Plane partitions and vertex algebras

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based on joint paper with B. Feigin, G. Merzon,
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  For $n=2$ $\mathrm{W}_2 \simeq H \oplus \mathrm{Vir}$.

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_n, L_m] = 0$$
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n).\delta_{n+m,0}$$

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- Verma modules $M_{\vec{a}}$ are parametrized by n numbers $a_1, \ldots, a_n$. For generic $a_1, \ldots, a_n$ the character equals

$$\chi(V) = \frac{q^{\Delta}}{\prod_{k=1}(1-q^k)^n}$$

In other words M is Fock module over $H^{\oplus n}$ and has a basis labeled by n-tuple of partitions.

AGT relation (Geometric representation theory)

- There exist an action of the algebra W_n on the direct sum of equivariant cohomology spaces of the moduli space of $U(n)$ instantons on $\mathbb{C}^2$, $M(\vec{a}) \simeq \oplus_{N=0}^{\infty} H_{\mathrm{equiv}}(\mathcal{M}(n,N))$.
  [Schiffmann, Vasserot; Maulik, Okounkov; Braverman, Finkelberg, Nakajima]
  Highest weight of M is given in terms of equivariant parameters a of the framing (or vev in gauge theory). Central charge is given in terms of ϵ_1, ϵ_2 equivariant parameters on $\mathbb{C}^2$ (Ω deformation parameters).

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- Torus fixed points in $\bigsqcup_{N=0}^{\infty} \mathcal{M}(n, N)$ label special basis $J_{\vec{\alpha}} \in M(\vec{a})$, where $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ is n tuple of partitions.

$$\mathrm{W}_n\text{-Conformal block} = \sum_{\vec{\alpha}} Z_{\mathrm{Nek}}(\vec{\alpha}, \vec{a}, \vec{\mu}, \vec{\epsilon}).$$

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$$\mathrm{W}_n\text{-Conformal block} = \sum\nolimits_{\vec{\alpha}} Z_{\mathrm{Nek}}(\vec{\alpha}, \vec{a}, \vec{\mu}, \vec{\epsilon}).$$

- For special (resonance) parameters ϵ, a, μ functions $Z_{\mathrm{Nek}}(\vec{\alpha}, \vec{a}, \vec{\mu}, \vec{\epsilon})$ become ill defined. Like x/y where $x, y \to 0$. Corresponding representations become reducible. Irreducible quotient has a basis labeled by n-tuples of partitions with an additional condition.

Plane partitions

In the resonance $a^{(l)} - a^{(l+1)} = (r_l + 1)\epsilon_1 + (s_l + 1)\epsilon_2$, for $1 \leq l \leq n-1$ restriction on α looks

$$\{\alpha^{(1)}, \ldots, \alpha^{(n)} | \alpha_i^{(l)} + r_l \geq \alpha_{i+s_l}^{(l+1)}, 1 \leq l \leq n-1\}$$

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For such $\vec{\alpha}$ we can assign plane partition in $[1, n] \times [1, \infty) - \lambda$

						$\alpha_1^{(1)}+\nu_1$	$\cdots$
				$\alpha_1^{(2)}+\nu_2$	$\cdots$	$\alpha_{s_1+1}^{(2)}+\nu_2$	$\cdots$
		$\alpha_1^{(3)}+\nu_3$	$\cdots$	$\alpha_{s_2+1}^{(3)}+\nu_3$	$\cdots$	$\alpha_{s_1+s_2+1}^{(3)}+\nu_3$	$\cdots$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\alpha_1^{(n)}$	$\cdots$						

$$\nu_i = \sum_{l=i}^{n-1} r_l, \qquad \lambda_i = \sum_{l=i}^{n-1} s_l,$$

Plane partitions with "pit"

We study plane partitions satisfying

1. $\lim_{j\to\infty} a_{i,j} = \nu_i$, **2.** $\lim_{i\to\infty} a_{i,j} = \mu_i$, **3.** $a_{i,j} = \infty$, iff $(i,j) \in \lambda$,

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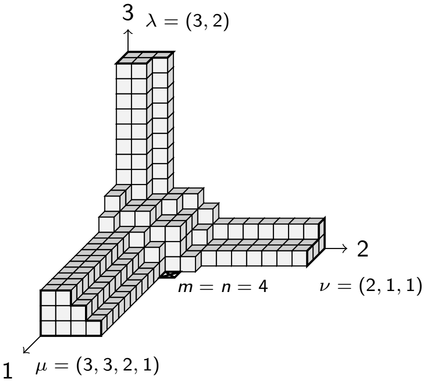

∞	∞	∞	4	4	3	2	2	2	$\cdots$	2
∞	∞	4	4	3	2	1	1	1	$\cdots$	1
5	4	4	3	3	1	1	1	1	$\cdots$	1
5	4	4	3	3	1	0	0	0	$\cdots$	0
4	3	3	2	⊠						
3	3	2	1							
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$\vdots$	$\vdots$	$\vdots$	$\vdots$							
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The corresponding W algebras we denote by $\mathrm{W}_{n|m}$.

