Vertex operator algebras, Higgs branches and modular differential equations

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Based on work with Chris Beem (to appear),
building on work with Beem, Lemos, Liendo, Peelaers and van Rees

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Physics and Math of $\mathcal{N} = 2$ SCFTs in $d = 4$

Vast landscape of $4d$ $\mathcal{N} = 2$ Super Conformal Field Theories (SCFTs). Still to be fully charted. Very rich physics.

$\mathcal{N} = 2$ is *half-maximal* susy.
Physics much more diverse than for $\mathcal{N} = 4$, but not as wild as for $\mathcal{N} = 1$.

At the center of many recent developments in physical mathematics.

The "category of $\mathcal{N} = 2$ SCFTs" does not admit (yet) a full mathematical definition, but many of its moving parts do.

Usual ploy: identify a set of axioms that capture a substructure.
Physics expectations $\rightarrow$ precise conjectures.

Today, I will focus on one such well-defined substructure.

To any $\mathcal{N}=2$ SCFT, one can canonically associate a vertex operator algebra

$$\text{4d } \mathcal{N}=2 \text{ SCFT} \longrightarrow \text{VOA}$$

The associated VOA $\mathcal{V}$ can be understood as a closed subalgebra of the full local operator algebra of the $4d$ theory.

It is a powerful invariant of the SCFT, but its relationship with more conventional protected quantities is still to be fully understood.

In this talk I will formulate a conjecture for how the Higgs branch $\mathcal{M}_H$ (as a complex affine variety) is encoded in $\mathcal{V}$.

This conjecture leads to surprising predictions, which I will test in a large class of examples.

Outline

- Introduction. Lightening review of $\mathcal{N} = 2$ SCFTs.
- (Algebraic) geometry of $\mathcal{N} = 2$ SCFTs:
  Higgs and Coulomb branches and the corresponding chiral rings.
- The VOA associated to an $\mathcal{N} = 2$ SCFT.
- How the VOA encodes the Higgs branch: a conjecture.
- Null states and modular properties for the Schur index.

Lightening review of $\mathcal{N} = 2$ SCFTs

Conventional Lagrangian theories specified by the following data:

- Gauge group (semisimple, reductive)

$$G = G_1 \times G_2 \times \ldots G_k$$

- Matter content, a pseudoreal finite-dim representation ρ of G.

For each simple factor G_i, must impose vanishing of the beta function,

$$\beta = -2h^\vee + C_2(\rho) = 0\,.$$

Lagrangian uniquely fixed by $\mathcal{N} = 2$ superconformal symmetry:
vector multiplets in Adj, (half-)hypermultiplets in ρ.

Complete combinatorial classification. Bardhway Tachikawa

Complexified gauge couplings $\{\tau_i\}$, $i = 1, \ldots k$, are the (only) continuous parameters. Exactly marginal deformations of the SCFT: they parametrize the conformal manifold $\mathcal{U}$ of the SCFT.

We have learnt that the landscape of $\mathcal{N} = 2$ SCFTs is much bigger.

Exotic “matter” in the form of isolated SCFTs with no conventional Lagrangian description.

Coupling them to gauge fields (by gauging a subgroup G of their flavor symmetry group, such that $\beta_G = 0$) leads to large continuous families of SCFT.

In all known examples, exactly marginal couplings arise from weak gauging of isolated SCFTs. Perhaps this is a general fact.

Notations

Superalgebra $\mathfrak{su}(2,2|2)$.

Bosonic subalgebra: $\mathfrak{su}(2,2) \times \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$.

$\mathfrak{su}(2,2) \cong \mathfrak{so}(4,2)$: conformal algebra in $d = 4$,
$\mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$: R-symmetry.

