Period integrals of algebraic manifolds and their differential equations

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Based on recent joint works with
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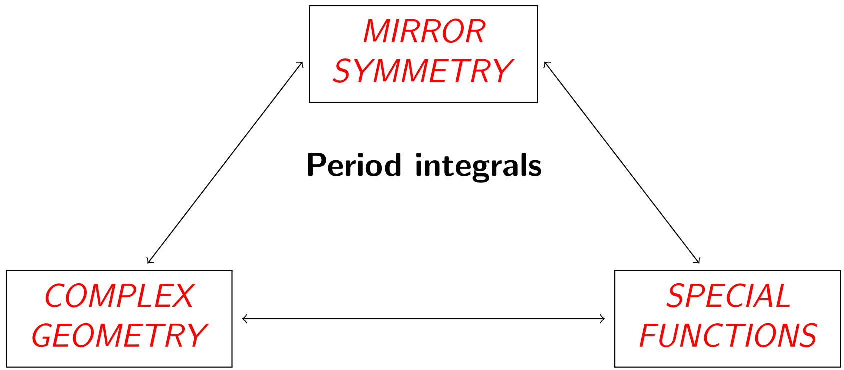
2. **Outline**

- Brief overview: classical theory of hypergeometric functions and elliptic integrals.
- Physics: mirror symmetry.
- Riemann-Hilbert problem for period integrals.
- Introduction to tautological systems.
- D-module description of tautological systems.
- Some applications.

The big picture

This will be a study on the interplay between



Let's begin with special functions ...

3. **Special function?**

Loosely defined, a special function is a local solution to a system of linear PDEs with **polynomial** coefficients in $\mathbb{C}^n$.

E.g. $\sin(z)$, $\cos(z)$, e^z, z^α, $\log(z)$,...

John Wallis first coined the term "**hypergeometric series**" in his book *Arithmetica Infinitorum* (1655) and gave the example

$(1+z)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k$.

But without further restrictions, there does not appear to be a coherent theory...

4. Euler-Gauss hypergeometric equation

- The EG hypergeometric equation is the ODE defined on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$:

$$z(1-z)\frac{d^2}{dz^2} + [c - (a+b+1)z]\,\frac{d}{dz} - ab = 0$$

  where $a, b, c \in \mathbb{C}$ are fixed parameters.
- Every second-order (homogeneous) linear ODE on $\mathbb{P}^1$ with three regular singular points can be transformed into this equation.

5. Euler-Gauss hypergeometric functions

- A EG hypergeometric function is a local solution to this equation. For $c \notin \mathbb{Z}_{\leq 0}$, around $z = 0$, it has a power series solution of the form

$$ {}_2F_1(a, b, c; z) := \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. $$

  Here $(\alpha)_n = \prod_{k=0}^{n-1} (\alpha + k) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.
- ${}_2F_1(a, b, c; z)$ has radius of convergence 1.
- Any other local solution can be obtained by analytic continuation of ${}_2F_1(a, b, c; z)$.

6. From geometry to EG functions

Figure: Portrait of Adrien-Marie Legendre (1752-1833) by Julien-Leopold Boilly

The first connection to geometry of the hypergeometric functions was attributed to Legendre, through the theory of elliptic integrals.

7. From geometry to EG functions

The Legendre family of elliptic curves:

$$Y_\lambda : \; y^2 = x(x-1)(x-\lambda), \; (x,y) \equiv [x,y,1] \in \mathbb{P}^2$$

parameterized by $\lambda \in B := \mathbb{C} - \{0,1\}$.

For $\lambda \in B$,

$$Y_\lambda \simeq^{homeo.} T^2.$$

For a given $\lambda_0 \in B$, we also have canonical identification

$$H^1(Y_\lambda, \mathbb{C}) \equiv H^1(Y_{\lambda_0}, \mathbb{C}) \equiv H^1(T, \mathbb{C}) \cong \mathbb{C}^2$$

if λ varies in any contractible neighborhood U of λ_0.

