} c_n(z_e) z_b^n. \tag{16}$$
We know in the local $z_b \to 0$ limit, there are two linearly independent solutions, which can be written as $E_4(\tilde{\tau})^{\frac{1}{4}}$ and $\tilde{t}E_4(\tilde{\tau})^{\frac{1}{4}}$, in terms of the modular parameter defined in (15). So the initial function $c_0(z_e)$ in the expansion is $E_4(\tilde{\tau})^{\frac{1}{4}}$ or $\tilde{t}E_4(\tilde{\tau})^{\frac{1}{4}}$.

- Plugging the ansatz, the first and second PF equations in (14) become

$$
\begin{aligned}
12z_e(6\theta_e+1)(6\theta_e+5)c_n(z_e) &= \theta_e(\theta_e-3n)c_n(z_e), \\
c_{n+1}(z_e) &= \frac{1}{(n+1)^3}\prod_{i=0}^{2}(\theta_e-3n-i)c_n(z_e)
\end{aligned} \tag{17}
$$

  where $\theta_{z_e} = z_e\partial_{z_e}$. We see that the second equation provides a recursion relation to compute the higher order coefficients $c_n(z_e)$.

- Furthermore, one can check that the two equations are consistent, i.e. the c_{n+1} from recursion in the second equation also satisfies the first equation for c_n with by replacing n with $n+1$. The recursion implies

$$
c_n(z_e) = \frac{1}{(n!)^3}\prod_{k=0}^{3n-1}(\theta_{z_e}-k)c_0(z_e) \tag{18}
$$

- Using Ramanujan identities (3), one can show by induction

$$\begin{aligned}\prod_{k=0}^{n-1}(\theta_{z_e}-k)E_4(\tilde{q})^{\frac{1}{4}} &= \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}}+E_6(\tilde{q}))^n}(a_nE_2(\tilde{q})+b_n),\\ \prod_{k=0}^{n-1}(\theta_{z_e}-k)[\tilde{t}E_4(\tilde{q})^{\frac{1}{4}}] &= \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}}+E_6(\tilde{q}))^n}\Big[\tilde{t}(a_nE_2(\tilde{q})+b_n)+12a_n\Big].\end{aligned}$$

  where a_n and b_n are modular forms of $\tilde{q}$ of weight $6n-2$ and $6n$.

- We can compute the mirror map for elliptic fiber $t_e=\frac{X_1}{X_0}$, where X_0 and X_1 are the solutions that start with $E_4(\tilde{\tau})^{\frac{1}{4}}$ and $\tilde{t}E_4(\tilde{\tau})^{\frac{1}{4}}$. We find

$$\partial_{E_2(\tilde{q})}(t_e-\tilde{t})^{-1}=\frac{1}{12}. \tag{19}$$

- From the useful formula (5), we derive the relation between derivatives

$$\partial_{E_2(\tilde{q})}P_k(q_e)=\partial_{E_2(q_e)}P_k(q_e)+\frac{k}{12}(t_e-\tilde{t})P_k(q_e), \tag{20}$$

  where P_k is a quasi-modular form of weight k.

- We will need to be careful when taking partial $E_2(q_e)$ derivative, by specifying the independent variables that are fixed. We can either keep z_b fixed or t_b fixed. To avoid confusion, we use the notation of the operator $\mathcal{L}_{E_2}$ for this first case and reserve the notation ∂_{E_2} only for the second case, i.e.

$$\mathcal{L}_{E_2} f := \partial_{E_2(q_e)} f(q_e, z_b), \qquad \partial_{E_2} f := \partial_{E_2(q_e)} f(q_e, q_b). \tag{21}$$

  Also the default argument of E_2 is q_e when omitted. The convention in previous literature uses ∂_{E_2} in our notation, while here we introduce the $\mathcal{L}_{E_2}$ as we shall see that it is more convenient for the derivation.

- We derive some useful formulas from differential equations

$$\mathcal{L}_{E_2} z_b = 0, \quad \mathcal{L}_{E_2} t_e = 0, \tag{22}$$

$$\mathcal{L}_{E_2} z_e = 0, \quad \mathcal{L}_{E_2} w_0 = 0, \quad \mathcal{L}_{E_2} t_b = \frac{1}{12} \partial_{t_b} P^{(0)}, \tag{23}$$

  where w_0 is the power series solution to the Picard-Fuchs equation, and $P^{(0)} \equiv P^{(0,0)}$ is the generating function of non-zero base degree instanton contributions in the prepotential.