Fermionic generators: Poincaré supercharges $\mathcal{Q}^{\mathcal{I}}_{\alpha}\,, \quad \tilde{\mathcal{Q}}_{\mathcal{I}\,\dot{\alpha}}$

Conformal supercharges $\mathcal{S}^{\alpha}_{\mathcal{I}}\,, \quad \tilde{\mathcal{S}}^{\mathcal{I}\,\dot{\alpha}}$

$\mathcal{I} = 1, 2$ $\mathfrak{su}(2)_R$ index; $\alpha = \pm$, $\dot{\alpha} = \dot{\pm}$ Lorentz spinor indices.

Vector complex scalar ϕ: $SU(2)_R$ singlet, $r(\phi) = -1$

Hypermultiplet complex scalars $Q^{\mathcal{I}}$, $SU(2)_R$ doublet, $r(Q^{\mathcal{I}}) = 0$

$\mathcal{N} = 2$ SCFTs and geometry

Two canonical manifolds of supersymmetric vacua:

- Higgs branch $\mathcal{M}_H$
  $SU(2)_R$ broken, $U(1)_r$ preserved.
  In Lagrangian theories, $\langle Q^I \rangle \neq 0$, $\phi = 0$.
  Hyperkähler. In particular, $\mathcal{M}_H$ is a holomorphic symplectic variety.
  Invariant of the conformal manifold $\mathcal{U}$.
- Coulomb branch $\mathcal{M}_C$
  $SU(2)_R$ preserved, $U(1)_r$ broken.
  In Lagrangian theories, $\langle Q^I \rangle = 0$, $\phi \neq 0$.
  Special Kähler geometry. Seiberg-Witten theory.
  Non-trivial dependence on $\mathcal{U}$.

(Mixed branches also interesting but less universal.)

More intrinsic characterization of $\mathcal{M}_H$ and $\mathcal{M}_C$ in terms of vevs of local (gauge-invariant) chiral operators

One defines:

- The Higgs chiral ring $\mathcal{R}_H$:
  cohomology of two supercharges of *opposite* chirality, $\mathcal{Q}^1_\alpha$ and $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$.
  $\hat{\mathcal{B}}_R$ multiplets. Scalar operators with $\Delta = 2R$, $r = 0$.
  In all examples, $\mathcal{R}_H$ is a *reduced* commutative $\mathbb{C}$-algebra.

- The Coulomb chiral ring $\mathcal{R}_C$:
  cohomology of two supercharges of the *same* chirality, $\tilde{\mathcal{Q}}_{1\dot{\alpha}}$ and $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$.
  $\bar{\mathcal{E}}_r$ multiplets. Scalar primaries with $\Delta = |r|$, $R = 0$.
  In all examples, $\mathcal{R}_C$ is freely generated.

One identifies each chiral ring with the coordinate ring of the branch,

$$\mathcal{R}_H \cong \mathbb{C}[\mathcal{M}_H]\,, \quad \mathcal{R}_C \cong \mathbb{C}[\mathcal{M}_C]\,.$$

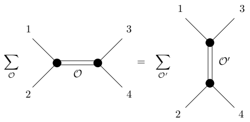
Easy to prove for Lagrangian theories.
Conjecturally true for all $\mathcal{N} = 2$ SCFTs.

The Higgs and Coulomb chiral rings are closed subalgebras of the full local operator algebra of the theory,

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k c_{12k}(x)\mathcal{O}_k(0)\,.$$

True operator equation, convergent expansion.

The full operator algebra looks (at present) much too complicated for a useful mathematical formalization.
Fortunately, one can carve out a very interesting closed subsector, isomorphic to a VOA. Though not a priori obvious, we will see that this VOA is a (vast) generalization of $\mathcal{R}_H$.



Aside: the constraints of superconformal symmetry, unitarity and associativity are very stringent and lead to *numerical* determination of non-protected operators dimensions and OPE coefficients. Beem Lemos Liendo LR van Rees

Meromorphy in $\mathcal{N} = 2$, $d = 4$ SCFTs

Fix a plane $\mathbb{R}^2 \subset \mathbb{R}^4$, parametrized by $(z, \bar{z})$.