8. From geometry to EG functions

The 1-form

$$\omega_\lambda := \frac{dx}{y}$$

is holomorphic on Y_λ, hence defines a (nonzero) cohomology class

$$[\omega_\lambda] \in H^1(T, \mathbb{C}) \equiv \mathbb{C}^2.$$

This vector varies holomorphically with $\lambda \in B$. In other words, it gives a canonical nonvanishing section of the flat bundle $R^1\pi_*\mathbb{C}$ over B, whose fibers are $H^0(Y_\lambda, \mathbb{C})$.

9. Period integrals

Fix a basis $\gamma_1, \gamma_2 \in H_1(T, \mathbb{Z}) = H^1(T, \mathbb{Z})^*$. Then

$$[\omega_\lambda] = \gamma_1^* \langle \gamma_1^*, \omega_\lambda \rangle + \gamma_2^* \langle \gamma_2^*, \omega_\lambda \rangle = \gamma_1^* \int_{\gamma_1} \omega_\lambda + \gamma_2^* \int_{\gamma_2} \omega_\lambda.$$

The coefficient functions $\int_{\gamma_i} \omega_\lambda \in \mathcal{O}_B(U)$ are called period integrals of the family Y_λ.

Remark: Even though they are defined locally on B, these integrals admit (multi-valued) analytic continuations along any path in B. Therefore the period integrals generate a local system on B.

10. Differential equations for period integrals

Proposition: The period integrals are precisely the solutions to the EG equation with parameters $a = b = \frac{1}{2}$, $c = 1$:

$$\mathcal{L}\varphi := \lambda(1-\lambda)\frac{d^2}{d\lambda^2}\varphi + (1-2\lambda)\frac{d}{d\lambda}\varphi - \frac{1}{4}\varphi = 0.$$

Proof. Check that

$$\mathcal{L}\omega_\lambda = \left(\frac{\partial}{\partial x}\frac{(x-1)^2x^2}{2y^3}\right)dx$$

Right side is an exact 1-form on Y_λ-finite set.

It follows that

$$\mathcal{L}\int_{\gamma_i}\omega_\lambda = \int_{\gamma_i}\mathcal{L}\omega_\lambda = 0$$

by Stoke's theorem. □

11. Computing period integrals

- The exercise effectively reduces an integration problem to a 2nd order ODE problem, which further reduces to one of determining the ‘initial conditions’.
- We can now use geometry of the curves Y_λ to solve this problem.
- For example, at $\lambda = 0$, the curve Y_λ develops a node. With a little more work - basically by studying how the form ω_λ develops a pole when $\lambda = 0$, we can determine the two constants as follows.

12. **Computing period integrals**

- If γ_1 is the basic 1-cycle on Y_0 that avoids the node, then the solution must be regular in λ. Thus

$$\int_{\gamma_1} \omega_\lambda = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, \lambda).$$

- If γ_2 is the basic 1-cycle that runs through the node, then it can be shown that the solution must have a leading log singularity. This gives

$$\int_{\gamma_2} \omega_\lambda = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, \lambda) \log \lambda + g_1(\lambda)$$

  where $g_1(\lambda)$ is a unique power series determined by the EG equation.
- Conversely, we can think of the EG hypergeometric functions as a way to parameterize variation of the vector $[\omega_\lambda]$ ('the complex structure') for the curve Y_λ.

13. **Integration problem vs. geometry**

- The EG hypergeometric equation reduces the (elliptic) integration problem

$$\int_{\gamma_i} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

to an ODE problem.

- The Legendre family gives a geometric interpretation of the integrals, and allows determination of initial conditions of the EG equation.

14. **Remarks**

- There is a similar story for hyper-elliptic integrals (Euler)

$$\int_\gamma \frac{x^k dx}{\sqrt{Q(x)}}$$

  where $Q(x)$ is square free polynomial.
- This interplay between special integrals and geometry will be the spirit in which we proceed to study
  higher dimensional analogues of elliptic integrals.

