Two mathematical applications of little string theory

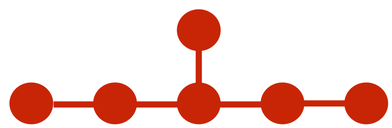
Mina Aganagic
UC Berkeley

Based on joint works with
Edward Frenkel and Andrei Okounkov, to appear
Andrei Okounkov, arXiv:1604.00423 [math.AG]
Nathan Haouzi, arXiv:1506.04183 [hep-th]

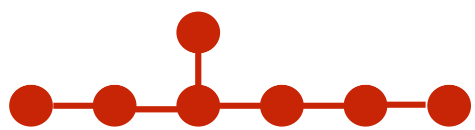
Let $\mathbf{g}$ be a simply-laced simple Lie algebra.

$\mathbf{g} = A_n$

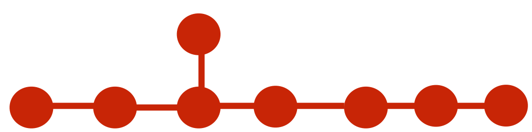
$\mathbf{g} = D_n$



$\mathbf{g} = E_6$



$\mathbf{g} = E_7$



$\mathbf{g} = E_8$

String theory predicts existence of a remarkable
quantum field theory in six dimensions
labeled by $\mathfrak{g}$.

This theory, often called theory X($\mathfrak{g}$),
is a conformal field theory with (2,0) supersymmetry.

Theory $X(\mathbf{g})$ is remarkable, in part,

because it is expected to play an important role in pure mathematics.

We have had evidence, for a while, that theory $X(\mathbf{g})$,

should have applications to:

Geometric Langlands

Program

Knot Categorification

Program

An obstacle to making progress is the fact that the theory $X(\mathbf{g})$ is hard to understand, even for physicists, because it has no classical limit.

AGT correspondence,

after Alday, Gaiotto and Tachikawa,

serves well to illustrate both the appeal of the theory,

and the difficulty of working with it.

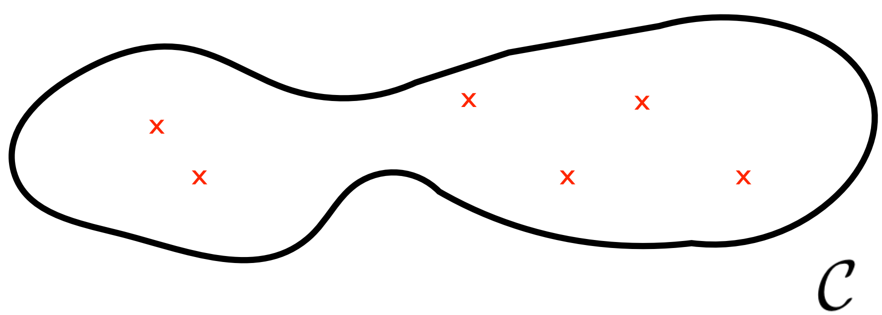
The AGT correspondence states
that the partition function of the theory $X(\mathbf{g})$,
on a six manifold of the form

$$M_6 = \mathcal{C} \times \mathbb{R}^4$$

is a conformal block on the Riemann surface $\mathcal{C}$
of a conformal vertex operator algebra
which is also labeled by $\mathbf{g}$:

$\mathcal{W}(\mathbf{g})$ -algebra

The correspondence further relates defects of the theory $X(\mathbf{g})$ to vertex operators of the $\mathcal{W}(\mathbf{g})$-algebra, inserted at points on $\mathcal{C}$.



If one is to take the conjecture at its face value,
it is hard to make progress on it.

Since we do not know how to describe the theory $X(\mathbf{g})$,
we cannot formulate or evaluate its partition function in any generality.

We do not even have
a well defined statement of the correspondence,
except in rare examples.

(Regardless, it's physical implications are extremely important.)

It turns out that there is an embedding of both sides
of the AGT correspondence
into bigger theories,
which allows us to make
the correspondence mathematically precise in a very general setting,
and to prove it.

On the W-algebra side, one replaces the ordinary W-algebra by its “q-deformation”.

The deformed W-algebra is the one defined by Frenkel and Reshetikhin in the 90’s.

One replaces the 6-dimensional theory $X(\mathbf{g})$,

which is a point particle theory,

by a 6-dimensional string theory which contains it,

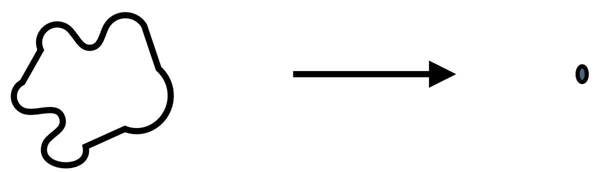
"the little string theory."

The little string theory is labeled by

the same ADE lie algebra g.

It can be viewed as a one parameter deformation

of theory $X(\mathbf{g})$



The parameter is the characteristic size of the string.

In the limit this vanishes, one recovers the theory $X(\mathbf{g})$,

which is a point particle theory.

On each side of the correspondence, one replaces a theory with conformal symmetry,

with its mass deformation.

The conformal symmetry is broken in either case,

but in a canonical way.

The correspondence is between

q-conformal blocks of the deformed $\mathcal{W}(\mathbf{g})$ -algebra on $\mathcal{C}$,

and

the partition function of the

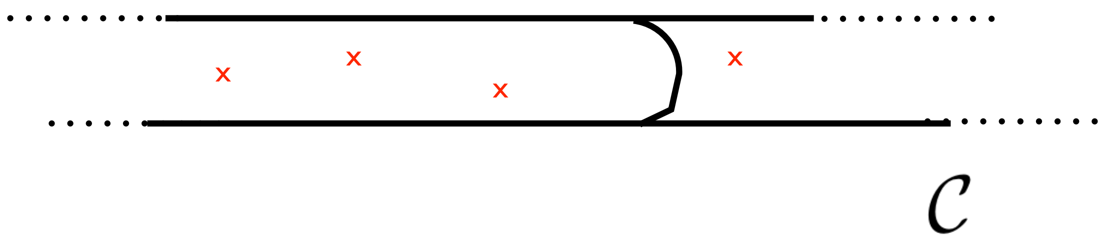
6d little string theory on

$$M_6 = \mathcal{C} \times \mathbb{R}^4$$

where the Riemann surface $\mathcal{C}$

can be taken to be either a cylinder, or a torus.

