Monopoles, Vortices, and Vermas

with T.Dimofte, D.Gaiotto, J.Hilburn, H-C.Kim

Mathew Bullimore

Mathematical Institute, University of Oxford

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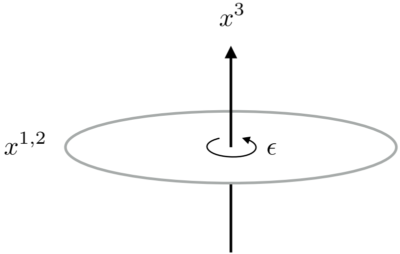
Introduction

Today's setup:

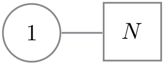


- A three-dimensional $\mathcal{N}=4$ supersymmetric gauge theory with an Ω-deformation in the $x^{1,2}$-plane.
- I will describe this system as an $\mathcal{N}=4$ supersymmetric quantum mechanics on the x^3-axis.

Example: SQED

$U(1)$ gauge theory with N fundamental hypermultiplets ($G = \mathbb{C}^*$, $N = \mathbb{C}^N \oplus \mathbb{C}^N$).

$t > 0$



$m_1, \ldots, m_N$

- Vectormultiplet contains a real scalar σ and a complex scalar φ.
- Hypermultiplet has complex scalars (X_j, Y_j) of charge $(+1, -1)$.

I will turn on parameters:

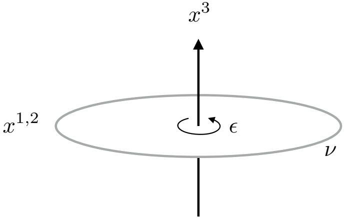
- Real FI parameter $t > 0$.
- Complex masses $(m_1, \ldots, m_N)$ for hypermultiplets.

With both parameters, there are isolated massive vacua ν_i

$$\sigma = 0 \qquad \varphi = -m_i \qquad X_j = \sqrt{t}\,\delta_{ij} \qquad Y_j = 0\,.$$

Omega Background

Introduce an Ω deformation with parameter ϵ in $x^{1,2}$-plane [1].



- Isolated massive vacuum ν at infinity in $x^{1,2}$-plane.
- This system preserves the same supersymmetry as an $\mathcal{N} = 4$ quantum mechanics on the x^3-axis.

[1] *Dimensional reduction of Nekrasov 2003, Nekrasov-Shatashvili 2009*

Hilbert Space 1

To find the Hilbert space, we study half-BPS configurations that are invariant under translations in x^3 direction.

They are solutions of vortex equations

$$F_{z\bar{z}} + \sum_{j=1}^{N} (|X_j|^2 - |Y_j|^2) = t$$

$$D_{\bar{z}} X_j = 0 \qquad D_{\bar{z}} Y_j = 0$$

$$\sum_{j=1}^{N} X_j Y_j = 0$$

where $z = x^1 + ix^2$.

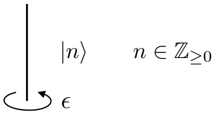
Solutions must tend to isolated vacuum at $|z| \to \infty$.

Hilbert Space 2

The Hilbert space is the equivariant cohomology of the moduli space of solutions to x^3-independent BPS equations:

$$\mathcal{H}_\nu = \bigoplus_{n \geq 0} H_{T_\nu}(\mathcal{M}_n)$$

- T_ν are symmetries preserved by vacuum ν
- The equivariant parameters are ϵ, $m_1, \ldots, m_N$
- Solutions classified by a vortex number: $n = \frac{1}{2\pi} \int \mathrm{Tr}(F)$
- Denote fundamental class of $\mathcal{M}_n$ by $|n\rangle$: state of n vortices at origin of $x^{1,2}$-plane.



Coulomb branch

In the absence of omega deformation and FI parameter, there is a Coulomb branch $\mathcal{M}_C$ of supersymmetric vacua.

Coulomb branch chiral ring [2]:

$$u^+u^- = P(\varphi) := \prod_j (\varphi + m_j)$$

- φ : complex scalar
- $u^\pm$: half-BPS monopole operators
- Complex masses m_j are deformation parameters.

This is the coordinate ring of deformed $\mathbb{C}^2/\mathbb{Z}_N$.

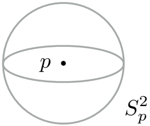
Holomorphic symplectic form: $\Omega = d\varphi \wedge d \log u^+$.

[2] *Borokhov, Kapustin, Wu, 2002*

Operators 1

Coulomb branch operators on x^3-axis become half-BPS operators in the $\mathcal{N} = 4$ quantum mechanics.