Claim : $\exists$ subsector $\mathcal{V} = \{\mathcal{O}_i(z_i, \bar{z}_i)\}$ with meromorphic

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \ldots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = f(z_i)\,.$$

Rationale: $\mathcal{V} \equiv$ cohomology of a nilpotent $\mathbb{Q}$,

$$\mathbb{Q} = \mathcal{Q}^1_- + \tilde{\mathcal{S}}^{2\dot{-}}\,,$$

$\mathcal{Q}$ Poincaré, $\mathcal{S}$ conformal supercharges.

$\bar{z}$ dependence is $\mathbb{Q}$-exact: cohomology classes $[\mathcal{O}(z, \bar{z})]_{\mathbb{Q}} \rightsquigarrow \mathcal{O}(z)$.

Much richer than the chiral ring, where cohomology classes $[\mathcal{O}(x)]_{\tilde{\mathcal{Q}}_{\dot{\alpha}}}$ are x-independent.

At the origin of $\mathbb{R}^2$, $\mathbb{Q}$-cohomology easy to describe. $\mathcal{O}(0,0) \in \mathcal{V} \leftrightarrow \mathcal{O}$ obeys

$$\frac{\Delta - (j_1 + j_2)}{2} = R$$

These are precisely the operators counted by the Schur limit of the superconformal index. Gadde LR Razamat Yan

To define $\mathbb{Q}$-closed operators $\mathcal{O}(z, \bar{z})$ away from origin, need to twist the right-moving generators by $SU(2)_R$,

$$\widehat{L}_{-1} = \bar{L}_{-1} + \mathcal{R}^- , \quad \widehat{L}_0 = \bar{L}_0 - \mathcal{R} , \quad \widehat{L}_1 = \bar{L}_1 - \mathcal{R}^+ .$$

Cohomology classes define vertex operators $\mathcal{O}_i(z)$, with conformal weight $h = R + \ell$. closed under OPE,

$$\mathcal{O}_1(z)\mathcal{O}_2(0) = \sum_k \frac{c_{12k}}{z^{h_1+h_2-h_k}} \mathcal{O}_k(0) .$$

$\mathcal{V}$ has the structure of a VOA

Schur operators

Multiplet	$\mathcal{O}_{\text{Schur}}$	h	Lagrangian "letters"
$\hat{\mathcal{B}}_R$	$\Psi^{11\dots1}$	R	Q, $\tilde{Q}$
$\mathcal{D}_{R(0,j_2)}$	$\tilde{\mathcal{Q}}^1_{\dot{+}}\,\Psi^{11\dots1}_{\dot{+}\dots\dot{+}}$	$R+j_2+1$	Q, $\tilde{Q}$, $\tilde{\lambda}^1_{\dot{+}}$
$\bar{\mathcal{D}}_{R(j_1,0)}$	$\mathcal{Q}^1_{+}\,\Psi^{11\dots1}_{+\dots+}$	$R+j_1+1$	Q, $\tilde{Q}$, λ^1_{+}
$\hat{\mathcal{C}}_{R(j_1,j_2)}$	$\mathcal{Q}^1_{+}\,\tilde{\mathcal{Q}}^1_{\dot{+}}\,\Psi^{11\dots1}_{+\dots+\,\dot{+}\dots\dot{+}}$	$R+j_1+j_2+2$	$D_{+\dot{+}}$, Q, $\tilde{Q}$, λ^1_{+}, $\tilde{\lambda}^1_{\dot{+}}$

- $\hat{\mathcal{B}}_R$: Higgs branch chiral ring operators
- $\mathcal{D}_{R(0,j_2)}/\bar{\mathcal{D}}_{R(j_1,0)}$: Additional $\mathcal{N}=1$ (anti-)chiral ring operators. "Hall-Littlewood" chiral ring.
- $\hat{\mathcal{C}}_{R(j_1,j_2)}$: Other less familiar semi-short operators. $\hat{\mathcal{C}}_{0(0,0)}$ is the stress-tensor multiplet, also containing R-symmetry currents.
- Coulomb branch $\frac{1}{2}$ BPS operators (such as $\operatorname{Tr}\phi^k$) not Schur.