We will take $\mathcal{C}$ to be a cylinder, since the torus case follows by additional identifications.



The partition function of the g-type little string on $\mathcal{C} \times \mathbb{R}^4$

with arbitrary collections of defects,

turns out to have a precise mathematical formulation.

For co-dimension two defects at points on $\mathcal{C}$

the partition function of $\mathfrak{g}$-type little string theory on

$$M_6 = \mathcal{C} \times \mathbb{R}^4$$

it is the equivariant, K-theoretic instanton partition function

of a certain $\mathfrak{g}$-type quiver gauge theory on $\mathbb{R}^4$.

Which quiver gauge theory we get on $\mathbb{R}^4$ depends on the choice of co-dimension two defects of the six dimensional theory on $\mathcal{C}$

The 6d theory, turns out to localize to
the theory on its defects.

The quiver gauge theory
is simply the theory
on the defects of little string theory,
at points on $\mathcal{C}$ and filling the $\mathbb{R}^4$.

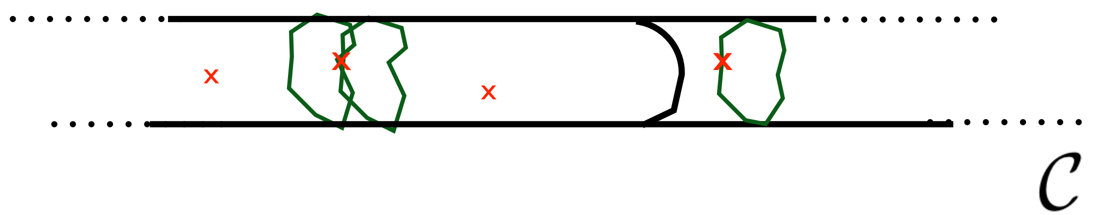
A consequence is that the
partition function of the 6d little string
is the K-theoretic Nekrasov partition function
of the gauge theory on its defects.

One gets the K-theoretic instanton partition function,

rather than the more obvious cohomological one, due to a stringy effect.

In a string theory,

one has to include the winding modes of strings around $\mathcal{C}$



These turn the theory on the defects

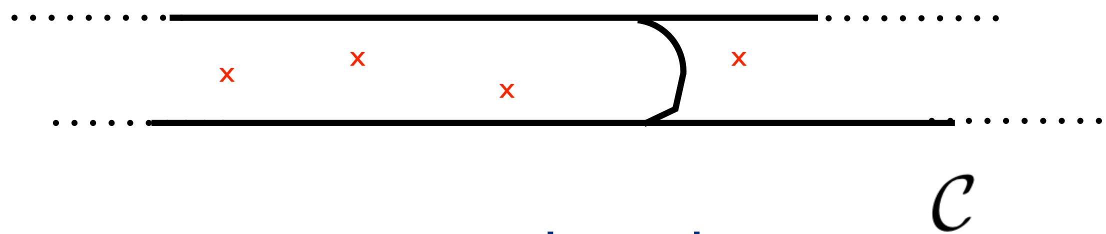
to a five dimensional gauge theory

on $S^1 \times \mathbb{R}^4$.

(the S^1 is the T-dual of the circle in $\mathcal{C}$).

Recall that the choice of defects

in the little string theory,



corresponds to the

choice vertex operators

of the deformed $\mathcal{W}(\mathbf{g})$ -algebra.

Thus, the mathematical statement of the correspondence

is between:

q-deformed conformal blocks

of the $\mathcal{W}(\mathbf{g})$ -algebra with collection of vertex operators

at points on $\mathcal{C}$,

and

K-theoretic instanton partition function

of a corresponding $\mathbf{g}$ -type quiver gauge theory on $\mathbb{R}^4$

This relation can be both spelled out and proven.

This fact relies on using little string theory,

rather than the theory $X(\mathbf{g})$ to which it reduces in the conformal limit.

In the point particle limit,

the theory on the defects

has no known direct description.

In particular, its partition function is not computed

by instanton counting of any known kind.

In the rest of the talk, I will describe this in more detail.

Then, I will describe another application of little string theory,
to the geometric Langlands program.

To define the $\mathbf{g}$ -type little string theory on

$$\mathcal{C} \times \mathbb{R}^4$$

one starts with the 10-dimensional IIB string theory on

$$Y \times \mathcal{C} \times \mathbb{R}^4$$

where Y is the ADE surface singularity of type $\mathbf{g}$

Y , the ADE surface of type $\mathbf{g}$,

is asymptotically, locally a flat complex 2-fold,

obtained by resolving a

$$\mathbb{C}^2/\Gamma$$

singularity.

Γ is the discrete subgroup of $SU(2)$,

related to $\mathbf{g}$ by McKay correspondence.

The resolution of the singularity at the origin of

$$\mathbb{C}^2/\Gamma$$

results in a collection of vanishing 2-cycles intersecting according to the Dynkin diagram of $\mathbf{g}$

$\mathbf{g} = D_n$

In the one’s first course on string theory,

one learns how to formulate the 10d string theory in this setting.

The 6d little string theory,

is a six dimensional string theory

by taking the limit of IIB string theory on

$$Y \times \mathcal{C} \times \mathbb{R}^4$$

where one keeps only the degrees of freedom

supported near the singularity

at the origin of Y.

The limit involves sending the string coupling constant to zero,

but keeping the characteristic size of the

IIB string finite.

