Cohomological Hall algebras actions and Kac polynomials

StringMath 16, Paris

Olivier Schiffmann (based on joint work with T. Bozec, F. Sala, E. Vasserot)

28 Juin 2016

Université de Paris-Sud Orsay, CNRS

I. Moduli spaces and correspondences (heuristics)

We work over the field $k = \mathbb{C}$.

Let $\mathcal{M}, \mathcal{M}'$ be smooth algebraic varieties, and $Z' \subset \mathcal{M} \times \mathcal{M}'$ a closed subvariety. Assume that the projection $Z \to \mathcal{M}'$ is proper. Using the maps

$$\mathcal{M} \xleftarrow{p_1} \mathcal{M} \times \mathcal{M}' \xrightarrow{p_2} \mathcal{M}'$$

we can define a convolution operator

$$\begin{aligned} H_*(\mathcal{M}, \mathbb{Q}) \otimes H_*(Z) &\to H_*(\mathcal{M}', \mathbb{Q}), \\ c \otimes \alpha &\mapsto p_{2*}(p_1^*(c) \cap \alpha). \end{aligned}$$

One class of potential examples :

$\mathcal{M} = \mathcal{M}'$ is the moduli space of (stable) objects (with extra structure) in some category $\mathcal{A}$ and

$$Z = \{(O, O') \in \mathcal{M} \times \mathcal{M}' \mid O \subseteq O'\}$$

Let $\mathcal{X}$ be the moduli stack of objects in $\mathcal{A}$. There is a map

$$\pi : Z \to \mathcal{X}, \qquad (O, O') \mapsto O'/O$$

which yields a pullback morphism $\pi^* : H_*(\mathcal{X}, \mathbb{Q}) \to H_*(Z, \mathbb{Q})$ and thus a map

$$H_*(\mathcal{M}, \mathbb{Q}) \otimes H_*(\mathcal{X}, \mathbb{Q}) \to H_*(\mathcal{M}, \mathbb{Q}). \tag{1}$$

I. Cohomological Hall algebras (still heuristics)

The stack $\mathcal{X}$ carries its own Hecke correspondences

$$\mathcal{X} \times \mathcal{X} \xleftarrow{\;q\;} \mathcal{H}ecke \xrightarrow{\;p\;} \mathcal{X}$$

where

$$\mathcal{H}ecke = \{(O, O') \mid O \subseteq O'\}$$

but now

$$q(O, O') = (O'/O\,,\; O), \qquad p(O, O') = O'.$$

The COHA associated to $\mathcal{A}$ is the induced convolution algebra structure on $H_*(\mathcal{X}, \mathbb{Q})$:

$$H_*(\mathcal{X}, \mathbb{Q}) \otimes H_*(\mathcal{X}, \mathbb{Q}) \to H_*(\mathcal{X}, \mathbb{Q}), \qquad c \star c' = p_* q^*(c \boxtimes c').$$

In nice cases, (1) defines an action of $H_*(\mathcal{X}, \mathbb{Q})$ on $H_*(\mathcal{M}, \mathbb{Q})$.

The stack $\mathcal{X}$ carries its own Hecke correspondences

$$\mathcal{X} \times \mathcal{X} \xleftarrow{q} \mathcal{H}ecke \xrightarrow{p} \mathcal{X}$$

where

$$\mathcal{H}ecke = \{(O, O') \mid O \subseteq O'\}$$

but now

$$q(O, O') = (O'/O\,,\ O), \qquad p(O, O') = O'.$$

The COHA associated to $\mathcal{A}$ is the induced convolution algebra structure on $H_*(\mathcal{X}, \mathbb{Q})$:

$$H_*(\mathcal{X}, \mathbb{Q}) \otimes H_*(\mathcal{X}, \mathbb{Q}) \to H_*(\mathcal{X}, \mathbb{Q}), \qquad c \star c' = p_* q^* (c \boxtimes c').$$

In nice cases, (1) defines an action of $H_*(\mathcal{X}, \mathbb{Q})$ on $H_*(\mathcal{M}, \mathbb{Q})$.