- From the chain rule for computing derivatives and the last equation in (23), we can relate these two derivatives

$$\begin{aligned} \mathcal{L}_{E_2} f &= \partial_{E_2} f + [\partial_{t_b} f(t_e, t_b)] \mathcal{L}_{E_2}(t_b) \\ &= \partial_{E_2} f + \frac{1}{12} (\partial_{t_b} f)(\partial_{t_b} P^{(0)}), \end{aligned} \tag{24}$$

- Analyzing the double logarithmic solution and use (20), we derive the modular anomaly equation

$$\mathcal{L}_{E_2(q_e)} \partial_{t_b} P^{(0)} = 0. \tag{25}$$

We convert to the ∂_{E_2} notation, integrate once and we arrive at the genus zero modular anomaly equation

$$\partial_{E_2} P^{(0)} = -\frac{1}{24} (\partial_{t_b} P^{(0)})^2. \tag{26}$$

Modular anomaly: higher genus

- We will also consider refined topological strings, which have two indices
$$F = \sum_{n,g=0}^{\infty} F^{(n,g)}(t_i)(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}, \tag{27}$$
where ϵ_1, ϵ_2 are gravi-photon field strength. The unrefined limit is $\epsilon_1+\epsilon_2=0$, while the Nekrasov-Shatashvilli limit is $\epsilon_2=0$. We use the calligraphy to denote B-model amplitude so that the holomorphic limit of $w_0^{2n+2g-2}\mathcal{F}^{(n,g)}$ is $F^{(n,g)}_{A-model}$.

- The refined holomorphic anomaly equation is proposed. The results for non-compact toric geometry agree with the calculations from refined topological vertex.
D. Krefl and J. Walcher, [arXiv:1007.0263];
M. x. Huang and A. Klemm, [arXiv:1009.1126]

- The genus one formulas are known in the literature

$$
\begin{aligned}
\mathcal{F}^{(1,0)} &= \frac{1}{24}[\log(\Delta_1\Delta_2) - \sum_{i=1}^{h^{1,1}} c_i \log(z_i)] + c_0 K, \\
\mathcal{F}^{(0,1)} &= \frac{1}{2}(3 + h^{1,1} - \frac{\chi}{12})K + \frac{1}{2}\log\det G^{-1} - \frac{1}{12}\log(\Delta_1\Delta_2) \\
&\quad -\frac{1}{24}\sum_{i=1}^{h^{1,1}} s_i \log(z_i),
\end{aligned}
\tag{28}
$$

- The only modular anomaly comes from the determinant $\det G$. We can compute

$$
\begin{aligned}
\mathcal{L}_{E_2}\mathcal{F}^{(1,0)} &= 0, \\
\mathcal{L}_{E_2}(\mathcal{F}^{(0,1)}) &= \frac{1}{2}\mathcal{L}_{E_2}\log(\det(\partial_{t_\alpha} z_\beta)) = \frac{1}{2}\sum_{\alpha,\beta}(\partial_{z_\beta} t_\alpha)\mathcal{L}_{E_2}(\partial_{t_\alpha} z_\beta) \\
&= -\frac{1}{24}\partial_{t_b}^2 P^{(0)},
\end{aligned}
\tag{29}
$$

where we use the formula for $\mathcal{L}_{E_2}(\partial_{t_\alpha} z_\beta)$ without providing the details here.

- Taking into the classical contribution, we can write the modular anomaly equation for the positive base degree contributions $P^{(1,0)}$ and $P^{(0,1)}$ as

$$\mathcal{L}_{E_2}P^{(1,0)}=-\frac{89}{1728}\partial_{t_b}P^{(0)},\quad \mathcal{L}_{E_2}P^{(0,1)}=\frac{1}{8}\partial_{t_b}P^{(0)}-\frac{1}{24}\partial_{t_b}^2P^{(0)}.$$

  The equation for $P^{(0,1)}$ agrees with previous literatures, while the equation for refined case $P^{(1,0)}$ is somewhat different.