The VOA associated to free SCFTs

- Free hypermultiplet $\longrightarrow$ a pair of symplectic bosons $q(z), \tilde{q}(z)$,

$$q(z)\,\tilde{q}(0) \sim \frac{1}{z}\,, \quad c_{2d} = -1\,.$$

- Free vector multiplet $\longrightarrow$ bc ghost system of weights (1, 0),

$$b(z)c(0) \sim \frac{1}{z}\,, \quad c_{2d} = -2\,.$$

Gauging prescription

Start with $4d$ SCFT $\mathcal{T}$, with flavor symmetry G_F.

We can generate a new SCFT $\mathcal{T}_G$ by gauging $G \subset G_F$, provided $\beta_G = 0$.

If we already know the VOA $\chi[\mathcal{T}]$, can we find $\chi[\mathcal{T}_G]$?

Extra $4d$ vector multiplet $\Rightarrow$ extra $(b^A c_A)$ ghost system, in the adjoint of G.
We must also restrict to gauge singlets.
This is the correct answer at zero gauge coupling. But at finite coupling, some states are lifted and the VOA must be smaller.

Elegant prescription to find quantum chiral algebra. Pass to the cohomology of

$$Q_{\mathrm{BRST}} := \oint \frac{dz}{2\pi i}\, j_{\mathrm{BRST}}(z)\,, \quad j_{\mathrm{BRST}} := c_A \left[J^A - \frac{1}{2} f^{AB}{}_C\, c_B b^C \right],$$

where J^A is the G affine current of $\chi[\mathcal{T}]$.

$Q^2_{\mathrm{BRST}} = 0$ precisely when the $\beta_G = 0$, which amounts to $k_{2d} = -2h^\vee$.

Structural properties

- Both Bose and Fermi statistics are allowed, so we have a vertex operator *super* algebra.
- Virasoro enhancement of $\mathfrak{sl}(2)$, with $T(z)$ arising from a component of the $SU(2)_R$ conserved current, $T(z) := [\mathcal{J}_R(z, \bar{z})]_{\mathbb{Q}}$, with

$$c_{2d} = -12\, c_{4d} \,,$$

  where c_{4d} is one of the conformal anomaly coefficient.
  So we have a *conformal* VOA.
- The VOA is either $\mathbb{Z}$ graded or $\mathbb{Z}/2$ graded by L_0, but there is *no* spin-statistics connection.
- The graded components of $\mathcal{V}$ are finite-dimensional.

NB: I will use physics notations for labelling modes,

$$a(z) \equiv Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-h_a} \,,$$

- Affine symmetry enhancement of global flavor symmetry, with $J(z)$ arising from the moment map operator, $J(z) := [M(z,\bar{z})]_{\mathbb{Q}}$, with

$$k_{2d} = -\frac{k_{4d}}{2}\,.$$

- Generators of $\mathcal{R}_H \Rightarrow$ (strong) generators of $\mathcal{V}$.
  Higgs branch relations encoded in null states of the chiral algebra!
  (Crucial that k_{2d}, c_{2d} take special negative values).

In several examples however, there are *additional* generators unrelated to the Higgs branch. Indeed HL chiral ring generators are always generators. The stress tensor is *usually* a generator (except when it is given by the Sugawara construction). In some examples, $\hat{C}$ multiplets give further generators.

Can one distill $\mathcal{R}_H$ (and hence $\mathcal{M}_H$) from the VOA?

Higgs branch reconstruction conjecture

There is a natural commutative algebra associated to $\mathcal{V}$, the "C_2-algebra" $\mathcal{R}_{\mathcal{V}}$.