The defects of little string theory originate as
D-branes of the ten dimensional string,
which survive the limit.

In string theory on

$$Y \times \mathcal{C} \times \mathbb{R}^4$$

the defects we need are D5 branes supported on

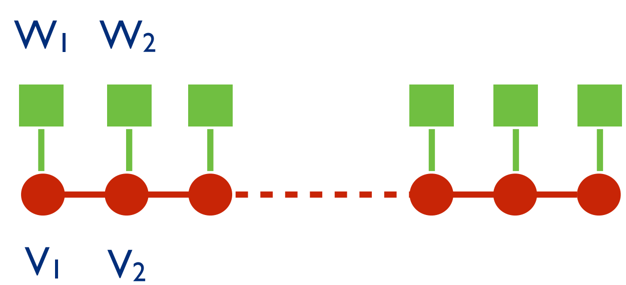
(non-)compact 2-cycles in Y,

at points on $\mathcal{C}$, and fill $\mathbb{R}^4$.

The (low energy) theory on the D5 branes is a

quiver gauge theory

with quiver diagram based on the Dynkin diagram of g:

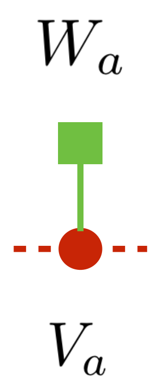


$$\mathbf{g} = A_n$$

In the quiver,

for each node of the Dynkin diagram of $\mathbf{g}$

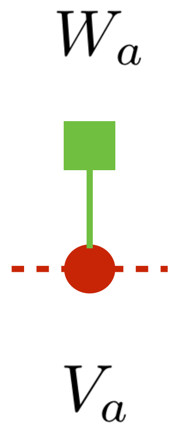
one gets a pair of vector spaces, V_a and W_a



$$\dim(W_a) = m_a$$

$$\dim(V_a) = d_a$$

The dimension vectors



$$\dim(W_a) = m_a$$

$$\dim(V_a) = d_a$$

are determined by the classes of 2-cycles in Y which support the D5 branes, Recall that aspects of geometry of the ADE surface Y

are captured by representation theory of g:

$H_2(Y, \mathbb{Z})$ viewed as a lattice with an inner product coming from the intersection form, is the same as root lattice of g

$H_2(Y, \partial Y, \mathbb{Z})$ is the same as weight lattice of g

V_a come from D5 branes supported on the a-th vanishing two cycle,

$$[S_a] \in H_2(Y, \mathbb{Z})$$

which in turn corresponds to a simple root of g.

W_a come from D5 branes wrapping the dual non-compact two-cycle,

$$[S_a^*] \in H_2(Y, \partial Y, \mathbb{Z})$$

corresponding to a fundamental weight of g:

$$\#(S_a, S_b^*) = \delta_{ab}$$

The gauge group of the quiver gauge theory

$$G = \prod_a U(d_a)$$

originates from the gauge theory on D5 branes supported on compact two-cycles in Y;

The matter fields come from strings at the intersections of the branes.

The flavor symmetry group

$$G_F = \prod_{a=1}^{n} U(m_a)$$

comes from the gauge symmetry group of the non-compact D5 branes.

We need not an arbitrary quiver gauge theory,

but rather those that describe defects in the 6d little string on

$$\mathcal{C} \times \mathbb{R}^4$$

which preserve 4d conformal invariance

in the (low energy) limit.

The answer can be stated as follows….

To get a single puncture on $\mathcal{C}$ in the conformal limit,
pick a collection of n+1 weights of g

$$\omega_i$$

$$i = 0, \ldots, n \leq \mathrm{rk}(\mathbf{g})$$

such that every ω_i is in the Weyl group orbit
a fundamental weight of g, and

$$\sum_{i=0}^{n} \omega_i = 0$$

If one wishes to get a "full puncture", the collection must in addition
span the weight lattice.

Corresponding to this is a collection

of (non-compact) 2-cycles in Y

on which the D5 branes are supported.

This leads to a quiver gauge theory with the gauge and global symmetry groups

$$G = \prod_{a=1}^{n} U(d_a) \quad , \quad G_F = \prod_{a=1}^{n} U(m_a)$$

determined from the equation:

$$\sum_a d_a e_a = \sum_a m_a \omega_a - \sum_i \omega_i$$

m_a is the number of ω_i 's from the orbit of the fundamental weight ω_a and one solves for d_a

If we consider several defects on $\mathcal{C}$ instead of one, the ranks of the gauge and flavor symmetry group simply add.

Nekrasov formulated the

K-theoretic equivariant instanton partition function

of the gauge theory on $M_4 = \mathbb{R}^4$,

as the equivariant Euler characteristic

of an appropriate bundle over the moduli space of G instantons on M_4

Localization with respect to

$$(\mathbb{C}^*)^2 \;\; \times \;\; G \;\; \times \;\; G_F$$

lets one express the partition function as a

sum over the fixed points in instanton moduli space,

labeled by rk(g)-tuples of 2d Young diagrams,

$$\{R\} = \{R_{a,i}\}$$

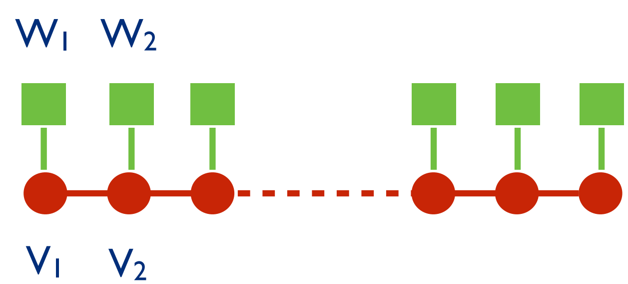
$$a = 1, \ldots \mathrm{rk}(\mathbf{g}) \qquad i = 1, \ldots d_a$$

with one 2d Young diagram, for each U(1) factor in G.

