

3d SUPERSYMMETRIC GAUGE THEORIES AND HILBERT SERIES

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Based on:

arXiv: 1309.2657 with Hanany, Zaffaroni

1403.0585
2384

H, Mekareeya, Z

1408.6835

Ferlito, H, M

1410.1584

H, M, Z

} 3d
N=4

1505.02409

} 3d N=2
(subclass)

1607.?

M, Z

} 3d N=3,2

MODULI SPACE OF SUPERSYMMETRIC VACUA \mathcal{M}

[Luty, Taylor '95]

- 4d $\mathcal{N}=1$ GAUGE THEORY:
 - Gauge group G VECTOR MULTIPLETS V^a (real)
 - Representation \mathcal{R} CHIRAL MULTIPLETS X^i (cplx)
 - G -invariant poly $W(X)$ SUPERPOTENTIAL

- Scalar potential: $V = \sum_i |F_i|^2 + \frac{1}{2} g^2 \sum_a D^a D^a$
 - $F_i = \frac{\partial W}{\partial X^i}$ "F-TERMS"
 - $D^a = \sum_i X_i^\dagger (T^a)^i_j X^j$ "D-TERMS"
 - ($D = \sum_i q_i |X_i|^2 - \xi$ for $G=U(1)$)

- MODULI SPACE OF SUSY VACUA:

$$\mathcal{M} = \left\{ X, X^\dagger \mid F_i(X) = 0 \ \forall i, D^a(X, X^\dagger) = 0 \ \forall a \right\} / G \cong \left\{ X \mid \frac{\partial W(X)}{\partial X^i} = 0 \right\} / G^c$$

symplectic quotient

holomorphic quotient

4d $\mathcal{N}=1$ SUSY $\Rightarrow \mathcal{M}$ KÄHLER

(Today: \mathcal{M} as cplx algebraic variety)

THE CHIRAL RING

[Lerche, Vafa, Warner '89]

• GAUGE INVARIANT CHIRAL OPERATORS $\mathcal{O}_i(x)$: $\boxed{\bar{Q}_\alpha \mathcal{O}_i(x) = 0}$

(Q_α, \bar{Q}_α supercharges)

Expectation values $\langle \mathcal{O}_i \rangle$ are HOLOMORPHIC FUNCTIONS on \mathcal{M} .

FACTS: 1) $\langle \bar{Q}_\alpha(\dots) \rangle = 0$

$$\mathcal{O}_i \sim \mathcal{O}_i + \bar{Q}(\dots)$$

2) $\langle \mathcal{O}_{i_1}(x_1) \dots \mathcal{O}_{i_n}(x_n) \rangle = \langle \mathcal{O}_{i_1} \rangle \dots \langle \mathcal{O}_{i_n} \rangle$

Spacetime constant

• CHIRAL RING:

$$\boxed{\mathcal{O}_i \mathcal{O}_j = c_{ij}^k \mathcal{O}_k + \bar{Q}_\alpha(\dots)^\alpha}$$

→ COORDINATE RING of \mathcal{M} : $\mathbb{C}[\mathcal{O}_1, \dots, \mathcal{O}_n] / \mathcal{I}$

- GENERATORS
- RELATIONS

4d $N=1$ gauge theories: \mathcal{O}_i are G -invariant polynomials in X .

THE HILBERT SERIES OF \mathcal{M}

[Pouliot '99; Benvenuti, Feng, Hanany, He '06]

$$H(t, \hat{x}) = \text{Tr}_{\mathcal{H}} \left(t^R \prod_{\hat{i}} \hat{x}_{\hat{i}}^{\hat{Q}_{\hat{i}}} \right)$$

R : R-charge
 $\hat{Q}_{\hat{i}}$: FLAVOR charges

$$[R, \bar{Q}] = \bar{Q}, \quad RW = 2W.$$

$$[\hat{Q}_{\hat{i}}, \bar{Q}] = 0, \quad \hat{Q}_{\hat{i}} W = 0.$$

$$\mathcal{H} = \bigoplus_r \mathcal{H}_r, \quad \mathcal{H}_r = \{ \mathcal{O}_i \mid \bar{Q} \mathcal{O}_i = 0, R \mathcal{O}_i = r \mathcal{O}_i \}$$