15. **Remarks**

- Consideration of other special functions (often with physics motivations) have led to development of more general hypergeometric functions: Kummer, Legendre, Hermit, Bessel, H. Schwarz, Pochammer, Appell,...
- Modern theory (1990's): Gel'fand school initiated a systematic study of hypergeometric functions of several variables.
- In parallel, consideration of period integrals have also led to development of modern Hodge theory: Riemann, Hodge, Griffiths, Schmid, Simpson,...

16. Period integrals from Physics

- Mirror Symmetry Postulate: For a given CY threefold Y (i.e. $K_Y \simeq \mathcal{O}_Y$), there should be another CY threefold $\tilde{Y}$, such that the 'type IIA' physics on Y is equivalent to the 'type IIB' physics on $\tilde{Y}$.
- This is highly nontrivial (if true) because Dine-Wen-Witten have shown that 3-point functions of type IIA can receive quantum corrections from worldsheet instantons in every degree. They can be thought of as counting holomorphic curves in a given homology class in Y.
- At the time of the discovery, worldsheet instantons are considered completely intractable ...
- Type IIB 3-point functions do not have instanton corrections, and are believed to be computable, at least in principle ...

17. Type IIB 3-point functions

- Given a CY3 Y and 3 complex moduli tangent directions a, b, c of Y, the (unnormalized) 3-point function is:

$$\langle a, b, c \rangle := \int_Y \nabla_a \nabla_b \nabla_c \Omega \wedge \Omega$$

where Ω = hol. 3-form on Y, and ∇ is the GM connection.

- One way to locally parameterize the moduli dependence of Ω: choose homology basis of 3-cycles γ on Y and write

$$\Omega = \sum_\gamma \omega_\gamma \gamma \quad \text{where} \quad \omega_\gamma = \int_\gamma \Omega$$

are the **period integrals** of Y.

- Thus all 3-point functions can be trivially written as combinations of the functions ω_γ.

18. **The First Example of Mirror Symmetry**

After Candelas-de la Ossa-Green-Parkes and Greene-Plesser...

- Consider the family of quintic hypersurfaces:

$$Y_\psi : \quad f(\psi) := x_0^5 + \cdots + x_4^5 + 5\psi x_0 \cdots x_4 = 0, \ \psi \in B \subset \mathbb{C}.$$

- $\Gamma = (\mathbb{Z}/5\mathbb{Z})^4$ acting by scaling leaves each Y_ψ invariant. Each "orbifold" Y_ψ/Γ resolves to a CY threefold $\tilde{Y}_\psi$ called the mirror quintic of Y_ψ.
- The Poincaré residue

$$\Omega = Res \frac{1}{f(\psi)} \sum_{i=0}^{4} (-1)^{i+1} x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_4.$$

gives a canonical holomorphic 3-form on $\tilde{Y}_\psi$.

19. Computing instantons by period integrals

- The period integrals $\int_\gamma \Omega$ of the family of mirror quintics are exactly the solutions to the ODE operator

$$\theta^4 + 5z(5\theta + 1) \cdots (5\theta + 4)$$

  where $\theta := z\frac{d}{dz},\ z := (5\psi)^{-5}$.
- The singular point $z = 0$ is called the large complex structure limit of the family. This is where instanton computation can be carried out explicitly.

20. **Computing instantons by period integrals**

- Candelas et al computed the 3-point function $\langle \partial_z, \partial_z, \partial_z \rangle$ near $z = 0$, constructed the explicit mirror map in terms of the period integrals, and compute all genus 0 instantons. They gave the famous predictions that the number of rational curves of each degree in Y are

$$2875, \ 609\,250, \ 317\,206\,375, \cdots$$

- They were later shown to be in complete agreement with counting by algebraic geometry (Givental, Lian-Liu-Yau).
- Physicists Bershadsky-Cecotti-Ooguri-Vafa has also generalized the instanton predictions for curves of all genera. This was partially confirmed by algebraic geometry only recently (J. Li, A. Zinger).