Generating functions

For any set $\mathcal{A}$ of plane partitions define its generating function by $\sum_{a\in\mathcal{A}} q^{|a|}$.
Denote by $\chi^{n,m}_{\lambda,\nu,\mu}(q)$ the generating function of plane partitions which satisfy **1,2,3,4**.

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- [Okounkov Vafa Reshetikhin 2003]

$$\chi_{\lambda,\nu,\mu}(q) = s_{\nu'}(q^{-\rho}) \sum_{\eta} s_{\lambda'/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu'-\rho}) \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^k}$$

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- [MacMahon 1915]

$$\chi^{n,0}_{\varnothing,\varnothing,\varnothing}(q) = \frac{1}{\prod_{k=1}^{n}(1-q^k)^k \prod_{k=n+1}^{\infty}(1-q^k)^n},$$

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Generalization

$$\chi^{n,0}_{\varnothing,\nu,\lambda}(q) = q^{\sum_{i=1}^{n}(\lambda_i+n-i)(\nu_i+n-i)} a_{\nu+\rho_n}(q^{-\lambda-\rho_n}) \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^n},$$

where $a_{\lambda+\rho_n}(x_1,\ldots,x_n) = \det\left(x_i^{\lambda_j+n-j}\right)$.

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$\chi^{n,0}_{\varnothing,\nu,\lambda}(q)$ are characters of W_n algebras.

Answer 1

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- Due to "pit" condition
  $\nu = (\nu_1, \ldots, \nu_n) \longleftrightarrow$ finite dimension representation of $\mathfrak{gl}_n$
  $\mu = (\mu_1, \ldots, \mu_m) \longleftrightarrow$ finite dimension representation of $\mathfrak{gl}_m$
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- Denote $N_i = \nu_i + n - i$, $M_i = \mu_i + n - i$, $A_i = \alpha_i + n - i$. Then

$$\chi^{n,n}_{\varnothing,\nu,\mu}(q) = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)^{2n}} \det\Big(\sum\nolimits_{A\geq 0}(-1)^a q^{\binom{A}{2}} q^{(N_i-M_j)A}\Big) =$$

$$= \sum\nolimits_{\alpha}(-1)^{\sum A_i} q^{\sum \binom{A_i+1}{2}} \frac{a_N(q^A)a_M(q^{-A})}{\prod_{k=1}^{\infty}(1-q^k)^{2n}},$$

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- Formulas for $\mathfrak{gl}_{m|n}$ characters

$$s_{\varnothing}(x|y)=1=\frac{\prod(1+y_j/x_i)}{\prod(x_i-x_j)\prod(y_i-y_j)}\det\Big(\sum\nolimits_{A\geq 0}(-1)^A x_j^{-A-1+m}y_i^A\Big)$$
$$=\sum\nolimits_{\alpha}(-1)^{\sum\alpha_i}s_{-\alpha}(x)s_{\alpha}(y)\prod(1+y_j/x_i)\,.$$

[Moens, van der Jeugt 2003], [Cheng Kwon Lam, 2008],

Answer 2

Denote $r = \min\{t|\lambda_{n-t} \geq m-t\}$, $P_i = \lambda_i + (n-m) - i$, $Q_j = \lambda'_j + (m-n) - j$.

$$\chi^{n,m}_{\mu,\nu,\lambda}(q) = \frac{(-1)^{mn-r}q^{\Delta}}{\prod_{k=1}^{\infty}(1-q^k)^{m+n}} \det \begin{pmatrix} \left(\sum_{a\geq 0}(-1)^a q^{\binom{a+1}{2}} q^{(N_j-M_i)a}\right)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} & \left(q^{-M_iQ_j}\right)_{\substack{1\leq i\leq m\\ 1\leq j\leq m-r}} \\ \left(q^{-N_j(P_i+1)}\right)_{\substack{1\leq i\leq n-r\\ 1\leq j\leq n}} & 0 \end{pmatrix}$$

$$= (-1)^{r(m+n)}q^{\Delta}\sum_{\alpha}(-1)^{\sum A_i}q^{\sum\binom{A_i+1}{2}}\frac{a_N(q^A, q^{-P-1})a_M(q^{-A}, q^{-Q})}{\prod_{k=1}^{\infty}(1-q^k)^{m+n}},$$

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$$= \sum\nolimits_{\alpha}(-1)^{\sum\alpha_i}s_{\pi+m-r,-\alpha}(x)s_{\alpha,\kappa}(y)\prod\left(1+y_j/x_i\right).$$

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Conjecturally there exist an equivalence of Drinfeld–Kohno or Kazhdan–Lusztig type between $\mathrm{W}_{n|m}$ and product of quantum groups $U_q\mathfrak{gl}_{n|m}\otimes U_{q'}\mathfrak{gl}_n\otimes U_{q''}\mathfrak{gl}_m$ for certain q, q', q''.

