Quantized Coulomb branches of 3d N=4 gauge theories and difference operators

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- ·1503.03676 Nakajima
- ·1601.03586 Braverman-Finkelberg-Nakajima
- ·1604.03586 ________________________
- · its appendix Braverman-Finkelberg-Kamnitzer-Kodera-Nakajima-Webster-Weekes
- ·1606.02002 Nakajima-Takayama
- · to appear Kodera-Nakajima

Coulomb branches of 3D N=4 SUSY gauge theories

(review for mathematicians)

Take G_c : a compact Lie group, G its complexification

M : a quaternionic representation of G_c

(a symplectic representation of G)

Physics

$\Rightarrow$ 3D N=4 SUSY gauge theory (4D N=2 as well)

$\rightsquigarrow$ $\mathcal{M}_C \equiv \mathcal{M}_C(G, M)$: Coulomb branch

a noncompact hyperKaehler manifold

possibly with singularities

and SU(2)-action rotating complex structures

Very roughly

gauge theory $\cong$ 3D σ-model with target $\mathcal{M}_C$

(when $\mathcal{M}_H = \{0\}$ and $\mathcal{M}_C$ is smooth)

Higgs branch = hyperKaehler quotient $M /\!/\!/ G_c$

1996 Seiberg-Witten

$\mathcal{M}_C(G_c, M)$ = Atiyah-Hitchin manifold

$\phantom{\mathcal{M}_C(}\shortparallel \quad \shortparallel$

$\phantom{\mathcal{M}_C}SU(2), \; 0$ = moduli of centered SU(2) charge 2

magnetic monopoles on $\mathbb{R}^3$

Then many subsequent works computing $\mathcal{M}_C$ in

examples......

But the definition of $\mathcal{M}_C$ was not clear to mathematicians (e.g., me).

17 years later

2013 Cremonesi-Hanany-Zaffaroni

combinatorial expression (monopole fomula) of

$$\mathrm{ch}_{\mathbb{C}^*} \underline{\mathbb{C}[\mathcal{M}_C]}$$

coordinate ring of $\mathcal{M}_C$ (chiral ring)

The formula is mathematically meaningful !

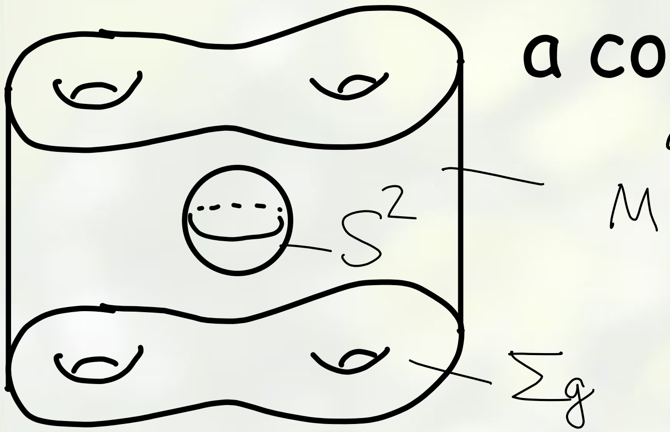
It motivated me to look for a mathematical definition.

A proposal of a mathematical definition

Idea : Suppose we have a TQFT given by twisting of
a SUSY gauge theory.

Then $Z(S^2)$ = (Hilbert space for S^2) is

a commutative ring acting on $Z(\Sigma_g)$ as



$\partial M = S^2 \sqcup \Sigma_g \sqcup -\Sigma_g$

$\therefore Z(M) \in \mathrm{Hom}(Z(S^2) \otimes Z(\Sigma_g), Z(\Sigma_g))$

Hence $\mathbb{C}[\mathcal{M}_c] = Z(S^2)$

Moreover, in mathematical works, $Z(\Sigma_g)$ is defined by

cohomology groups of moduli spaces $G_c = SU(2), M = 0$ $\quad G_c = U(1), M = \mathbb{H}$

of a nonlinear PDE (e.g., flat connection,Seiberg-Witten)

Combining it with heuristic consideration and monopole formula, we arrive at the following:

Assume $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$.

$\mathcal{R}$ = the space of Hecek correspondences with sections

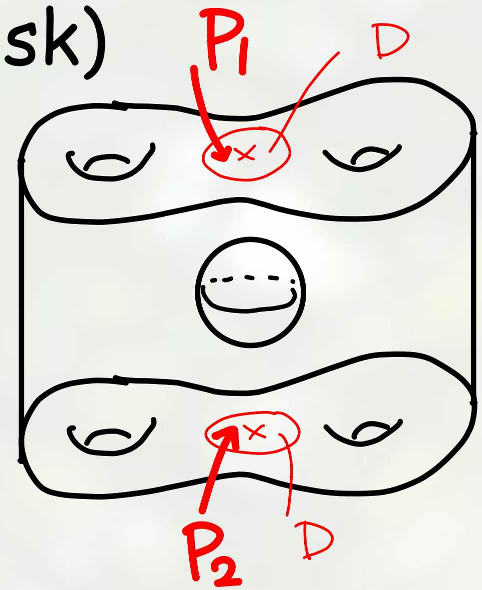
= moduli stack of (P_1, P_2, φ, s)