21. **Higher dimensions**

Let B connected complex manifold (parameter space).

Let $E \to B$ be a vector bundle equipped with a flat connection

$$\nabla : \mathcal{O}(E) \to \mathcal{O}(E) \otimes \Omega^1_B.$$

Let

$$\langle\ ,\ \rangle : \mathcal{O}(E) \otimes \mathcal{O}(E^*) \to \mathcal{O}_B$$

be the usual pairing.

Fix global section $s^* \in \Gamma(B, E^*)$.

Definition: The **period sheaf**

$$\mathbf{\Pi} \equiv \mathbf{\Pi}(E, s^*) \subset \mathcal{O}_B$$

is the image of the map

$$\mathcal{O}(E) \supset \ker \nabla \to \mathcal{O}_B,\ \ \gamma \mapsto \langle \gamma, s^* \rangle.$$

22. **Period sheaves from Complex Geometry**

Let $\pi : \mathcal{Y} \to B$ be a family of d-dimensional compact complex manifolds, with $Y_b := \pi^{-1}(b)$.

The cohomology groups of fibers $H^k(Y_b, \mathbb{C})$ form the vector bundle $E^* := R^k\pi_*\mathbb{C}$ over B; dual bundle $E = E^{**}$ has fibers $H_k(Y_b, \mathbb{C})$, and

$$\langle\ ,\ \rangle : \mathcal{O}(E) \otimes \mathcal{O}(E^*) \to \mathcal{O}_B$$

is the Poincaré pairing; E is equipped with the GM connection ∇.

Fix $s^* \in \Gamma(B, E^*)$, and represent $s^*(b) \in H^k(Y_b, \mathbb{C})$ by a closed form on Y_b. Represent section $\gamma \in \ker \nabla$ by cycle on Y_b. So, a section $f \in \mathbf{\Pi}(U)$ is an integral

$$f(b) = \langle \gamma, s^*(b) \rangle = \int_\gamma s^*(b).$$

We call this a **period integral** of $\mathcal{Y}$ with respect to s^*.

23. **Problem**

Fix a compact Kähler manifold X^{d+1}, and assume

$$\pi : \mathcal{Y} \to B$$

is a family of hypersurfaces in X. Consider the associated flat bundle $E^* = R^d\pi_*\mathbb{C}$.

The subspaces

$$\Gamma(Y_b, K_{Y_b}) \subset H^d(Y_b, \mathbb{C}).$$

form a subbundle $\mathbb{H}^{top} \subset E^*$.

Theorem: (Lian-Yau) If the Y_b are CY, then the line bundle $\mathbb{H}^{top}$ admits a canonical trivializing section; we denote it by Ω.

24. Riemann-Hilbert problem for period integrals

Problem: Construct an explicit *complete* system τ of linear partial differential equations for the sheaf $\mathbf{\Pi}(E, \Omega)$. That is a system τ whose analytic solutions are exactly the period integrals

$$\int_\gamma \Omega \in \mathbf{\Pi}(E, \Omega).$$

General goals & applications:

- To compute explicitly periods $\int_\gamma \Omega$ as power series
- To understand local monodromy of periods
- To count curves in algebraic varieties by (homological) mirror symmetry
- To construct interesting new classes of special functions

25. **Explicit realizations and applications**

- Physics: computing instanton corrections (of all genera) using period integrals requires explicit power series solutions to their differential equations: Candelas, de la Ossa, Green, Parkes, Greene, Hosono, Katz, Klemm, Morrison, Plesser, Theisen, Yau, etc.
- Monodromy problem: explicit differential equations can be used to compute local monodromy of period integrals near singularities. Examples: Kontsevich and Beukers conjecture explicit formulas for monodromy operators for toric hypersurfaces.
- Degenerations of CY manifolds: to construct certain degenerations of CY manifolds. For examples, large complex structure limit points with maximally unipotent monodromy: Bloch, Candelas, de la Ossa, Green, Parkes, Hosono, Huang, Katz, Lian, Morrison, Srinivas, Yau, etc. Other singularities include: conifold and orbifold points.