* EXAMPLES ($G = \{1\} \Rightarrow \mathcal{M} = \mathcal{F}$)

- 1 chiral X w/ $W=0$:



$$H = 1 + \tau + \tau^2 + \dots = \frac{1}{1-\tau} = \text{PE}[\tau]$$

$$\tau = t^{R[X]}$$

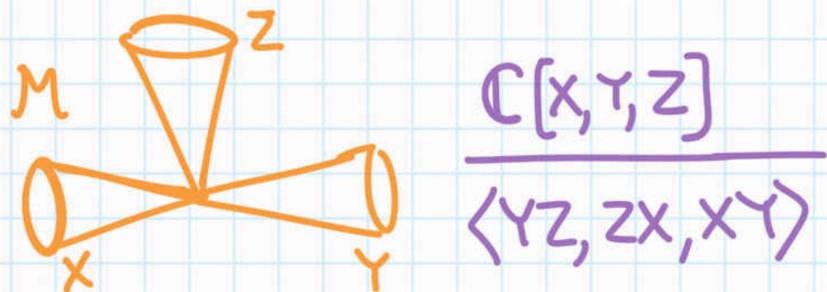
- 1 chiral X w/ $W = X^{N+1}$:



$$H = 1 + \tau + \tau^2 + \dots + \tau^{N-1} = \frac{1-\tau^N}{1-\tau} = \text{PE}[\tau - \tau^N]$$

$$\tau = t^{\frac{2}{N+1}}$$

- 3 chirals X, Y, Z w/ $W = XYZ$:



$$H = \frac{1}{1-\tau x} + \frac{1}{1-\tau y} + \frac{1}{1-\tau z} - 2 =$$

$$= \frac{1-\tau^2(yz+xz+xy) + 2\tau^3xyz}{(1-\tau x)(1-\tau y)(1-\tau z)}$$

$$\tau = t^{2/3}$$

$$xyz = 1$$

• GAUGING G:

$$H_M(t, \hat{x}) = \oint d\mu_G(x) H_F(t, \hat{x}, x)$$

$$\oint d\mu_G(x) \equiv \left(\prod_{j=1}^r \oint_{|x_j|=1} \frac{dx_j}{2\pi i x_j} \right) \prod_{\alpha \in \Delta_+} \left(1 - \prod_{i=1}^r x_i^{\alpha_i} \right)$$

$r \equiv \text{rk}(G)$

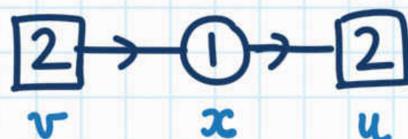
* EXAMPLES

- G with 1 adjoint Φ :
(→ Coulomb branch of 4d $N=2$)

$$H(\tau) = \oint d\mu_G(x) \text{PE}[\tau \chi_{\text{ad}}^G(x)] = \prod_{i=1}^r \frac{1}{1 - \tau^{d_i(G)}}$$

Casimir invariants
 $\mathcal{C}[g]^G = \mathcal{C}[u_1, \dots, u_r]$
 $\text{deg } u_i = d_i(G)$

- $G=U(1)$ with 2 flavors:



$$H_B(\tau, u, v) \equiv g_1(\tau, u, v; B) = \oint \frac{dx}{2\pi i x} x^{-B} \text{PE}[\tau x(u + \frac{1}{u}) + \tau \frac{1}{x}(v + \frac{1}{v})]$$

$$= \begin{cases} \sum_{n=0}^{\infty} [n+B; n]_{u,v} \tau^{2n+B} & , B \geq 0 \\ \sum_{n=0}^{\infty} [n; n+|B|]_{u,v} \tau^{2n+|B|} & , B \leq 0 \end{cases}$$