The contribution of each fixed point

$$I_{\{R\}}$$

can be read off from the quiver,



$$\mathbf{g} = A_n$$

as a product of contributions of the nodes and the arrows.

In the end we sum over all the fixed points.

$$Z = \sum_{\{R\}} e^{\tau \cdot R}\, I_{\{R\}}$$

The answer

$$\sum_{\{R\}} e^{\tau \cdot R} \, I_{\{R\}}$$

depends, in addition to the instanton counting parameter e^{τ},

on the equivariant parameters:

$$I_{\{R\}} = I_{\{R\}}(q, t; g, x)$$

q, t for rotations of two complex planes in $M = \mathbb{C}^2$

g for maximal torus of the gauge group G

x for maximal torus of the global symmetry group G_F

The parameters which enter the gauge theory partition function

$$\sum_{\{R\}} e^{\tau \cdot R} \, I_{\{R\}}(q, t; g, x)$$

have a geometric interpretation in string theory.

The equivariant parameters we labeled by x

associated to G_F,

are the positions, on $\mathcal{C}$, of D5-branes on non-compact cycles.

The equivariant parameters labeled by g

associated to G,

are the positions, on $\mathcal{C}$, of D5-branes on compact cycles.

q, t are the equivariant parameters associated to rotations

of $M_4 = \mathbb{C} \times \mathbb{C}$

τ are the moduli of Y
associated with sizes of vanishing 2-cycles.

As soon as we resolve the singularities of Y,
by giving the two-cycles non-zero area,
all the relevant dynamics of the little string theory
is localized on the D5-branes.

The instanton partition function
is the partition function of little string theory on M_6,
with the corresponding collection of defects on $\mathcal{C}$

Now, let me describe the deformed $\mathcal{W}(\mathbf{g})$ -algebra.

corresponding to a simple Lie algebra $\mathbf{g}$.

It is defined by Frenkel and Reshetikhin

in the "free-field formalism".

We will specialize to the simply laced case,

relevant for now.

One starts with a

deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathbf{g})$,

depending on two parameters q, and t, with generators

$$e_a[k], \quad k \in \mathbb{Z}$$

for each node of the Dynkin diagram, $a = 1, \ldots, \mathrm{rk}(\mathbf{g})$

with commutation relations

$$[e_a[k], e_b[m]] = \frac{1}{k}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})C_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}})\delta_{k,-m}$$

where $C_{ab}(q,t)$ is a deformed Cartan matrix

For each weight $\mu \in \mathbb{C}^{\mathrm{rk}(\mathbf{g})}$ of the Cartan subalgebra,
we get the Fock representation of the Heisenberg algebra
with state $|\mu\rangle$ as a generator:

$e_a[k]|\mu\rangle = 0$ for $k > 0,$ $e_a[0]|\mu\rangle = \mu_a|\mu\rangle$

The $\mathcal{W}_{q,t}(\mathbf{g})$ algebra itself is defined as the set of vertex operators which commute with the screening charges

$$Q_a = \int dx \, S_a(x)$$

acting on the Heisenberg vacuum $|\mu\rangle$

where

$$S_a(x) =: \exp\Big(\sum_{k\neq 0} \frac{e_a[k]}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}} e^{kx}\Big) :$$

are the screening vertex operators.

Taking the limit

$$t = q^{\beta}, \qquad q \to 1$$

the deformed $\mathcal{W}(\mathbf{g})$ algebra becomes the ordinary one

$$\mathcal{W}_{q,t}(\mathbf{g}) \qquad \to \qquad \mathcal{W}_{\beta}(\mathbf{g})$$

containing the Virasoro algebra as a subalgebra,
with central charge depending on β

General q-conformal blocks of the W-algebra
are expected to be correlators of vertex operators,

$$\langle\mu|V_{\alpha_1}(z_1)\dots V_{\alpha_k}(z_k)\ \prod_{a=1}^{n} Q_a^{N_a}|\mu'\rangle$$

where $V_\alpha(z)$ are built out of Heisenberg algebra generators

There is as of now, neither a math nor physics definition of what it means to be a q-deformed chiral vertex operator algebra.

Correspondingly, Frenkel and Reshetikhin did not define q-deformations of general vertex operators of the W-algebra.

My student Nathan Haouzi and I showed the following….

Given a collection of weights

$$\{\,(\omega_i; x_i)\,\}_{i=0}^{n}$$

satisfying the conditions given earlier,

with weight ω_i associated to a point x_i on the Riemann surface,

consider the operator

$$V_\alpha(z) \equiv : \prod_{i=0}^{n} V_{\omega_i}(x_i) :$$

where $V_{\omega_i}(x)$ are normal ordered products of simple root

$$E_a(x) =: \exp\Big(\sum_{k\neq 0} \frac{e_a[k]}{(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})} e^{kx} \Big) :$$

and fundamental weight operators,

$$W_a(x) =: \exp\Big(\sum_{k\neq 0} \frac{w_a[k]}{(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})} e^{kx} \Big) :$$

which naturally "quantize" the classical Lie algebra relations, e.g.

$$e_a[k] = \sum_{b=1}^{n} C_{ab}(q^k, t^k) w_b[k]$$

In particular, in the limit in which the q-deformation goes away,

$$t = q^{\beta}, \qquad q \to 1$$

the vertex operators

$$V_{\alpha}(z) \equiv : \prod_{i=0}^{n} V_{\omega_i}(x_i) :$$

with α and z fixed

$$x_{a+1}/x_a = q^{\alpha_a}, \qquad x_0 = z$$

becomes the primary vertex operators

of the conformal $\mathcal{W}_{\beta}(\mathbf{g})$ algebra, in free field formalism.

We expect there is a natural definition
of the deformed chiral vertex operator algebra
under which these are the primary operators.

(This natural definition should be based on quantum K-theory,
and a class of three dimensional gauge theories
which will appear later in the talk.)