26. **Approaches to the RH Problem**

- One approach is by the so-called Riemann-Hilbert correspondence (between regular holonomic D-modules and perverse sheaves) of Deligne, Kashiwara and Mebkhout. However, it's unclear how to give explicit description of the differential equations this way, let alone solving them.
- Another approach is by direct construction using the reduction-of-poles method of Dwork-Griffiths. It works well for 1-parameter families of hypersurfaces.
- **Example:** The Legendre family of elliptic curves in $\mathbb{P}^2$. More generally, Lefschetz pencils of hypersurfaces of the form

$$f + tg = 0, \quad t \in \mathbb{P}^1$$

in $\mathbb{P}^{d+1}$, where $f = 0$ is smooth.

27. **Approaches to the RH Problem**

- For multi-parameter families, it is much more difficult to apply the reduction-of-pole method.
- For given a candidate PDE system for $\mathbf{\Pi}(E, \Omega)$, the hard part is to decide completeness:

When does the solution sheaf coincide with the period sheaf?

- We will consider a recently developed approach – the '*symmetry-constraints method*' – that provides a more explicit realization of differential equations.
- We will also study the completeness problem for systems constructed by this approach.

28. RH Problem for Period Integrals: Toric hypersurfaces

For X a toric manifold with $c_1(X) \geq 0$, and $\pi : \mathcal{Y} \to B$ is the universal family of CY hypersurfaces in X, both a PDE system and its solutions are known explicitly. Let's briefly review this.

- Let $T \subset X$ be the dense torus, $\hat{\mathfrak{t}}$ the Lie algebra of $T \times \mathbb{C}^\times$. Then T action on X induces a representation

$$\hat{\mathfrak{t}} \to End\ H^0(K_X^{-1}),\ y \mapsto Z_y.$$

- Let $\beta : \hat{\mathfrak{t}} \to \mathbb{C}$ be the character $\beta(\mathfrak{t}, 0) = 0$, $\beta(0, 1) = 1$.
- There is a canonical basis of $H^0(K_X^{-1})$ given by Laurent monomials $x^{\mu_0}, .., x^{\mu_p}$, where $\mu_0, .., \mu_p \in \mathbb{Z}^n$ are T-characters.

29. **Toric hypersurfaces**

- Period integrals of the CY family $\mathcal{Y}$ in X are solutions to the following Gel'fand-Kapranov-Zelevinski PDE system

$$\Box_l \varphi = 0, \; (Z_y + \beta(y))\varphi = 0, \; y \in \hat{\mathfrak{t}}$$

where $l \in \mathbb{Z}^{p+1}$ satisfies $\sum_i l_i \mu_i = 0$, $\sum_i l_i = 0$, and

$$\Box_l := \prod_{l_i > 0} (\frac{\partial}{\partial a_i})^{l_i} - \prod_{l_i < 0} (\frac{\partial}{\partial a_i})^{-l_i}.$$

- This GKZ system is regular holonomic, hence every solution is analytic, and the solution sheaf has finite rank.
- This GKZ system is *never* complete - solution sheaf always larger than period sheaf.

30. **Toric hypersurfaces**

- The point $a_0 = \infty$ is a large complex structure limit (LCSL) point of this family (Hosono, Lian, Yau). The degenerate CY Y_∞ =union of all T-invariant toric divisors in X.
- Explicit power series formula near $a_0 = \infty$ can be given in terms of the "mirror toric manifold" $X^\vee$: there is combinatorially defined function

$$B : \{|a_0| >> 1\} \to H^*(X^\vee)$$

such that the solution sheaf of the GKZ system is generated by $\langle B, \alpha \rangle$, $\alpha \in H_*(X^\vee)$.