$B=0$: $H(\tau, u, v; 0) = \text{PE}[[1; 1]_{u,v} \tau^2 - \tau^4]$

Conifold $\frac{\mathcal{C}[x, y, z, w]}{\langle xy - zw \rangle}$

$$\text{PE}[f(t_1, \dots, t_n)] \equiv \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k, \dots, t_n^k)\right) \Rightarrow \text{PE}\left[\sum_i a_i \prod_j t_j^{b_{ij}}\right] = \prod_i \left(1 - \prod_j t_j^{b_{ij}}\right)^{-a_i} \quad \Bigg| \quad [\alpha; \beta]_{u,v} \equiv \chi_{[\alpha]}(u) \chi_{[\beta]}(v)$$

• Moduli spaces of vacua of 4d ($N \geq 1$) SUSY gauge theories and Hilbert series:

MATH — (Hyper)Kähler quotients

PHYS — mostly CLASSICAL

(quantum: [Seiberg '93-'94;...])

• In the rest of my talk, I will discuss a different construction

MATH — (Hyper)Kähler

PHYS — QUANTUM

which is based on the physics of 3d ($N \geq 2$) SUSY gauge theories.

[Intriligator, Seiberg '96; de Boer, Hori, Ooguri, Oz '96;
Aharony, Hanany, Intriligator, Seiberg, Strassler '97; de Boer,
Hori, Oz '97; Dorey, Tong '99; Tong '00; Intriligator, Seiberg '13]
...

NOVELTY — 't Hooft monopole operators: extra chiral operators

subject to quantum relations not derived from a superpotential.

'T HOOFT MONOPOLE OPERATORS

['t Hooft '78; Borokhov, Kapustin, Wu '02]

- 3d $N=2$ "BARE" MONOPOLE OPERATOR $V_m(x)$:

Euclidean path integral over field configs with Dirac monopole singularity at x : $U(1) \hookrightarrow G$. ^{$\frac{1}{2}$ -BPS}



$$A_{\pm} \underset{r \rightarrow 0}{\sim} \frac{m}{2} (\pm 1 - \cos \theta) d\varphi$$

$$\sigma \sim \frac{m}{2r}$$

VECTOR
MULTIPLY

A_{μ}, σ, \dots
 \mathbb{R}



CHIRAL
MULTIPLY

V_m

$m \in h/w$
Cartan Weyl

MAGNETIC
CHARGE

Dirac
quantization

$$e^{2\pi i m} = \mathbb{1}_G$$

$$\Rightarrow m \in \Gamma_G / w$$

[Goddard, Nuyts, Olive '77]

- Can do the same for GLOBAL non-R SYMMETRIES : BACKGROUND MAGNETIC CHARGES .

Monopole operators $V_{m; \hat{m}, B}$

$m \leftrightarrow \sigma$	real scalar	GAUGE
$\hat{m} \leftrightarrow \hat{\sigma}$	real mass	FLAVOR
$B \leftrightarrow \xi$	FI parameter	TOPOLOGICAL

CHARGES OF BARE MONOPOLE OPERATORS

CLASSICAL: • TOPOLOGICAL SYMMETRY $G_J = Z(G^\vee)$

For $G = U(N)$:

$$J(m) = \text{Tr } m = \sum_{i=1}^N m_i$$

$$(m_i) \in \mathbb{Z}^N / S_N$$

• CHARGES FROM CHERN-SIMONS COUPLINGS

e.g. $\frac{k_{AB}}{4\pi} \int A_A \wedge dA_B + \dots$ for $U(1)$'s

$$Q_A^{\text{cl}}(M) = - \sum_B k_{AB} M_B$$

$\{M_A\} \equiv \{m_i, \hat{m}_i, B_i, 0\}$
 gauge ↑
 flavor ↑
 topological ↑
 R

QUANTUM:

$$Q_A^{\text{q}}(M) = - \frac{1}{2} \sum_{\text{fermi } \psi_a} Q_A(\psi_a) |m_a^{\text{eff}}(M)|$$

[Borokhov, Kapustin, Wu '02
 Imamura, Yokoyama '11
 Benini, Closset, SC '11]

$$m_a^{\text{eff}}(M) \equiv \sum_A Q_A(\psi_a) M_A$$

"EFFECTIVE MASS"