The corresponding q-correlators

$$\langle \mu | V_{\alpha_1}(z_1) \dots V_{\alpha_k}(z_k) \prod_{a=1}^{n} Q_a^{N_a} | \mu' \rangle$$

are in fact contour integrals, since

$$Q_a = \int dy \; S_a(y)$$

To specify the q-conformal block, we need to specify the contour.

One can show that the choices of contours

(which one can make explicit),

are parameterized by choices of splitting the numbers of screening charges

$$\{N_a\}_{a=1}^{n} \quad \rightarrow \quad \{N_{a,I}\}_{a,I=1}^{n,d_a} \qquad\qquad \sum_{I=1}^{d_a} N_{a,I} = N_a$$

such that the q-conformal block

$$\langle \mu | V_{\alpha_1}(z_1) \ldots V_{\alpha_k}(z_k) \ \prod_{a=1}^{n} Q_a^{N_a} | \mu' \rangle$$

when evaluated by residues….

… equals the gauge theory partition function,

$$\sum_{\{R\}} e^{\tau \cdot R} \, I_{\{R\}}(q,t;g,x)$$

One identifies the instanton counting parameter with the weight of the Verma module $|\mu\rangle$

$$\tau = \mu$$

The choice of contour corresponds to a choice of parameterization of the G-equviariant parameters of the instanton partition function as

$$g_{a,I} = t^{N_{a,I}}$$

The relation between the

q-conformal blocks

of the $\mathcal{W}(\mathbf{g})$ -algebra on $\mathcal{C}$,

and

K-theoretic instanton partition function

of the corresponding $\mathbf{g}$ -type quiver gauge theory

is simple and direct.

The sum over the poles

$$\langle\mu|V_{\alpha_1}(z_1)\ldots V_{\alpha_k}(z_k)\ \prod_{a=1}^{n} Q_a^{N_a}|\mu'\rangle \equiv \int \prod_{a=1}^{n}\ dy^{N_a}\ y^{\mu}\ \mathcal{I}(q,t;y,x)$$

in the contour prescription to evaluate the conformal block

$$\sum_{\{R\}} e^{\tau\cdot R}\ I_{\{R\}}(q,t;g,x)$$

is the sum over instantons, term by term:

$$\mathrm{res}_{\{R\}}\mathcal{I}(q,t;y,x) = I_{\{R\}}(q,t;g,x)$$

Another application of little string theory is to geometric Langlands correspondence.

The geometric Langlands correspondence
was formulated by
Beilinson and Drinfeld
in early ’90s preprint.

In the same work, they explained that
one can phrase the correspondence
in the language of 2d conformal field theory.

(This was further developed by Frenkel, Feigin and others.)

Let ${}^{L}\mathbf{g}$ and $\mathbf{g}$ be a Langlands dual pair of Lie algebras.

Geometric Langlands correspondence

can be interpreted as the relation

between conformal blocks on a Riemann surface $\mathcal{C}$:

The electric side are the conformal blocks of the affine current algebra

$$^{L}\hat{\mathbf{g}}$$

at the critical level $k = -h$ (infinite coupling).

On the magnetic side, we get conformal blocks of the

$$\mathcal{W}_{\beta}(\mathbf{g})$$

algebra in the classical, $\beta \to \infty$, limit.

Aspects of the geometric Langlands correspondence

were proven in this context by

Belinson and Drinfeld,

in their original paper

and

in later works of

Frenkel with Gaitsgory and Vilonen.

There are two ways in which one may try to generalize this.

First, it is natural to deform away from
the critical level k or equivalently, to finite β.

Second, it is natural to replace the conformal chiral algebras by their
q-deformed counterparts.

The first deformation

is the "quantum Langlands correspondence."

In the abelian case, it was proved by Polishchuk and Rothstein.

When $\mathbf{g} = A_1$

some partial results were obtained by

Feigin, Frenkel and Stoyanovsky, Teschner, and others.

It turns out that one can
implement both generalizations,
and moreover it is easiest to do it
at the same time.

In a joint work with E. Frenkel and A. Okounkov,

we formulate the quantum q-Langlands correspondence,

for any Lie algebra $\mathbf{g}$ and its Langlands dual ${}^L\mathbf{g}$

We outline the proof of the correspondence,

in the simply laced case.

We conjecture the correspondence between
deformed conformal blocks of the
quantum affine current algebra

$$U_{\hbar}({}^{L}\hat{\mathbf{g}})$$

corresponding to ${}^{L}\hat{\mathbf{g}}$ at level k ,
and the q-conformal blocks of the deformed W-algebra

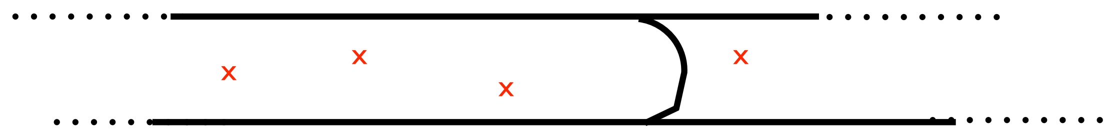
$$\mathcal{W}_{q,t}(\mathbf{g})$$

The parameters are related by:

$$t = \hbar, \qquad q = \hbar^{(k+h)}$$

Take the Riemann surface $\mathcal{C}$ which is a cylinder,

with punctures at distinct points x_i



To a puncture at x_i associate

a finite dimensional representation ${}^L\rho_i$ of ${}^L\mathfrak{g}$

For simplicity, take ${}^L\rho_i$ to be one of the fundamental representations of ${}^L\mathfrak{g}$, labeled by the nodes of its Dynkin diagram.

On the electric, affine current algebra side,

one considers the usual q-conformal blocks of the quantum affine algebra,

$$\langle\lambda| \prod_i \Phi_i(x_i) \, |\lambda'\rangle$$

$\Phi_i(x_i)$ corresponds to the finite dimensional representation ${}^L\rho_i$ of ${}^L\mathbf{g}$ attached to x_i .