The C_2 subspace is defined as

$$C_2[\mathcal{V}] := \mathrm{Span}\{S^i_{-h^i-1}\phi\,,\ \phi \in \mathcal{V}\}\,,$$

where h_i is the dimension of S^i. Then

$$\mathcal{R}_{\mathcal{V}} := \mathcal{V}/C_2[\mathcal{V}]\,.$$

with the natural commutative product. $\mathcal{R}_{\mathcal{V}}$ has also a Poisson structure, but it is *not* in general reduced.

Our conjecture (C. Beem, LR)

$$\mathcal{R}_H = (\mathcal{R}_{\mathcal{V}})_{\mathrm{red}}\ \ .$$

This means that $\mathcal{M}_H$ is the "associated variety" of $\mathcal{V}$ defined by Arakawa. Highly non-trivial, true in all examples.

Modular differential equations for the Schur index

Immediate consequence of the conjecture

$$(L_{-2})^r \Omega \quad \in \quad C_2(\mathcal{V})$$

so the Verma module must have a null vector of the form

$$\mathcal{N} = (L_{[-2]})^r \Omega - v \; , \qquad v \in C_{[2]}(\mathcal{V}) \; .$$

The zero mode of $\mathcal{N}$ must vanish in the torus partition function

$$\mathrm{STr}_{\mathcal{V}} \left(o(\mathcal{N}) q^{L_0 - \frac{c}{24}} \right) = 0 \; .$$

Under additional technical assumption (which we cannot prove in general), this leads to a monic, holomorphic (twisted) modular differential equation for the vacuum character of $\mathcal{V}$ (the Schur index $\mathcal{I}(q)$ of the SCFT).

We distinguish an untwisted case ($\mathbb{Z}$ grading), which must be covariant under the full modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, and and a twisted case ($\mathbb{Z}/2$ grading), covariant under the congruence subgroup $\Gamma^0(2)$,

$$\Gamma^0(2) \equiv \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \ , \qquad b \equiv 0 \mod 2 \right\}$$

The differential operator takes the form

$$\mathcal{D}^{(k)} \equiv D_q^{(k)} + \sum_{r=2}^{k} f_r(q) D_q^{(k-r)} \ ,$$

$$f_r(q) \in M_{2r}(\Gamma, \mathbb{C}) \ \ \text{or} \ \ f_r(q) \in M_{2r}(\Gamma^0(2), \mathbb{C})$$

Here

$$D_q^{(k)} f(q) \equiv \partial_{(2k-2)} \circ \cdots \circ \partial_{(2)} \circ \partial_{(0)} f(q) \ .$$

where $\partial_{(k)}$ are Serre derivatives mapping modular forms of weight k to modular forms of weight $k+2$,

$$\partial_{(k)} f(q) = \left(q\partial_q - \frac{k}{12} E_2(q) \right) f(q) \ .$$

A corollary is that the Schur index $\mathcal{I}(q)$ should transform as an element of a vector-valued modular form.

This allows to control its $q = e^{-\beta} \to 1$ limit in terms of $h_{\min}$, the smallest weight that enters the vector valued form,

$$\log \mathcal{I}(q) \sim \frac{\pi c_{\text{eff}}}{12} \frac{2\pi}{\beta}\,, \quad c_{\text{eff}} = c - 24 h_{\min}\,.$$

Using the formalism of Di Pietro and Komargodski, one can in turn relate this asymptotic to the $4d$ anomaly coefficients. All in all,

$$h_{\min} = -\frac{5}{2} c_{4d} + 2 a_{4d}\ .$$

This allows to compute a_{4d} entirely from the VOA data.

We have found the expected modular differential equation in all examples that we have checked.