We will count monopole operators according to their (classical + quantum) charges:

$$Q_A(M) = - \sum_B k_{AB}^{\text{eff}}(M) M_B, \text{ where}$$

$$k_{AB}^{\text{eff}}(M) = k_{AB} + \frac{1}{2} \sum_{\text{fermi } \psi_a} Q_A(\psi_a) Q_B(\psi_a) \text{sign } m_a^{\text{eff}}(M)$$

EFFECTIVE CS LEVELS

DRESSED MONOPOLE OPERATORS

[SC, Hanany, Zaffaroni '13
SC '15; SC, Nekrasova, '16]

- BARE monopole operator defined via VECTOR multiplets:

← Gauge/Flavor/Top.
 $G \times \hat{G} \times G_f$

$$G \xrightarrow{m} G_m$$

$$G_m \cdot m = 0$$

"RESIDUAL GAUGE GROUP"

$$\rightarrow \prod_{\alpha \in \Delta_+} t^{-|\alpha(m)|} (1-x^\alpha)^{\delta_{\alpha(m),0}}$$

- Can be DRESSED by matter fields (CHIRAL multiplets) s.t.

← Representation
 $(R, \hat{R}, 1)$

$$\boxed{m_{e,\hat{e}}^{\text{eff}}(m, \hat{m}) = \rho(m) + \hat{\rho}(\hat{m}) = 0} \quad \text{"RESIDUAL MATTER FIELDS"}$$

$$\rightarrow \prod_{e,\hat{e}} \left(t^{r-1} x^e \hat{x}^{\hat{e}} \right)^{-\frac{1}{2} |\rho(m) + \hat{\rho}(\hat{m})|} \text{PE} \left[\delta_{\rho(m) + \hat{\rho}(\hat{m}), 0} t^r x^e \hat{x}^{\hat{e}} \right] \quad (\text{if } W=0)$$

Dress bare monopole op. by G_m -invariants of residual matter fields, then average over W .

* Kronecker δ : adjoint Higgs mechanism.

* Prefactor: quantum correction to monopole charges

DRESSED MONOPOLE OPERATORS AND HILBERT SERIES

DRESSING governed by "RESIDUAL GAUGE THEORY" $T_{m; \hat{m}, B}$:

- RESIDUAL GAUGE GROUP G_m (FLAVOR GROUP $\hat{G}_{\hat{m}}$)
- RESIDUAL MATTER FIELDS in rep. of $G_m \times \hat{G}_{\hat{m}}$
- RESIDUAL SUPERPOTENTIAL
- BACKGROUND ELECTRIC CHARGES $Q_i(m, \hat{m}, B)$ for G_m

Hilbert series
 $H_{Q(m, \hat{m}, B)}^{T_{m; \hat{m}, B}}(t, \hat{x})$
 "DRESSING FACTOR"

• HILBERT SERIES: $H(t, \hat{x}, z; \hat{m}, B) = \text{Tr}_{\mathbb{H}_{\hat{m}, B}} \left(z^J t^R \prod_i \hat{x}_i^{\hat{Q}_i} \right)$

$$H(t, \hat{x}, z; \hat{m}, B) = \sum_{m \in \Gamma_q} z^{J(m)} t^{R(m, \hat{m}, B)} \prod_i \hat{x}_i^{\hat{Q}_i(m, \hat{m}, B)} \cdot H_{Q(m, \hat{m}, B)}^{T_{m; \hat{m}, B}}(t, \hat{x})$$

[SC '15; SC, Mekaraya, Zaffaroni '16 to appear]

$\Gamma_q = \Gamma_{G^v}/W$ or a sublattice thereof (due to nonperturbative effects).