Φ 's are the vertex operators of $U_\hbar({}^L\hat{\mathbf{g}})$ constructed by I. Frenkel and Reshetikhin.

On the magnetic, $\mathcal{W}$ -algebra side,

one considers q-correlators of the form

$$\langle\lambda| \prod_i V_i^\vee(x_i) \prod_a \left(Q_a^\vee\right)^{d_a} |\lambda'\rangle$$

$V_i^\vee(x_i)$ is the magnetic, degenerate, vertex operator of the $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra

defined by E. Frenkel and Reshetikhin.

It is labeled by the fundamental co-weight of $\mathbf{g}$,

which is, by Langlands duality,

the highest weight of the representation ${}^L\rho_i$ of the electric group ${}^L\mathbf{g}$,

we attached to x_i

The choice of numbers of magnetic screening charge $Q_a^\vee$ -insertions

$$\langle\lambda| \prod_i V_i^\vee(x_i) \prod_a \left(Q_a^\vee\right)^{d_a} |\lambda'\rangle$$

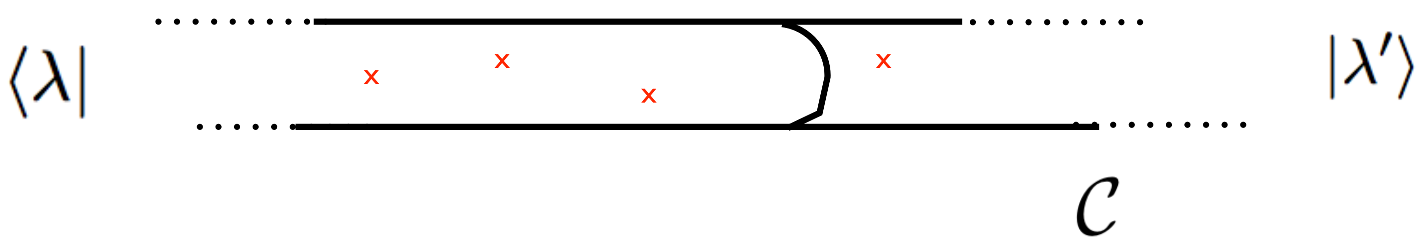
in the $\mathcal{W}$ -algebra q-conformal block corresponds to restricting the

$$\langle\lambda| \prod_i \Phi_i(x_i) |\lambda'\rangle$$

$\otimes_a \rho_a^{\otimes m_a}$-valued $U_\hbar({}^L\hat{\mathbf{g}})$ block to the subspace of the weight

$$\sum_a m_a \, {}^L w_a - \sum_a d_a \, {}^L e_a$$

The states $|\lambda\rangle$ and $|\lambda'\rangle$,



generate Verma module representations of the algebras.

Their weights are not independent:

$$\lambda - \lambda' = \sum_a m_a\, {}^L w_a - \sum_a d_a\, {}^L e_a$$

To specify the q-conformal block, on the W-algebra side,
we need to choose the contour of integration.

On the affine current algebra side, we need to specify the
the Verma modules one gets in the intermediate channels.

The q-conformal blocks of

electric

$U_{\hbar}({}^L\hat{\mathbf{g}})$

magnetic

$\mathcal{W}_{q,t}(\mathbf{g})$

chiral algebras

arize as partition functions of little string theory,

with defects.

For the current purpose,

the relevant objects in little string theory

are not the co-dimension two defects of the theory

we studied so far,

but rather the co-dimension four defects.

For now, let $\mathbf{g}$ be simply laced, so that

$$ {}^{L}\mathbf{g} = \mathbf{g} $$

Take the little string theory associated

to the simply laced Lie algebra $\mathbf{g}$

on

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$$

just as before.

The defects are strings supported at points on $\mathcal{C}$ and on one of the two complex planes in

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$$

In the present notation, this is the plane rotated by q.

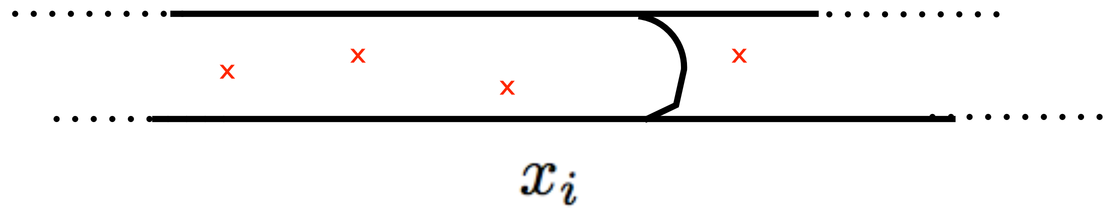
From perspective of IIB string on

$$Y \times M_6$$

the defect strings come from D3 branes,

supported on 2-cycles in Y , and the chosen 2-plane in M_6.

The insertion points of vertex operators,



are the positions of non-compact D3 branes on $\mathcal{C}$

The class of a cycle in Y that supports the D3 brane is the highest weight of the corresponding representation.

$$w_a = [S_a^*] \in H_2(Y, \partial Y, \mathbb{Z})$$

The weight of the Verma module $|\lambda\rangle$ is the Kahler modulus of Y

The partition function of the 6d little string theory with

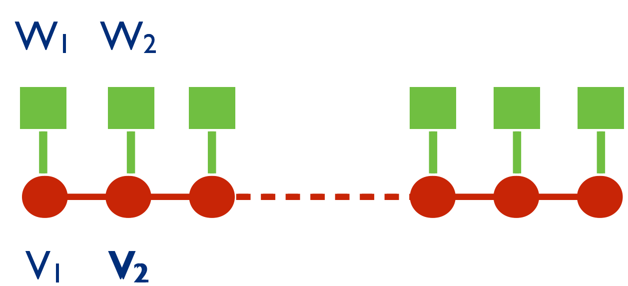
these classes of defects, for generic λ ,

localizes to the partition function of the gauge theory on the defect.