The sexy slogan : "The COHA of the moduli *stack* of objects in $\mathcal{A}$ acts on the cohomology of the moduli *space* of *stable* objects in $\mathcal{A}$"

I. Cohomological Hall algebras (still heuristics)

The stack $\mathcal{X}$ carries its own Hecke correspondences

$$\mathcal{X} \times \mathcal{X} \xleftarrow{q} \mathcal{H}ecke \xrightarrow{p} \mathcal{X}$$

where

$$\mathcal{H}ecke = \{(O, O') \mid O \subseteq O'\}$$

but now

$$q(O, O') = (O'/O\,,\ O), \qquad p(O, O') = O'.$$

The COHA associated to $\mathcal{A}$ is the induced convolution algebra structure on $H_*(\mathcal{X}, \mathbb{Q})$:

$$H_*(\mathcal{X}, \mathbb{Q}) \otimes H_*(\mathcal{X}, \mathbb{Q}) \to H_*(\mathcal{X}, \mathbb{Q}), \qquad c \star c' = p_* q^*(c \boxtimes c').$$

In nice cases, (1) defines an action of $H_*(\mathcal{X}, \mathbb{Q})$ on $H_*(\mathcal{M}, \mathbb{Q})$.

The sexy slogan : "The COHA of the moduli *stack* of objects in $\mathcal{A}$ acts on the cohomology of the moduli *space* of *stable* objects in $\mathcal{A}$"

The attempt to wake up physicists in the audience : "The algebra of BPS states"

I. Something concrete at last

One example (Alday-Gaiotto-Tachikawa correspondence, SUSY, N=2, $U(r)$ gauge theory, pure, without matter)

Here, we have

$$\mathcal{A} = Coh(\mathbb{C}^2),$$

$$\mathcal{M} = \left\{ (\mathcal{E}, \phi) \mid {}^{\mathcal{E} \text{ torsion-free coherent sheaf on } \mathbb{P}^2}_{\phi \,:\, \mathcal{E}_{|\mathbb{P}^1_\infty} \xrightarrow{\sim} \mathcal{O}^{\oplus r}_{\mathbb{P}^1_\infty}} \right\} = \bigsqcup_n M(r, n)$$

is the moduli space of rank r instantons on $\mathbb{P}^2$ and

$$\mathcal{X} = \underline{Coh}(\mathbb{C}^2)_0 = \bigsqcup_d [\mathcal{C}_d / GL_d]$$

where $\mathcal{C}_d = \{(x, y) \in gl(d) \mid [x, y] = 0\}$, is the moduli stack of zero-dimensional sheaves on $\mathbb{C}^2$.

I. Something concrete at last

Let $T = (\mathbb{C}^*)^2$ act on $\mathbb{C}^2$, hence also on $\mathcal{M}$ and $\mathcal{X}$. Put

$$C = (\bigoplus_d H_*^{T\times GL_d}(\mathcal{C}_d, \mathbb{Q}), \star)$$

Theorem (S-Vasserot)

i) The COHA C is isomorphic to $Y_h^+(\widehat{gl}(1))$, the (positive half of the) Yangian of $\widehat{gl}(1)$
ii) C acts faithfully and cyclically on $\bigoplus_n H^{T\times GL_r}(M(r,n), \mathbb{Q})$.