"QUANTUM
MAGNETIC
LATTICE"

COULOMB BRANCH OF 3d N=4 GAUGE THEORIES

G • (N=4 VECTOR) = (N=2 VECTOR) \oplus (N=2 CHIRAL $\Phi \in \text{adj}$)

R • (N=4 HYPER) = (N=2 CHIRAL $\in \mathcal{R}$) \oplus (N=2 CHIRAL $\in \overline{\mathcal{R}}$)

M_c

HyperKähler
 $\dim_{\mathbb{H}} M_c = r$

M_H

HK quotient

$$R(m, \hat{m}) = -\frac{1}{2} \sum_{\alpha} |\alpha(m)| + \frac{1}{2} \sum_{\rho, \hat{\rho}} |\rho(m) + \hat{\rho}(\hat{m})|$$

[Borokhov, Kapustin, Wu '02
Gaiotto, Witten '08; ...]

HILBERT SERIES of M_c :

(good/ugly theories)
cf [Gaiotto-Witten '08]

$$H(\tau, z; \hat{m}) = \sum_{m \in \Gamma_G / W} z^{J(m)} \tau^{2R(m, \hat{m})} P_G(\tau^2; m)$$

[SC, Hanany, Zaffaroni '13
" + Mekareeya '14]

$(\tau^2 = t)$

$P_G(t; m) = \prod_{i=1}^r \frac{1}{1 - t^{d_i(m)}}$ Casimirs of G_m

REMARKS:

- $\hat{m} = 0$: M_c cone vs. $\hat{m} \neq 0$: (partial) resolution
- (Very computable) limit of SUSY indices [Razamat, Willett '14; Closset, Kim '16].
- Can often deduce charges of generators / relations. Complementary to [Bullimore, Dimofte, Gaiotto '15].

EXAMPLES

($\hat{m}=0$)



$$H(\tau, z) = \frac{1}{1-\tau^2} \sum_{m \in \mathbb{Z}} z^m \tau^{N|m|} = \frac{1-\tau^{2N}}{(1-\tau^2)(1-z\tau^N)(1-z^{-1}\tau^N)}$$

$$\mathcal{M}_c = \mathbb{C}^2 / \mathbb{Z}_N$$

$$\mathbb{C}[\Phi, v_+, v_-] / \langle v_+ v_- - \Phi^N \rangle$$

cf [Intriligator-Seiberg '96]



$$H(\tau, z) = \sum_{m_1 \geq \dots \geq m_k} z^{\sum_i m_i} \tau^{-2 \sum_{i < j} |m_i - m_j| + N \sum_i |m_i|} P_{U(k)}(\tau^2; \mathbf{m})$$

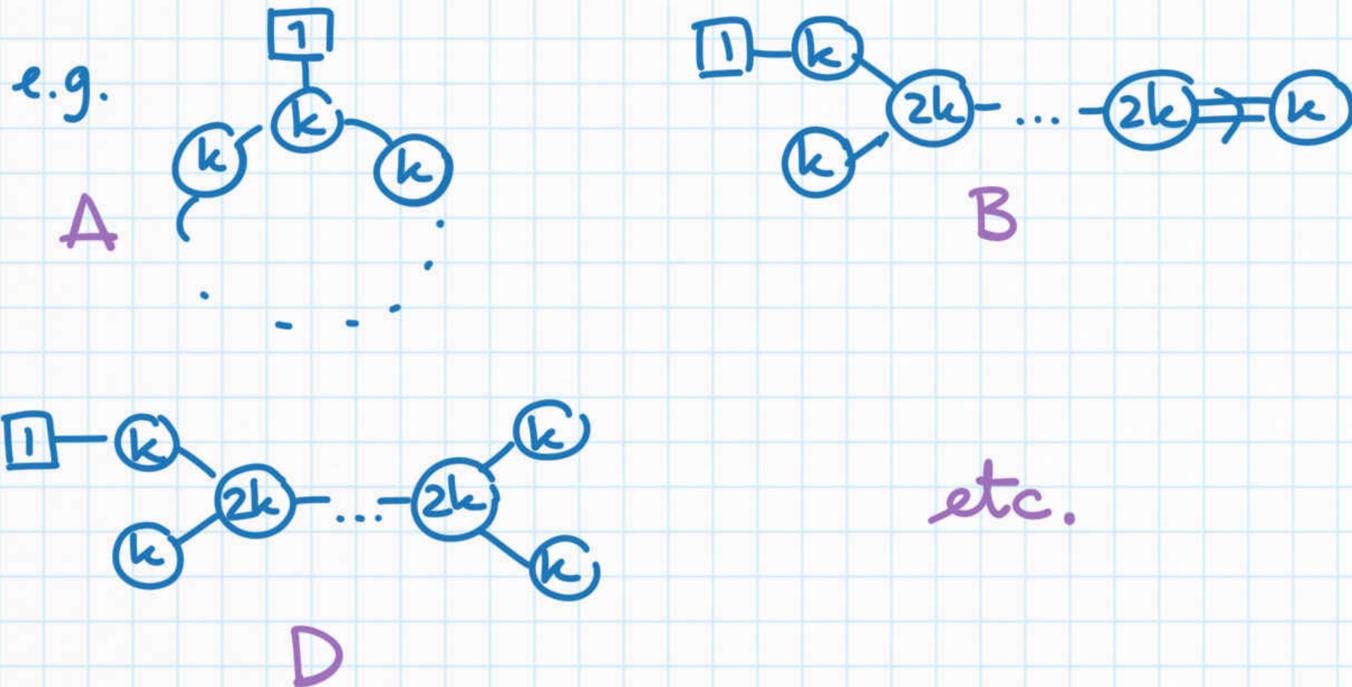
$$= \text{PE} \left[\sum_{j=1}^k \left(\tau^{2j} + (z+z^{-1}) \tau^{N-2(k+j)} - \tau^{2(N-k-j)} \right) \right]$$