The theory on these defects is

a tree dimensional quiver gauge theory, with N=4 supersymmetry

on $\mathbb{C} \times S^1$.



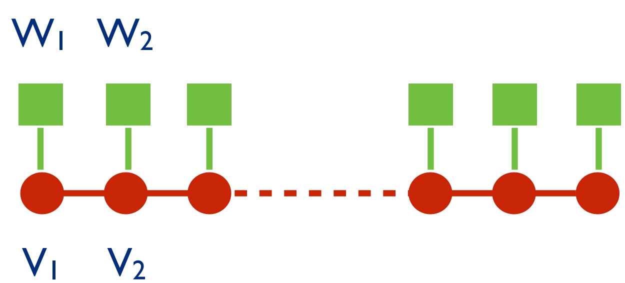
$$\mathfrak{g} = A_n$$

The quiver is based on the Dynkin diagram of g.

The quiver gauge group and the matter content are determined by the weight

$$\sum_a m_a\, {}^L w_a - \sum_a d_a\, {}^L e_a$$

of the subspace the q-conformal blocks of $U_\hbar({}^L\hat{\mathbf{g}})$ live in



$$(\dim V_a, \dim W_a) = (d_a, m_a)$$

The dimension vectors ,

$$(\dim V_a, \dim \mathrm{W}_a) \;=\; (d_a, m_a)$$

are the classes of 2-cycles in Y

$$[S^*] = \sum_a m_a [S_a^*], \qquad [S] = \sum_a d_a [S_a]$$

which support the D3 branes.

The partition function

of the 3d N=4 quiver gauge theory on

$$\mathbb{C} \times S^1$$

is computed by

quantum K-theory

of a Nakajima quiver variety X:

X is the Higgs branch of the gauge theory.

Quantum K-theory of symplectic resolutions,
including Nakajima quiver varieties,
was developed recently by Okounkov with Maulik and Smirnov.

Roughly, the theory counts quasi-maps to X,
working equivariantly with respect to

$$G_F \times \mathbb{C}^*_q \times \mathbb{C}^*_\hbar$$

The fact that quantum K-theory computes the partition function of little string theory on

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$$

with 2d defects on $\mathbb{C}$

is analogous to the fact that the partition function of little string theory with 4d defects on $\mathbb{C} \times \mathbb{C}$

is computed by

K-theoretic instanton counting.

The most basic object of quantum K-theory is the "vertex function".

$$\mathbf{V}$$

Vertex function is a vector which "counts" (quasi-)maps from $\mathbb{C}$ to X .

The choice of a component of $\mathbf{V}$ has to do with the conditions imposed on the maps at infinity of $\mathbb{C}$.

The components of the vertex function of X are the

$\mathcal{W}_{q,t}(\mathbf{g})$ algebra q-conformal blocks

$$\mathbf{V} = \langle\lambda| \prod_i V_i^\vee(x_i) \prod_a \left(Q_a^\vee\right)^{d_a} |\lambda'\rangle$$

To define the W-algebra blocks requires the choice of contour of integration, coming from $Q_a^\vee = \int dx\, S_a^\vee(x)$,

There is a choice such that, computing the integral by residues, we get the vertex function components.

A way to characterize the

conformal blocks of the affine Lie algebra

$$^L\hat{g}_k$$

is as solutions to Kniznik-Zamolodchikov equation

$$(k+h)\, x_\ell \frac{\partial}{\partial x_\ell}\, \Psi = (\sum_{j \neq \ell} r_{\ell i}(x_\ell/x_j) + r_{\ell 0} + r_{\ell \infty})\, \Psi$$

analytic in a chamber

$$\mathfrak{C}: \qquad |x_1| > |x_2| > \ldots$$

corresponding to

how we take the insertion points x_i to infinity.

The deformation of this equation,
corresponding to the quantum affine algebra
is a difference equation,
the quantum Kniznik-Zamolodchikov equation,
discovered by I. Frenkel and Reshetikhin.

A key result of quantum K-theory,

due to Maulik and Okounkov,

is that the vertex function of X

solves the quantum Kniznik-Zamolodchikov equation

corresponding to

$$U_\hbar({}^L\hat{\mathbf{g}})$$

q-conformal block

More precisely, from

$$\mathbf{V}$$

(by differentiating with respect to x_i 's,
or placing insertions at $0 \in \mathbb{C}$),
we can generate the fundamental solution
to the qKZ equation.

The solutions the vertex function

produces are not themselves the $U_\hbar({}^L\hat{\mathbf{g}})$ conformal blocks,

because $\mathbf{V}$

is not holomorphic in any one chamber $\mathfrak{C}$,

$$\mathfrak{C}: \qquad |x_1| > |x_2| > \ldots$$

of the parameter space.

Instead, there is a linear change of basis,

depending on a chamber $\mathfrak{C}$,

that does the job.

$$\mathbf{V}_{\mathfrak{C},\ell} = \sum_{\ell'} \mathfrak{P}_{\mathfrak{C},\ell}{}^{\ell'} \, \mathbf{V}_{\ell'}$$

such that $\mathbf{V}_{\mathfrak{C}}$ are holomorphic in $\mathfrak{C}$ and generate

$U_{\hbar}({}^{L}\hat{\mathbf{g}})$ conformal blocks.

The choice of basis we need to get

$$\mathbf{V}_{\mathfrak{C}}$$

turns out to have a geometric meaning:

It is the

elliptic stable basis

of the Nakajima quiver variety X,

recently discovered in a joint work with Andrei Okounkov.

The elliptic stable basis

gives a basis of elliptic cohomology of X,

generalizing the stable basis in cohomology and K-theory

due to Maulik and Okounkov.

The elliptic stable envelope provides an explicit map between the q-conformal blocks of

$$U_\hbar({}^L\hat{\mathbf{g}}) \qquad \text{and} \qquad \mathcal{W}_{q,t}(\mathbf{g})$$

algebras.