<u>Remarks:</u>
i) By construction, the *same* COHA C acts for all values of r,
ii) Some (Lagrangian) variants : $\mathcal{X} = \underline{Coh}_L(\mathbb{C}^2)_0$, zero-dimensional sheaves supported on a curve $L \subset \mathbb{C}^2$,
iii) Works equally well with K-theory, Chow groups, ... (see Yang, Zhao)
iv) Maulik-Okounkov constructed an action of the same Yangian by a different approach (stable enveloppes, r-matrix)

II. Cohomological Hall algebras and Nakajima quiver varieties

Let $Q = (I, \Omega)$ be any (finite) quiver. For instance,

$$Q = S_g =$$

To $(v, w) \in \mathbb{N}^I \times \mathbb{N}^I$ is associated a triple $(\mathcal{M}(v, w), \mathcal{M}_0(v, w), \pi)$, where

- $\mathcal{M}(v, w)$ is a smooth quasi-projective symplectic variety
- $\mathcal{M}_0(v, w)$ is an affine (singular) Poisson variety
- $\pi \ : \mathcal{M}(v, w) \to \mathcal{M}_0(v, w)$ is a projective map (semismall)

Heuristically, $\mathcal{M}(v, w)$ may be thought of as $T^*\mathcal{N}(v, w)$, where $\mathcal{N}(v, w)$ is a moduli space of *framed* v-dimensional representations of Q

$$\mathcal{N}(v, w) = \{(M, \phi) \mid M \in Rep_v Q, \phi : P^w \to M\}$$

for some projective representation P^w of Q.

Example : if $Q = S_1$ then $\mathcal{M}(v, w) = M(w, v)$, moduli space of rank w instantons on $\mathbb{P}^2$ with $c_2 = v$.

The relevant COHA is constructed as follows. Let $\overline{Q} = (I, \overline{\Omega})$ be the doubled quiver, where $\overline{\Omega} = \Omega \sqcup \Omega^*$. Let $\mathbb{C}\overline{Q} = \mathbb{C}\langle x_\gamma, x_{\gamma^*}\rangle_{\gamma \in \Omega}$ be the path algebra of $\overline{Q}$, and let

$$\pi_Q = \mathbb{C}\overline{Q} \ / \ \sum_\gamma [x_\gamma, x_\gamma^*]$$

be the *preprojective algebra* of Q.

Set $T = \{(z_\gamma, z_{\gamma^*}) \in (\mathbb{C}^*)^{\overline{\Omega}} \mid \forall\, \gamma, \sigma \in \Omega,\ z_\gamma z_{\gamma^*} = z_\sigma z_{\sigma^*}\}$

Example : for $Q = S_g$, $\pi_Q = \mathbb{C}\langle x_1, y_1, \ldots, x_g, y_g\rangle / \sum_i [x_i, y_i] = 0$ and $T \simeq (\mathbb{C}^*)^{g+1}$.

Let $Rep_v \pi_Q$ be the stack of v-dimensional representations of π_Q.

Theorem (SV)

i) There is an algebra structure on $C := \bigoplus_v H_*^T(Rep_v \pi_Q, \mathbb{Q})$,
ii) C acts on $\bigoplus_v H_*^{T \times G_w}(\mathcal{M}(v, w), \mathbb{Q})$.

II. Some more pleasant (Lagrangian) correspondences

Definition: a representation $(x_\gamma, x_{\gamma^*})_\gamma \in Rep_v \pi_Q$ is *semi-nilpotent* if there exists a (graded, partial) flag of subrepresentations

$$W_1 \subset W_2 \cdots \subset W_s = \mathbb{C}^v$$

such that for all $\gamma \in \Omega$ and all i

$$x_\gamma(W_i) \subset W_{i-1}, \qquad x_{\gamma^*}(W_i) \subset W_i.$$

Denote by $\Lambda_v \subset Rep_v \pi_Q$ the closed substack of semi-nilpotent representations.

Theorem (Bozec)
For any v, Λ_v is a Lagrangian substack of $Rep_v \pi_Q$.

A subtle point : in the presence of oriented cycles in Q, we get different notions of semi-nilpotency depending upon whether we require W_i/W_{i-1} to be supported at a single vertex in Q or not. Hereafter we consider the notion in which W_i/W_{i-1} is required to be supported at a single vertex.