3k generators
k relations

see also [Bullimore, Dimofte, Gaiotto '15]

• EXTENDED AFFINE "QUIVERS" for INSTANTON MODULI SPACES:

[SC, Ferlito, Hanany, Mekareeya '14]



$$H_{\mathcal{M}_{g,k}} = \text{PE} \left[\sum_{p=0}^{k-1} \left([p+1; 0] t^{p+1} + [p; \text{Adj}] t^{p+2} \right) + \dots t^{k+2} + \dots \right] \text{ for any } g$$

$\begin{matrix} \text{SU}(2) \\ \downarrow \\ [p+1; 0] \\ \uparrow \\ g \end{matrix}$

see [Braverman, Finkelberg, Nakajima '16]

MODULI SPACES OF 3d N=2 THEORIES

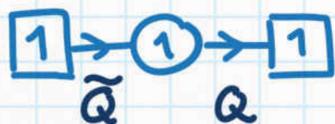
• YANG-MILLS THEORIES ($W=0$ for simplicity, $\hat{m}=B=0$)

[SC '15; Hanany et al '15]

$$H(t, z, \hat{x}) = \sum_{m \in \Gamma_q} z^{J(m)} \prod_{i=1}^r \oint_{2\pi i x_i} \prod_{\alpha \in \Delta_+} (1-x^\alpha)^{\delta_{\alpha(m),0}} t^{-|\alpha(m)|} \prod_{e \in \hat{e}} (t^{r_e \hat{e}^{-1}} x^{e \hat{x} \hat{e}})^{-\frac{1}{2} |e(m)|} \text{PE}[\delta_{e(m),0} t^{r_e \hat{e}} x^{e \hat{x} \hat{e}}]$$

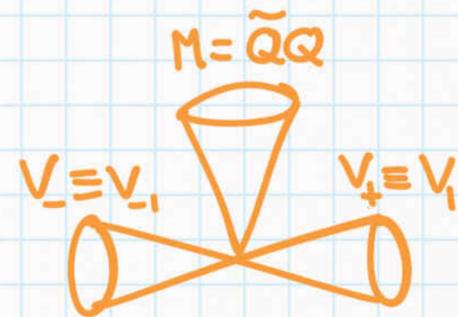
$$\Gamma_q = \left\{ m \in \Gamma_G / W \mid \sum_e \rho(\alpha_i) \text{sign} \rho(m) \neq 0 \vee \alpha_i(m) = 0 \ \forall i=1, \dots, r \right\}$$