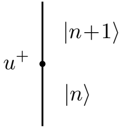
A monopole operator of charge n at a point p requires

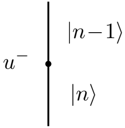


$$\frac{1}{2\pi}\int_{S^2_p} F = n$$

Monopole operators create and destroy vortices:

$$u^+|n\rangle \propto |n+1\rangle \qquad u^-|n\rangle \propto |n-1\rangle$$





Operators 2

Action of half-BPS operators on the Hilbert space may be computed by supersymmetric localization:

$$
\begin{aligned}
\varphi|n\rangle &= (-m_i - (n+\tfrac{1}{2})\epsilon)|n\rangle \\
u^+|n\rangle &= P(-m_i - n\epsilon)|n+1\rangle \\
u^-|n\rangle &= |n-1\rangle
\end{aligned}
$$

- The operators generate a quantization of the Coulomb branch:

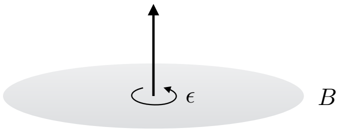
$$
\begin{aligned}
[u^\pm, \varphi] &= +\epsilon\, u^\pm & \qquad u^+u^- &= P(\varphi + \tfrac{\epsilon}{2}) \\
[u^-, \varphi] &= -\epsilon\, u^- & \qquad u^-u^+ &= P(\varphi + \tfrac{\epsilon}{2})\ .
\end{aligned}
$$

- This is spherical part of rational Cherednik algebra.

- The Hilbert space $\mathcal{H}_\nu$ is a 'highest weight' Verma module.

Boundary Conditions

Add a boundary condition in the $x^{1,2}$-plane preserving 2d $\mathcal{N} = (2,2)$ supersymmetry [3].



- This defines a state in the supersymmetric quantum mechanics.
- Example: Neumann for the vectormultiplet with

$$D_3 X_j = 0 \qquad Y_j = 0 \qquad j = 1, \dots, N$$

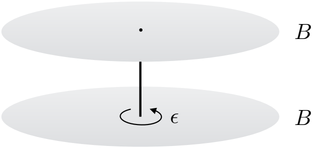
  and boundary FI parameter τ.
- This gives a coherent state of vortices

$$u^- |B\rangle = e^{-\tau} |B\rangle \qquad\qquad |B\rangle \propto \sum_{n \geq 0} e^{-n\tau} |n\rangle \quad .$$

[3] *MB, Dimofte, Gaiotto, Hilburn 2016*

Intervals

Now compactify on an interval with boundary conditions leads to a 2d $\mathcal{N} = (2, 2)$ theory.



- Example: Neumann boundary conditions at top and bottom.
- $U(1) + N$ chiral multiplets $X_1, \ldots, X_N$.
- It's vortex partition [4] function is an overlap of boundary states:

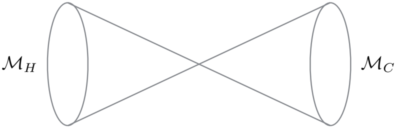
$$Z = \sum_{n \geq 0} \frac{e^{-\tau(m_i/\epsilon + n + \frac{1}{2})}}{\prod_{j=1}^{N} \prod_{d=1}^{n} (-m_i + m_j - d\epsilon)} = \langle B | B \rangle$$

[4] *Shadchin 2007, Dimofte, Gukov, Hollands 2011*

General Setup 1

We assume a 3d $\mathcal{N} = 4$ gauge theory that

- Flows to an SCFT in the infrared.
- Generic mass and FI parameters that leave only isolated massive vacua.



Moduli spaces:

- Higgs branch $\mathcal{M}_H$ is resolved by real FI parameter t.
- Coulomb branch $\mathcal{M}_C$ is deformed by complex mass parameter m [5].
- Both parameters: isolated massive vacua ν.

[5] *Nakajima 2015 and Braverman, Finkelberg, Nakajima 2016. See also MB, Dimofte, Gaiotto 2015*

Summary 1

Expected structure of $\mathcal{N} = 4$ supersymmetric quantum mechanics:

- Hilbert space:

$$\mathcal{H}_\nu = \bigoplus_{n=0}^{\infty} H_{T_\nu}(\mathcal{M}_n)$$

  $\mathcal{M}_n$ = moduli space of based holomorphic maps $\mathbb{CP}^1 \to [\mathcal{M}_H]$ of degree n.

  ν = T_ν-equivariant fixed point of $\mathcal{M}_H$

- Half-BPS operators : algebra

$$\mathcal{A}_C = \mathbb{C}_\epsilon[\mathcal{M}_C]$$

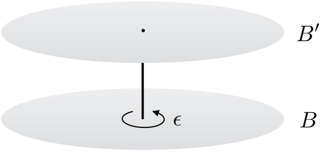
  is quantization of the Coulomb branch $\mathcal{M}_C$ with period m.

- For generic m, $\mathcal{H}_\nu$ are the highest weight Verma modules for $\mathcal{A}_C$.

A 'Finite' AGT Correspondence

Summary:

- Neumann-type boundary conditions define generalized Whittaker vectors for $\mathcal{A}_C$.
- Let $\mathcal{T}_{B,B'}$ be 2d $\mathcal{N} = (2,2)$ theory obtained from a 3d $\mathcal{N} = 4$ theory $\mathcal{T}$ on an interval with Neumann boundary conditions B and B'.
- It's vortex partition function is an overlap of Whittaker vectors: $Z_{\mathcal{T}_{B,B'}} = \langle B | B' \rangle$.



This generalizes results of Braverman, Feigin, Rybnikov, Finkelberg 2011, and Nakajima 2011.

Systematic construction for all hypertoric examples ($G = (\mathbb{C}^*)^r$, N arbitrary).

Questions

Thank you for your attention!