II. Examples of Lagrangians correspondences

For $Q = S_1$, we have

$$\Lambda_v = \bigsqcup_{\lambda} T^* O_\lambda$$

where λ runs among all partitions of v, and $O_\lambda \subset gl_v$ is the nilpotent orbit of type λ;

For $Q = S_g, g > 1$ we have

$$Irr(\Lambda_v) \simeq \{\nu, \text{composition of } v\}$$

and for instance

$$\Lambda_{(v)} = [\{0\}^g \times (gl_v)^g / GL_v] \subset Rep_v \pi_Q.$$

Set

$$C_\Lambda := \bigoplus_v H_*^T(\Lambda_v, \mathbb{Q}).$$

As before, there is an algebra structure on C_Λ and it acts on $\bigoplus_v H_*^{T\times G_w}(\mathcal{M}(v,w),\mathbb{Q})$.

Let I^{re} be the set of vertices without edge loops (real vertices) and $I^{im} = I \backslash I^{re}$ (imaginary vertices).

Theorem (SV)

i) For any v, $H_*^T(\Lambda_v, \mathbb{Q})$ is equivariantly pure, even and torsion free over $H^*_{T\times G_v}$,
ii) C_Λ acts faithfully on $\bigoplus_{v,w} H_*^{T\times G_w}(\mathcal{M}(v,w),\mathbb{Q})$,
iii) C_Λ is generated by the fundamental classes $[\Lambda_{\epsilon_i}]$ for $i \in I^{re}$ and $[\Lambda_{(l\epsilon_i)}]$ for $i \in I^{im}$ and $l \geq 1$.

Corollary (of i))

C_Λ admits an explicit combinatorial realization as a shuffle algebra inside $\bigoplus_v H_*^{T\times G_v}(pt)$.

III. Character formula for the Lagrangian COHA

It turns out that one can explicitly compute the characters of C_Λ and C for any quiver. The answer is given in terms of (variants of) *Kac A polynomials*.

For any finite field $\mathbb{F}_q$ let $\mathcal{A}_v(\mathbb{F}_q)$ be the number of isomorphisms classes of absolutely indecomposable representations of Q of dimension v over $\mathbb{F}_q$.

Let $\mathfrak{g}_Q$ be the Borcherds-Kac-Moody algebra associated to the underlying graph of Q, and let $\{\alpha_i\}_{i\in I}$ denote its set of simple roots, and $\Delta_\mathfrak{g}$ its root system.

Theorem (Kac, 81)

There exists a unique polynomial $A_v(t) \in \mathbb{Z}[t]$ such that $\mathcal{A}_v(\mathbb{F}_q) = A_v(q)$ for any finite field $\mathbb{F}_q$. Moreover, $A_v(t) \neq 0$ if and only if $\sum_i v_i\alpha_i \in \Delta^+_\mathfrak{g}$.

In fact, as recently shown by Hausel, Letellier and Rodriguez-Villegas, $A_v(t) \in \mathbb{N}[t]$.

III. Examples of Kac polynomials

Examples :

- If Q is of finite Dynkin type then $A_d(t) = 1$ if $d \in \Delta_{\mathfrak{g}}^{+}$ and $A_d(t) = 0$ otherwise,
- If Q is of affine Dynkin type then $A_d(t) = 1$ if $d \in \Delta_{\mathfrak{g}}^{+,re}$, $A_d(t) = t + rank(\mathfrak{g})$ if $d \in \Delta_{\mathfrak{g}}^{+,im}$ and $A_d(t) = 0$ otherwise,
- If $Q = S_1$ then $A_v(t) = q$ for all v,
- If $Q = S_g$ then $A_3(t) = \frac{t^{9g-3} - t^{5g+1} - t^{5g} - t^{5g-1} + t^{3g-1} + t^{3g-2}}{(t^2-1)(t^3-1)}$.