- Higher genus $g \geq 2$. The BCOV holomorphic anomaly comes from boundaries of moduli space of string world sheet. M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, [arXiv:hep-th/9309140].
  The refined holomorphic anomaly equation is

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(n,g)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}[D_jD_k\mathcal{F}^{(n,g-1)} + (\sum_{n_1=0}^{n}\sum_{g_1=0}^{g})' D_j\mathcal{F}^{(n_1,g_1)}D_k\mathcal{F}^{(n-n_1,g-g_1)}], \tag{30}$$

- We will derive the modular anomaly equation from holomorphic anomaly equation. The topological string amplitude $\mathcal{F}^{(n,g)}$ is a polynomial of BCOV propagators S^{ij}, S^i, S, where the coefficients are rational functions of complex structure modulus parameters. So we have the anomaly

$$\mathcal{L}_{E_2}\mathcal{F}^{(n,g)} = \frac{\partial\mathcal{F}^{(n,g)}}{\partial S^{ij}}(\mathcal{L}_{E_2}S^{ij}) + \frac{\partial\mathcal{F}^{(n,g)}}{\partial S^{i}}(\mathcal{L}_{E_2}S^{i}) + \frac{\partial\mathcal{F}^{(n,g)}}{\partial S}(\mathcal{L}_{E_2}S). \tag{31}$$

  The partial derivatives $\frac{\partial\mathcal{F}^{(n,g)}}{\partial S^{ij}}, \frac{\partial\mathcal{F}^{(n,g)}}{\partial S^{i}}, \frac{\partial\mathcal{F}^{(n,g)}}{\partial S}$ are computed from (30).

- The BCOV propagators S^{ij}, S^i, S are defined by

$$\bar{\partial}_{\bar{i}} S^{jk} = \bar{C}_{\bar{i}}^{jk}, \quad \bar{\partial}_{\bar{i}} S^{j} = G_{\bar{i}k} S^{jk}, \quad \bar{\partial}_{\bar{i}} = G_{\bar{i}j} S^{j}, \tag{32}$$

where $G_{\bar{i}j} = \bar{\partial}_{\bar{i}} \partial_j K$ is the special Kahler metric of the Calabi-Yau moduli space. These equations can integrated and one fixes the holomorphic ambiguities. M. Alim and J. D. Lange, [arXiv:0708.2886] Using the integrated equations we compute the modular anomaly of BCOV propagators

$$\mathcal{L}_{E_2} S^{ij} = -\frac{1}{12 w_0^2} (\partial_{t_b} z_i)(\partial_{t_b} z_j), \tag{33}$$

$$\mathcal{L}_{E_2} S^{i} = -\frac{1}{12 w_0^2} (\partial_{t_b} z_i)(\partial_{t_b} \log w_0), \tag{34}$$

$$\mathcal{L}_{E_2} S = -\frac{1}{24 w_0^2} (\partial_{t_b} \log w_0)^2.. \tag{35}$$

- After some lengthy calculations, we finally derive the refined modular anomaly equations. For zero base degree we have

$$\partial_{E_2} P_0^{(0,2)} = -\frac{3}{32}, \qquad \partial_{E_2} P_0^{(1,1)} = \frac{89}{1152},$$
$$\partial_{E_2} P_0^{(2,0)} = -\frac{1}{24}(\frac{89}{144})^2 = -\frac{7921}{497664},$$
$$\partial_{E_2} P_0^{(n,g)} = 0, \text{for} \quad n+g>2. \tag{36}$$

For positive base degree amplitudes, we have

$$\mathcal{L}_{E_2} P^{(n,g)} = -\frac{1}{24}(\sum_{n_1=0}^{n}\sum_{g_1=0}^{g})'[\partial_{t_b} P^{(n_1,g_1)}][\partial_{t_b} P^{(n-n_1,g-g_1)}]$$
$$-\frac{89}{1728}\partial_{t_b} P^{(n-1,g)} + \frac{1}{8}\partial_{t_b} P^{(n,g-1)} - \frac{1}{24}\partial_{t_b}^2 P^{(n,g-1)}.$$

- This agrees with previous literature for unrefined limit $n=0$. We check the topological invariants with geometric calculations. We see it is fine for $n=1$ but not valid for $n \geq 2$.

Summary and Conclusion

- This can be generalized to more base manifolds, e.g. Hirzebruch surfaces $\mathbb{F}_n, n = 0, 1, 2$. There is an alternative derivation from Witten's equation, which interprets the topological string partition function as a wave function. See S. Katz's talk.

- The refined holomorphic anomaly equation needs correction in the compact case for $n \geq 2$.

- We shall discover more structures (e.g. gap condition near conifold point, weak Jacobi Form) in the topological string partition function, enable the complete solution on a compact Calabi-Yau three-fold.

Merci Beaucoup