* Examples

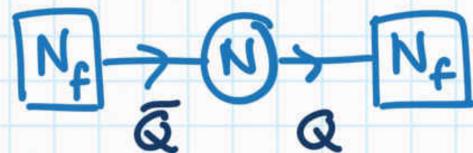


$$H(t; z, y) = \sum_{m \in \mathbb{Z}} z^m t^{(1-r)|m|} y^{-|m|} \oint \frac{dx}{2\pi i x} \text{PE}[\delta_{m,0} t^r y (x + \frac{1}{x})]$$

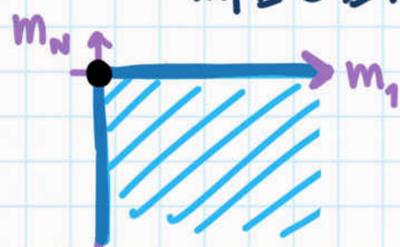
$$= \frac{1}{1-y^2 t^{2r}} + \frac{1}{1-zy^{-1}t^{1-r}} + \frac{1}{1-z^{-1}y^{-1}t^{1-r}} - 2$$



$\mathbb{C}[M, V_+, V_-] / \langle V_+ V_-, V_\pm M \rangle$
cf [Aharony et al, de Boer et al '97]



$$\Gamma_q = \left\{ m = (m_1, 0, \dots, 0, m_N) \in \mathbb{Z}^N \mid m_1 \geq 0 \geq m_N \right\}$$



- $m_1 = 0 = m_N$: $T_m = U(N)$ w/ N_f
- $m_1 > 0 = m_N$: $T_m = [U(N-1) \text{ w/ } N_f] \times U(1)$
- $m_1 > 0 > m_N$: $T_m = [U(N-2) \text{ w/ } N_f] \times U(1)^2$