P_i : holomorphic G-b'dle on D (formal disk)

$\varphi : P_1|_{D\setminus\{0\}} \xrightarrow{\cong} P_2|_{D\setminus\{0\}}$ isomorphism

s : holomorphic section of $P_1 \times_{\mathbb{C}} \mathbf{N}$

s.t. $\varphi(s)$ has no pole at 0



Lemma

commutative

$$\boxed{\mathbb{C}[\mathcal{M}_C] \overset{\text{def.}}{=} H_*^{BM}(\mathcal{R}) + \text{convolution product}}$$

* This gives a definition of $\mathcal{M}_C$ as an affine algebraic variety
* $\mathbb{C}^*$-action is given by homological degrees with correction
* $\mathrm{ch}_{\mathbb{C}^*}\mathbb{C}[\mathcal{M}_C]$ = monopole formula (when the degree $\geqq 0$)

Example $G = \mathbb{C}^*,\ N = 0$

$P_i \cong$ trivial line bundle $\quad \varphi = z^k \quad (k \in \mathbb{Z})$

$\therefore\ \mathcal{R} = \cdots \overset{-2}{\times} \quad \overset{-1}{\times} \quad \overset{0}{\times} \quad \overset{1}{\times} \quad \overset{2}{\times} \cdots$

point$/\mathbb{C}^* = B\mathbb{C}^*$

$\therefore\ H_*^{BM}(\mathcal{R}) = \bigoplus_{k \in \mathbb{Z}} H_*(B\mathbb{C}^*) \cong \mathbb{C}[w, x, x^{-1}]$

$\therefore\ \mathcal{M}_C = \mathbb{C} \times \mathbb{C}^* \quad (= T^*\mathbb{C}^*)$

Remark 1) When G = torus, N = arbitrary,

$\mathbb{C}[\mathcal{M}_C]$ has an explicit presentation.

2) $\pi_0(\mathcal{R}) \cong \pi_1(G)$ (in general)

Pontryagin dual

$\Rightarrow \mathcal{M}_C$ has an additional $\pi_1(G)^\wedge$-action.

Quantization

Consider $\mathbb{C}^*$-action on the formal disk D

The equivariant homology $H_*^{\mathbb{C}^*}(\mathcal{R})$

- is a deformation of $H_*(\mathcal{R})$ over $\operatorname{Spec} H^*_{\mathbb{C}^*}(pt) = \mathbb{C}$
- has a convolution product, but noncommutative !

Def. $\boxed{\mathcal{A}_\hbar = H_*^{\mathbb{C}^*}(\mathcal{R})}$: quantized Coulomb branch

Example $G = \mathbb{C}^*$, $N = 0$

$$\mathcal{A}_\hbar = \langle w, x, x^{-1} \rangle \qquad [x^{\pm}, w] = \hbar x^{\pm}$$

So $x^{\pm} = \mathcal{D}^{\pm} : f(w) \mapsto f(w \pm \hbar)$ difference operators

Lemma $\mathcal{A}_\hbar(G,N) \overset{\star}{\hookrightarrow} \mathcal{A}_\hbar(T,0)[\mathrm{root}^{-1}]$

difference operators with poles along root hyperplanes

When $G = GL(k)$ (more generally $G = \prod GL(k_i)$)

$$\mathcal{R} \supset \{ (E_1, E_2, \varphi, s) \mid E_1 \subset E_2 \subset E_1(0) \}$$
$$\text{or } E_2 \subset E_1 \subset E_2(0) \}$$

smooth closed subvariety

The image of its fundamental class under $\star$ can be computed by the fixed point fomrula !

cf. Bullimore, Dimofte, Gaiotto 1503.04817

Example $G = GL(k)$, $N = \mathfrak{gl}(k) \oplus (\mathbb{C}^k)^{\oplus \ell}$

$$\begin{array}{c} \mathbb{C}^k \circlearrowleft \\ \uparrow \\ \mathbb{C}^\ell \end{array}$$

$1 \leq n \leq k$

$$F_n = \sum_{\substack{I \subset \{1, \cdots, k\} \\ \# I = n}} \prod_{\substack{i \in I \\ j \notin I}} \frac{w_i - w_j + t}{w_i - w_j} \prod_{i \in I} \prod_{a=1}^{\ell} (w_i - z_a) \vartheta_i^{-1}$$

(generalized) Macdonald operators

Th. $\mathcal{A}_h \cong$ spherical part of

in this example cycltomic rational Cherednik algebra

$\mathfrak{S}_k \ltimes (\mathbb{Z}/\ell\mathbb{Z})^k$

Remark 1) $\mathcal{M}_C \cong \mathrm{Sym}^k(\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z}))$ de Boer, Hori, Ooguri, Oz 1997

2) finite ADE quiver $\Rightarrow$ $\mathcal{A}_h$ shifted Yangian

Future Problems

- Give a hyper-Kaehler metric (or twistor space)
- Determine (quantized) Coulomb branches of quiver gauge theory of affine ADE types

  Conj. moduli of instantons on multi Taub-NUT
- Construct TQFT (with/without S^1-action)
- Study representation theory of quantized Coulomb branches (e.g. symplectic duality of BLPW)
- Relation to 4d Coulomb branches ? (K-theory ver.)