III. Hua's formula

Hua gave an explicit formula for $A_v(t)$ for all quiver. Let $\nu = (\nu^i\,;\, i \in I)$ be an I-partition, i.e. an I-tuple of partitions. Consider the I-tuples of integers $|\nu|$ and ν_k in $\mathbb{N}^I$ given, for each $k = 1, 2, \ldots,$ by

$$|\nu| = (|\nu^i|), \quad \nu_k = (\nu^i_k).$$

Let $X(\nu, t) \in \mathbb{Q}(t)$ be given by

$$X(\nu, t) = \prod_k t^{\langle \nu_k, \nu_k \rangle} \, [\infty, \nu_k - \nu_{k+1}]_t,$$

where $\langle\, , \,\rangle$ is the Cartan bilinear. We define a power series $r(w, t, z)$ depending on $w \in \mathbb{Z}^I$ as

$$r(w, t, z) = \sum_\nu X(\nu, t^{-1}) \, t^{w \cdot \nu_1} \, z^{|\nu|}. \tag{2}$$

Then Hua's formula reads

$$\mathrm{Exp}\Big(\frac{1}{1-t} \sum_v A_v(t) \, z^v \Big) = \frac{1}{r(0, t, z)}, \tag{3}$$

where Exp is the *plethystic exponential*.

III. Variations on Kac polynomials

Definition: A representation $(x_\gamma)_\gamma$ of Q is *edge-nilpotent* if for any imaginary vertex $i \in I^{im}$, the maps x_γ, for γ a loop at i, are (simultaneously) nilpotent, i.e. if they generate a nilpotent algebra.

For any finite field $\mathbb{F}_q$ let $\mathcal{A}_v^{nilp}(\mathbb{F}_q)$ be the number of isomorphisms classes of absolutely indecomposable edge-nilpotent representations of Q of dimension v over $\mathbb{F}_q$.

Theorem (Bozec-S-V)

There exists a unique polynomial $A_v^{nilp}(t) \in \mathbb{Z}[t]$ such that $\mathcal{A}_v^{nilp}(\mathbb{F}_q) = A_v^{nilp}(q)$ for any finite field $\mathbb{F}_q$. Moreover, $A_v^{nilp}(1) = A_v(1)$ for any v.

Examples :

- . If Q has no edge loop then $A_d(t) = A_d^{nilp}(t)$ for all d,
- . If $Q = S_1$ then $A_v^{nilp}(t) = 1$ for all v,
- . If $Q = S_g$ then $A_3^{nilp}(t) = t^{2(g-1)} + \frac{t^{2(g-1)}-1}{t^2-1} \cdot \left(\frac{t^{g+1}-1}{t-1} + \frac{t^g-1}{t-1}\right)$

Remarks:

i) There exists a nilpotent analogue of Hua's formula.

ii) one can play the exact same game, with the (standard) condition of nilpotency, and define another nilpotent Kac polynomial, which will agree with $A_v^{nilp}(t)$ when Q has no *oriented cycle*.

III. Character formula for the Lagrangian COHA

Set $P(\Lambda_v, t) = \sum_i (-1)^i dim\, H_i(\Lambda_v, \mathbb{Q}) t^{i/2} = \sum_i dim\, H_{2i}(\Lambda_v, \mathbb{Q}) t^i$.

Theorem (BSV)

For any quiver Q, the Betti numbers of Λ_v are given by the following relation

$$\sum_v P(\Lambda_v, t) t^{\langle v,v \rangle} z^v = Exp\Big(\frac{1}{1 - t^{-1}} \sum_v A_v^{nilp}(t^{-1})\, z^v \Big) \quad (4)$$

Remarks:

i) The proof uses reduction to finite fields and Weil conjectures, but more importantly it hinges on a compactification of Λ_v constructed in terms of Nakajima quiver varieties together with Hausel's formula for the Betti numbers of the latter.

ii) A similar result holds for the other semi-nilpotency condition (and the other nilpotent Kac polynomial).