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$$g(z, w) = \frac{(z - q_1 w)(z - q_2 w)(z - q_3 w)}{(z - q_1^{-1} w)(z - q_2^{-1} w)(z - q_3^{-1} w)}.$$

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- The image of $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ in the $\mathrm{End}\left(\mathcal{F}_{u_1}^{(1)} \otimes \cdots \otimes \mathcal{F}_{u_n}^{(1)}\right)$ is q-deformed W_n algebra. [Feigin Hoshino Shibahara Shiraishi Yanagida 2010]

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- The proof is based on the fact that image $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ in the $\mathrm{End}\left(\mathcal{F}_{u_1}^{(1)} \otimes \mathcal{F}_{u_2}^{(1)}\right)$ commutes with two screening operators
$$S_+^{11} = \oint S_+^{11}(z)\mathrm{d}z, \quad S_-^{11} = \oint S_-^{11}(z)\mathrm{d}z.$$

For the case $\mathcal{F}_{u_1}^{(1)} \otimes \cdots \otimes \mathcal{F}_{u_n}^{(1)}$ we have two commuting systems of operators which corresponds corresponding to quantum group $\mathfrak{gl}_n$
$$\mathfrak{S}_1 = \left\{ \left(S_-^{11}\right)_{i,i+1} | 0 < i < n \right\}, \quad \mathfrak{S}_2 = \left\{ \left(S_+^{11}\right)_{i,i+1} | 0 < i < n \right\}.$$

q-deformed W algebras 2

- q-deformed $\mathrm{W}_{n|m}$ algebra is the image of $U_{\vec{q}}(\ddot{\mathfrak{gl}}_1)$ in the $\mathrm{End}\left(\mathcal{F}_{u_1}^{(1)} \otimes \cdots \otimes \mathcal{F}_{u_n}^{(1)} \otimes \mathcal{F}_{u_{n+1}}^{(2)} \otimes \cdots \otimes \mathcal{F}_{u_{n+m}}^{(2)}\right)$.
  For the tensor product $\mathrm{End}\left(\mathcal{F}_{u_1}^{(1)} \otimes \mathcal{F}_{u_2}^{(2)}\right)$ we have one fermionic screening operator $S^{12} = \oint S^{12}(z)\mathrm{d}z$.

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  Decompose the corresponding screening operators on 3 systems

$$\mathfrak{S}_1 = \left\{ \left(S^{11}_{-}\right)_{i,i+1} | 0 < i < n \right\}, \quad \mathfrak{S}_2 = \left\{ \left(S^{22}_{+}\right)_{j,j+1} | n < j < m+n \right\},$$
$$\mathfrak{S}_3 = \left\{ \left(S^{11}_{+}\right)_{i,i+1}, \left(S^{12}\right)_{n,n+1}, \left(S^{22}_{-}\right)_{j,j+1} | 0 < i < n,\ n < j < n+m \right\}.$$

  Operators from different $\mathfrak{S}_k$ commutes. Systems $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3$ corresponds to quantum groups $\mathfrak{gl}_n$, $\mathfrak{gl}_m$, $\mathfrak{gl}_{n|m}$.

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  $$\mathfrak{S}_1 = \left\{\left(S_{-}^{11}\right)_{i,i+1} | 0 < i < n\right\}, \quad \mathfrak{S}_2 = \left\{\left(S_{+}^{22}\right)_{j,j+1} | n < j < m+n\right\},$$
  $$\mathfrak{S}_3 = \left\{\left(S_{+}^{11}\right)_{i,i+1}, \left(S^{12}\right)_{n,n+1}, \left(S_{-}^{22}\right)_{j,j+1} | 0 < i < n,\ n < j < n+m\right\}.$$
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  Screening operators form three quantum groups $\mathfrak{gl}_{n|m}$, $\mathfrak{gl}_{m|k}$, $\mathfrak{gl}_{k|n}$.

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  Screening operators form three quantum groups $\mathfrak{gl}_{n|m}$, $\mathfrak{gl}_{m|k}$, $\mathfrak{gl}_{k|n}$.
- In the limit $q\to 1$ all this $W_{n|m|k}$ algebras are quotients of $W_\infty(c)$. There is no other quotients of $W_\infty(c)$ defined by screening construction. [Litvinov, Spodyneiko 2016]

Further questions

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- Relation to supersymmetric gauge theories.
  "Pit" condition means that we study monomial ideals $I \subset \mathbb{C}[x, y, z]$, such that $x^n y^m z^k \in I$.