Deligne exception series

Embedding the VOA into the full theory leads to novel unitarity bounds for central charges. Beem Lemos Liendo Peelaers LR van Rees, Lemos Liendo

$$k_{4d}(-180c_{4d}^2 + 66c_{4d} + 3\dim_G) + 60c_{4d}h^\vee - 22c_{4d}h^\vee \leqslant 0$$

$$\frac{1}{k_{4d}} \leqslant \frac{12c_{4d} + \dim_G}{24c_{4d}h^\vee}$$

G	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$	$\mathfrak{g}_2$	$\mathfrak{so}(8)$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
c_{4d}	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{7}{6}$	$\frac{5}{3}$	$\frac{13}{6}$	$\frac{19}{6}$	$\frac{31}{6}$
k_{4d}	$\frac{8}{3}$	3	$\frac{10}{3}$	4	5	6	8	12

Bounds simultaneously saturated by rank one SCFTs of type $H_1, H_2, D_4, E_6, E_7, E_8$, whose Higgs branches are one-instanton moduli spaces. We also find G_2, F_4: they complete the Deligne exceptional series of Lie algebras. Joseph relations of the moment map

$$(M \otimes M)|_{\mathcal{I}_2} = 0 \ , \qquad \mathrm{Sym}^2(\mathbf{adj}) = (2\,\mathbf{adj}) \oplus \mathcal{I}_2 \ .$$

When bounds are saturated, extra nulls in the affine Lie algebra: needed for correct Higgs branch and for $L_{-2}^2\Omega \in C_2$.

The corresponding characters obey a uniform second order modular differential equation Beem LR

$$\left(D^{(2)} - \frac{(h^\vee + 1)(h^\vee - 1)}{144} E_4(q)\right) \mathcal{I}(q) = 0 \ .$$

$\mathfrak{g}$	$h^\vee$	k_{2d} $(\frac{-h^\vee-6}{6})$	c_{2d} $(-2-2h^\vee)$	h_{min} $(-\frac{h^\vee}{6})$	a_{4d} $(\frac{5+3h^\vee}{24})$	r_{Coulomb}
$\mathfrak{a}_0$	$\frac{6}{5}$	$-\frac{6}{5}$	$-\frac{22}{5}$	$-\frac{1}{5}$	$\frac{43}{120}$	$\frac{6}{5}$
$\mathfrak{a}_1$	2	$-\frac{4}{3}$	-6	$-\frac{1}{3}$	$\frac{11}{24}$	$\frac{4}{3}$
$\mathfrak{a}_2$	3	$-\frac{3}{2}$	-8	$-\frac{1}{2}$	$\frac{7}{12}$	$\frac{3}{2}$
$\mathfrak{g}_2$	4	$-\frac{5}{3}$	-10	$-\frac{2}{3}$	$\frac{17}{24}$	$\frac{5}{3}$ (?)
$\mathfrak{d}_4$	6	-2	-14	-1	$\frac{23}{24}$	2
$\mathfrak{f}_4$	9	$-\frac{5}{2}$	-20	$-\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{2}$ (?)
$\mathfrak{e}_6$	12	-3	-26	-2	$\frac{13}{8}$	3
$\mathfrak{e}_7$	18	-4	-38	-3	$\frac{59}{24}$	4
$\mathfrak{e}_8$	30	-6	-62	-5	$\frac{95}{24}$	6

Argyres-Douglas theories, I

- (A_1, A_{2n}) : trivial Higgs branch.
  Associated $\mathcal{V}$ conjectured to be the $(2, 2n+3)$ Virasoro algebras
  Beem LR
  $\mathrm{Vir}_{p,q}$ has a null state over the vacuum at level $(p-1)(q-1)$, so

  $$\mathcal{R}_{\mathrm{Vir}_{(p,q)}} = \mathbb{C}[x]/\langle x^{\frac{1}{2}(p-1)(q-1)} \rangle \,.$$

  The reduced ring is trivial

  $$\left(\mathcal{R}_{\mathrm{Vir}_{(p,q)}}\right)_{\mathrm{red}} \cong \mathbb{C} \,,$$

  and the associated variety is just a point.