III. Relation to Kac's conjecture / Okounkov's conjecture

Taking the leading term of (4) we get

$$\sum_v dim\ H_{top}(\Lambda_v, \mathbb{Q}) z^v = \mathrm{Exp}\Big(\sum_v A_v^{nilp}(0)\, z^v \Big). \tag{5}$$

It is known (Bozec) that $\bigoplus_v H_{top}(\Lambda_v, \mathbb{Q})$ is isomorphic (as a graded vector space) to $U^+(\mathfrak{g}'_Q)$ where $\mathfrak{g}'_Q$ is the Bozec-Kac-Moody algebra associated to the underlying graph of Q. We deduce that

$$dim\ \mathfrak{g}'_Q[v] = A_v^{nilp}(0) \qquad \forall\ v \in \Delta^+$$

(an extension of Kac's conjecture (R.I.P, 2006, Hausel) to arbitrary quivers).

III. Relation to Kac's conjecture / Okounkov's conjecture

Assuming that C_Λ is a deformation of the envelopping algebra of a Lie algebra, the full of (4) shows that C_Λ is isomorphic to the Yangian of (i.e. a deformation of the envelopping algebra of the current algebra of) a *graded* Lie algebra $\overline{\mathfrak{g}}_Q$ satisfying

$$grdim\, \overline{\mathfrak{g}}_Q[v] = A_v^{nilp}(t^{-1}).$$

<u>Remark:</u> This may be viewed as a COHA variant of a conjecture of Okounkov concerning the Maulik-Okounkov Yangian.

III. Relation to cuspidal functions for quivers

Fix $k = \mathbb{F}_q$, and set

$$\mathcal{X} = Rep\ Q = \bigsqcup_v Rep_v Q$$

the moduli space of representations of the (non-doubled) quiver Q. Consider the (usual) Hall algebra $H_{\mathbb{F}_q} = (\bigoplus_v Fun(Rep_v Q, \mathbb{C}), *)$.

Proposition (trivial)

We have

$$\bigoplus_v dim\ H_{\mathbb{F}_q}[v] z^v = Exp_{q,z}\Big(\sum_v A_v(q) z^v \Big), \qquad (6)$$

Theorem (Sevenhant- Van den Bergh)

$H_{\mathbb{F}_q}$ is isomorphic to $U_q^+(\widetilde{\mathfrak{g}}_{Q,\mathbb{F}_q})$ for a suitable Borcherds-Kac-Moody algebra $\widetilde{\mathfrak{g}}_{Q,\mathbb{F}_q}$.

By construction, the Borcherds-Cartan datum is given by

$$(\widetilde{I}, \underline{m}) = \{(v, \mathcal{H}_v^{cusp}(\mathbb{F}_q) \mid v \in \mathbb{N}^I\}$$

where

$$\mathcal{H}_v^{cusp}(\mathbb{F}_q) = dim\ H_{\mathbb{F}_q}^{cusp}[v]$$

is the space of cuspidal (i.e. quasi-primitive) elements.

Proposition (BS)

For any v there exists a unique $H_v(t) \in \mathbb{Q}[t]$ such that $\mathcal{H}_v^{cusp}(\mathbb{F}_q) = H_v^{cusp}(q)$ for any finite field $\mathbb{F}_q$.

Comparing (6) to

$$\sum_v grdim\ U(\bar{\mathfrak{g}}_Q)[v]z^v = \mathrm{Exp}\Big(\sum_v A_v^{nilp}(t^{-1})z^v\Big) \tag{7}$$

suggests that $\bar{\mathfrak{g}}_Q$ is (close to) a (graded) BKM algebra whose (graded) Borcherds-Cartan datum may be deduced from the collection of polynomials $H_v^{cusp}(t)$.