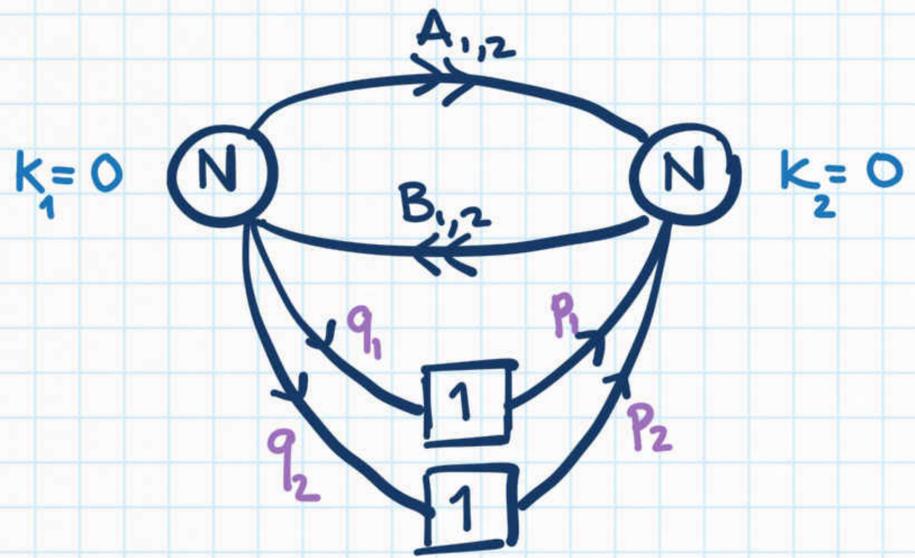


$$\mathbb{C}[M_{N_f \times N_f}, V_+, V_-]$$

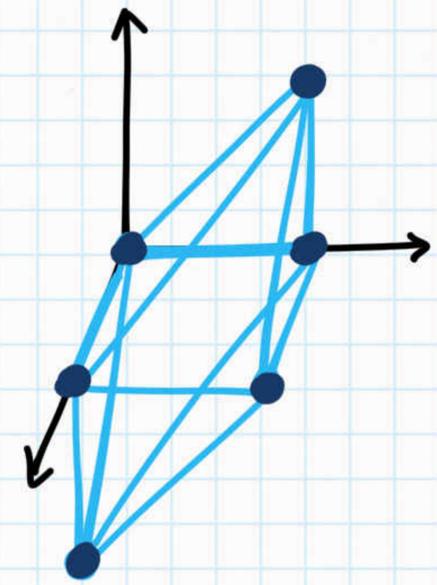
$$\left\{ \begin{array}{l} \text{minor}_{N+1}(M) = 0, \\ V_\pm \cdot \text{minor}_N(M) = 0, \\ V_+ V_- \cdot \text{minor}_{N-1}(M) = 0 \end{array} \right.$$

cf [Aharony '97]

* EXAMPLE OF CHERN-SIMONS QUIVER : M2-BRANES PROBING $C(Q^{1,1,1})$
toric CY_4



$$W = \text{Tr} (A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) + p_1 B_1 q_1 + p_2 B_2 q_2$$



[Benini, Closset, SC '09; Jafferis '09]

For $N=1$, $B = \hat{m} = 0$:
 $m_1 = m_2 = m$

$$k_1^{\text{eff}} = -k_2^{\text{eff}} = \text{sign}(m)$$

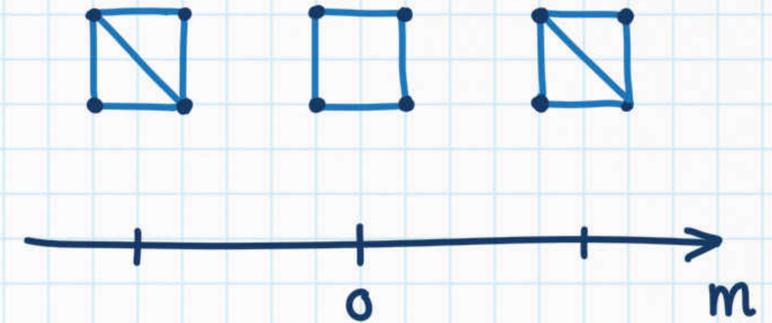
$$H(t, u, v, z) = \sum_{m \in \mathbb{Z}} z^m t^{\frac{1}{2}|m|} \oint \frac{dx}{2\pi i x} x^{-|m|} \text{PE} \left[t^{\frac{1}{2}} x \left(u + \frac{1}{u} \right) + t^{\frac{1}{2}} \frac{1}{x} \left(v + \frac{1}{v} \right) \right] =$$

$$= \sum_{m \in \mathbb{Z}} z^m t^{\frac{1}{2}|m|} \cdot g_1(t^{\frac{1}{2}}, u, v; |m|) = \sum_{n=0}^{\infty} [n; n; n]_{\alpha, \beta, \gamma} t^n$$

$SU(2)^3 \times U(1)_R$

$u = \alpha$
 $v = \gamma\beta$
 $z = \gamma/\beta$

[SC, Mekareeya, Zaffaroni] to appear



- $N > 1$: $M = \text{Sym}^N C(Q^{1,1,1})$.
- $B \neq 0, \hat{m} \neq 0$: resolutions of $C(Q^{1,1,1})$.

The HS formalism can be applied to a large class of M2-brane theories probing CY_4 cones.

OUTLOOK

I presented a general formalism to count gauge invariant chiral operators that parametrize moduli spaces of vacua of 3d $\mathcal{N} \geq 2$ gauge theories.

This simplifies the study of the chiral ring and in favorable cases determines it completely.

SOME OPEN PROBLEMS:

- Explicit chiral ring relations (like [Bullimore, Dimofte, Gaiotto '15] for $\mathcal{N}=4$)
- Mathematical definition (like [Nakajima '15; Braverman, Finkelberg, N. '16] for $\mathcal{N}=4$)
- Path integral derivation as an index (like [Razamat, Willett '14; Closset, Kim '16] for $\mathcal{N}=4$)
(if \mathcal{F} is a complete intersection)

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THANK YOU FOR YOUR ATTENTION!