To extend this to non-simply laced groups,

where

$$ {}^{L}\mathbf{g} \neq \mathbf{g} $$

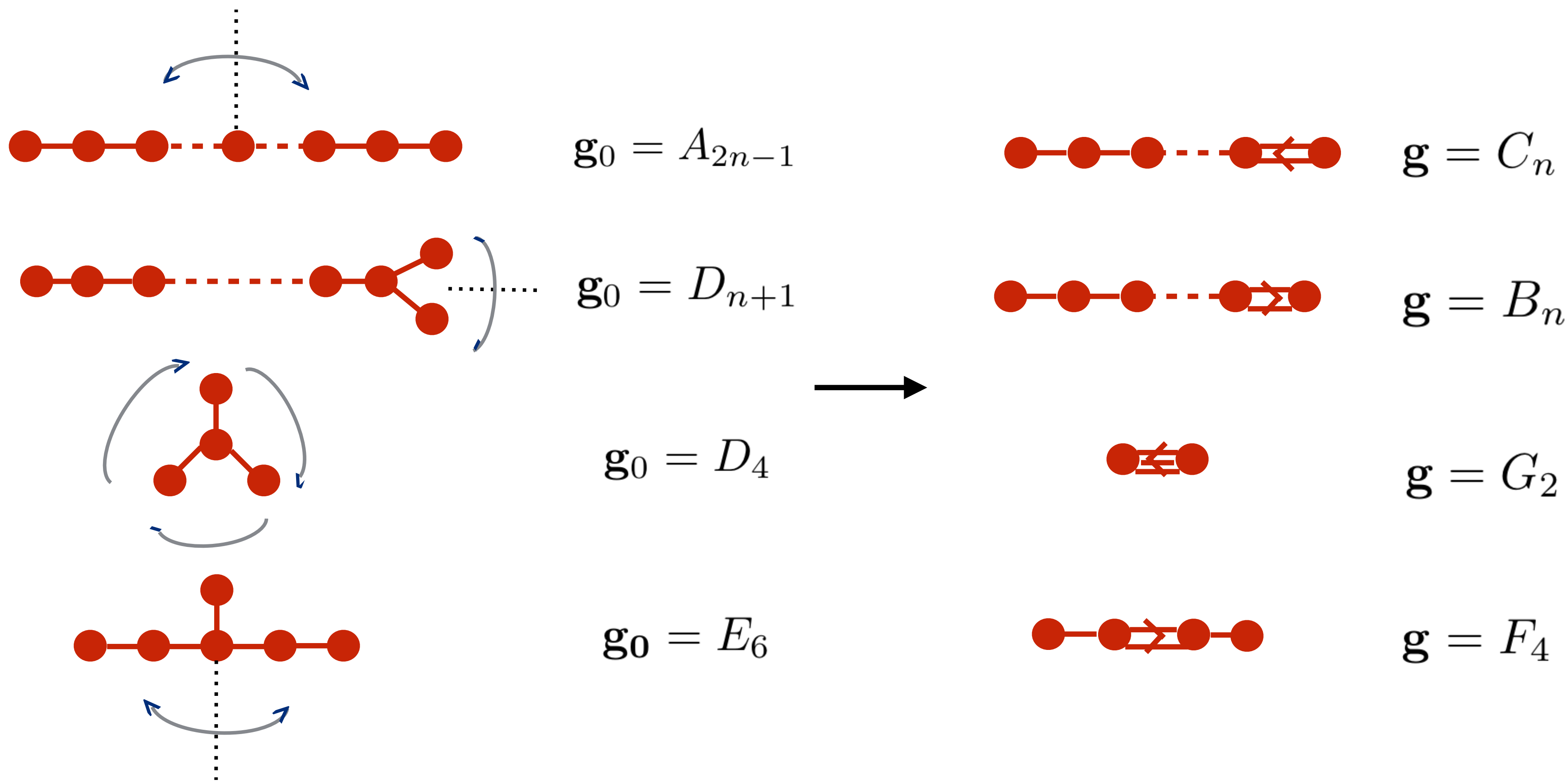
one uses the fact that the non-simply laced Lie algebra

can be obtained from a simply laced Lie algebra $\mathbf{g}_0$,

using an outer automorphism H of $\mathbf{g}_0$:

$$ (\mathbf{g}_0, H) \rightarrow \mathbf{g} $$

H acts as an involution of the Dynkin diagram of $\mathfrak{g}_0$



One studies little string theory of type $\mathbf{g}_0$

on

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$$

with a twist

which ends up permuting the nodes of the Dynkin diagram

by a generator of H,

as we go once around the origin of the complex $\mathbb{C}$ plane

which supports the defects.

This implies that correspondence between

q-conformal blocks of

$$U_{\hbar}({}^L\hat{\mathbf{g}}) \quad \text{and} \quad \mathcal{W}_{q,t}(\mathbf{g})$$

for non-simply laced

$${}^L\mathbf{g} \quad , \quad \mathbf{g}$$

should follow by generalizing

the elliptic stable basis to the H-equivariant setting.

An important generalization
of the geometric Langlands program
is to include ramifications.

This corresponds to including
D5 brane defects from the first half of the talk.

The relevant variety X
in this case
is a $\mathfrak{g}$ -type "hand-saw" quiver variety,
the Higgs branch of the 3d N=2 gauge theory
on D3 branes in presence of D5 branes.

As Witten and Kapustin explained,
the geometric Langlands correspondence
is related to
S-duality of
N=4 super-Yang-Mills theory.

Deep understanding of
the geometric Langlands correspondence
should come from
a deep understanding of S-duality.

Many aspects of S-duality

can be understood within N=4 SYM theory,

or using the six dimensional theory X(g)

compactified on a two-torus.

It was shown by Vafa in '97 that one can derive
S-duality of N=4 SYM theory
from T-duality in
(little) string theory.

This explains why one is able to make progress on the problem,
in the context of the
quantum q-Langlands correspondence.