Example : when Q is affine or S_1, define polynomials $K_v(t) \in \mathbb{N}[t]$ by

$$\sum_v K_v(t)z^v = \mathrm{Exp}_z \mathrm{Log}_{z,t}\Big(\sum_v H_v^{cusp}(t)z^v\Big)$$

Then $\bar{\mathfrak{g}}_Q$ is a (graded) BKM algebra with data

$$(\widetilde{I}, \underline{m}) = \{(v, K_v(t) \mid v \in \mathbb{N}^I\}.$$

There exists a fruitful analogy between representations of quivers and coherent sheaves on curves, in which we replace *Rep Q* by *Coh*(X), for X a smooth projective curve, and

$$(\Lambda_v)_{v\in\mathbb{N}^I} \qquad \text{by} \qquad (\mathcal{N}_{r,d})_{(r,d)\in\mathbb{N}\times\mathbb{Z}}$$

where $\mathcal{N}_{r,d} = \mu_{r,d}^{-1}(0)$, is the moduli *stack* of nilpotent Higgs sheaves and

$$\mu_{r,d} \; : Higgs_{r,d}(X) \to \mathbb{A}_r$$

is the Hitchin map.

Theorem (Sala-S, Minets for $r = 0$)

There is an algebra structure on $C_X = \bigoplus_{r,d} H_*^{\mathbb{C}^*}(\mathcal{N}_{r,d}, \mathbb{Q})$, *and on* $K_X = \bigoplus_{r,d} K^{\mathbb{C}^*}(\mathcal{N}_{r,d})$

When $X = \mathbb{P}^1$ there is an extra $\mathbb{C}^*$ action (coming from the action on X).

Conjecture (SS)

We have $K_{\mathbb{P}^1} \simeq U^+_{q,t}(\widehat{\widehat{\mathfrak{gl}_2}})$.

One can make similar conjectures for orbifold $\mathbb{P}^1_{a,b}$, $a, b \geq 2$.

IV. Actions on moduli spaces of sheaves on surfaces

According to the general slogan, one expects C_X or K_X to act on cohomology (or K-theory) of suitable moduli spaces of sheaves on surfaces.

Example : Let $k \geq 2$ and let $\pi : X_k \to \mathbb{C}^2/\mathbb{Z}_k$ be the resolution of singularities of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_k$. Let $M_w = \bigsqcup_v M_{v,w}$ be the moduli space of w-framed torsion free sheaves on $\overline{X_k}$ (a suitable compactification on X_k)

Theorem (SS)

For any irreducible component $C_l \subset \pi^{-1}(0)$, $l = 1, \ldots k-1$ there is an action of $K_{\mathbb{P}^1}$ on $K^T(M_w)$.

Conjecture (SS)

These glue together to give an action of $U^+_{q,t}(\widehat{\widehat{\mathfrak{gl}_k}})$ on $K^T(M_w)$.

Note that M_w is a Nakajima quiver variety, with a *non-dominant* stability parameter.

More generally, one can hope to use the same technique to construct actions of K_X on (suitable) moduli space of sheaves on a surface S whenever $X \subset S$ satisfies $X \cdot X = deg(\Omega_X)$.

Theorem (S)

For any g, r, d there exists a unique polynomial $A_{g,r,d}$ such that, for any $\mathbb{F}_q$ and any smooth projective curve X defined over $\mathbb{F}_q$

$$A_{g,r,d}(\sigma_X) = \#\{\mathcal{F} \in Coh_{r,d}(X),\ \mathcal{F} \text{ abs. indecomp.}\}.$$

Here σ_X is the set of Weil numbers of X. Moreover, if $gcd(r, d) = 1$ then

$$P(\Lambda^{st}_{r,d}, \mathbb{Q}) = \overline{A}_{g,r,d}(z)$$

where $\overline{A}_{g,r,d}(z) = t^{2(1+(g-1)r^2}A_{g,r,d}(z^{-1})$ is the Poincaré dual of $A_{g,r,d}$.

This strongly suggests that C_X is the Yangian of a certain $\mathbb{Z}^{g+1}$-graded Lie algebra $\mathfrak{g}_X$ satisfying

$$grdim\ \mathfrak{g}_X[r, d] = A_{g,r,d}(z^{-1}), \qquad (r, d).$$

For instance, if $g = 1$ we should get $\mathfrak{g}_X = \widehat{\widehat{\mathfrak{gl}(1)}}$ and hence C_X is a *triple* loop algebra !