Argyres-Douglas theories, II

- (A_1, D_{2n+1}) theories:
  $\mathbb{C}^2/\mathbb{Z}_2$ Higgs branches.

  Associated VOAs conjectured to be $\widehat{\mathfrak{su}(2)}_{\frac{-4n}{2n+1}}$ Cordova Shao

  We find a modular differential equation of order $n+1$.

- (A_1, A_{2n-1}) theories:

$$\mathcal{M}_H(A_1, A_{2n-1}) \cong \mathbb{C}^2/\mathbb{Z}_n \ .$$

  Associated VOAs are conjectured to be generalized Bershadsky-Polyakov algebras.

A_1 generalized quivers

$\mathcal{C}_{g,s}$	Order of Differential Operator	Invariance Group	Indicial roots
$\mathcal{C}_{0,3}$	1	$\Gamma^0(2)$	(-4)
$\mathcal{C}_{0,4}$	2	Γ	$(-14, 10)$
$\mathcal{C}_{0,5}$	4	$\Gamma^0(2)$	$(-24, 0_3)$
$\mathcal{C}_{0,6}$	6	Γ	$(-34, -10_3, 14)$
$\mathcal{C}_{0,7}$	13	$\Gamma^0(2)$	$(-44, -20_3, 4_5, \star_4)$
$\mathcal{C}_{0,8}$	16	Γ	$(-54, -30_4, -6_6, 18, \star_4)$
$\mathcal{C}_{1,1}$	2	$\Gamma^0(2)$	$(-3, 9)$
$\mathcal{C}_{1,2}$	4	Γ	$(-20, -12, 4_2)$
$\mathcal{C}_{1,3}$	6	$\Gamma^0(2)$	$(-30, -18, -6_3, 6)$
$\mathcal{C}_{1,4}$	9	Γ	$(-40_2, -16_5, 8_2)$
$\mathcal{C}_{2,0}$	6	Γ	$(-26_2, -2_4)$
$\mathcal{C}_{2,1}$	11	$\Gamma^0(2)$	$(-36, -12_3, 0_4, \star_3)$

Summary of modular differential operators that annihilate the Schur index of class $\mathcal{S}$ theories of type A_1.

$\mathcal{N} = 4$ SYM

$$\mathcal{D}^{\mathfrak{su}(2)} = D^{(2)} - \frac{(\theta_2(q)^4 + \theta_3(q)^4)}{6} D^{(1)} - \frac{3(\theta_2(q)^4 - \theta_3(q)^4)^2}{32} D^{(0)} ,$$

$$\mathcal{D}^{\mathfrak{su}(3)} = D^{(4)} - \frac{11}{36} E_4(q) D^{(2)} - \frac{5}{216} E_6(q) D^{(1)} .$$

Many open questions

- "Experimental" mathematics, but no theory yet that would *e.g.* predict the order of the modular differential equation
- Physical interpretation of other elements of the vector valued modular form? Natural to guess BPS defects, but in fact impossible to preserve $\mathbb{Q}$.
- Very intriguing, still mysterious connection with the BPS index defined on the Coulomb branch from wall-crossing invariant data. Experimentally

  $$\mathcal{I}_{\rm Schur} \cong (q)_\infty^{2\,\rm rank}\, {\rm Tr}\, \mathcal{O}(q)$$

  $\mathcal{O}(q)$ a certain order product of quantum Kontsevith-Soibelman factors. Cecotti Nietzke Vafa, Cordova Shao, Cecotti Son Vafa Yan, Cordova Gaiotto Shao
- Retaining the nulls, Higgs variety → Higgs scheme (?)
- What is the correct geometric framework to incorporate $\mathcal{D}$ multiplets?
- ......

Conclusion

The operator-algebraic viewpoint inspired by the conformal bootstrap is very fruitful.

Extremely rich structure already in a small protected subsector.

We can look forward to the new physical mathematics that will allow us to describe the full operator algebra.