31. B-series

Theorem [Hosono-Lian-Yau 1995]: Near the LCSL point $a_0 = \infty$ in $\bar{B}$, the complete set of solutions to the GKZ system is given by the following cohomology valued function:

$$B_X(t) = a_0^{-1} \sum_{d \in M_+} O_d \times \prod_{k=1}^{\langle [Y^\vee], d \rangle} ([Y^\vee] + k) e^{\langle d, t \rangle + t} \in H^*(X^\vee).$$

$t = \sum_i (\log a_i) D_i \in H^2(X^\vee)$ where $f(a) = \sum_i a_i x^{\mu_i}$ is the defining section of the CY hypersurface Y_f.

32. **Notations**

$X^\vee$: the mirror toric manifold, combinatorially associated to X (in the sense of Batyrev.)

$[Y^\vee] \in H^2(X^\vee, \mathbb{Z})$: the anticanonical class of $X^\vee$.

$M_+ \subset H_2(X^\vee, \mathbb{Z})$: the set of nonzero integral points in the Mori cone of $X^\vee$, i.e. the cone dual to the Kähler cone of $X^\vee$.

D_i: the invariant toric divisors in $X^\vee$.

$$O_d := \frac{\prod_{\langle D_i, d\rangle < 0} \prod_{k=0}^{-\langle D_i, d\rangle - 1} (D_i + k)}{\prod_{\langle D_i, d\rangle \geq 0} \prod_{k=1}^{\langle D_i, d\rangle} (D_i - k)}.$$

Can be shown that $B_X(t)$ converges for $|a_0| >> 1$;
$a_0 = \infty$ corresponds to a LCSL point.

33. **Remarks**

- This B-series above comes from a generalization of the classical Frobenius method for solving linear ODEs. This method was systematically developed for GKZ systems for CY complete intersections by Hosono, Klemm, Lian, Theisen, Yau, in a series of papers in 1993-1995. Here the toric divisors D_i of the mirror manifold $X^\vee$ play the role of the "Frobenius parameters".
- Stienstra has also considered solutions for more general GKZ systems (1998.)
- The formula above has been generalized to certain noncompact toric manifold by S.Li-Lian-Yau (2009.)

34. Example

- Take X to be the mirror $\mathbb{P}^n$: X is a certain resolution of $\mathbb{P}^n/(\mathbb{Z}_{n+1})^{n+1}$.
- Then our B-series reduces to

$$B_X(t) = a_0^{-1} \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{5d}(5H+k)}{\prod_{k=1}^{d}(H+k)^5} e^{d\langle t,H\rangle+H}$$

  where $H \in H^2(\mathbb{P}^n, \mathbb{Z})$ is the hyperplane generator. In this case H is nothing but the so-called "Frobenius parameter" in classical ODE theory!
- Expanding this in powers of H, the coefficients of $1, H, H^2, H^3$ are exactly the 4 independent periods of the mirror quintic threefolds.
- But the coefficient of H^4 is an extra solution which cannot come from a period, indicating that the GKZ system is incomplete in this case. As we shall see later, this extra solution corresponds to a topological 4−chain in the complement $X - Y$, rather than a 3-cycle in the CY hypersurface $Y \subset X$.

35. **Completeness**

There are two ways to deal with the completeness problem in this case.

- Consider $X = \mathbb{P}^n$. The T action on $H^0(K_X^{-1})$ enlarges to a $PSL_{n+1} \supset T$ action, and we get corresponding

$$Z : sl_{n+1} \oplus \mathbb{C} \to EndH^0(K_X^{-1})$$

  and character with $\beta(sl_{n+1}, 0) = 0$ and $\beta(0, 1) = 1$. We then get an extended GKZ system with the new symmetry group PSL_{n+1}.
- **Theorem** (Bloch,Huang,Lian,Srinivas,Yau): *The extended GKZ system is complete, i.e. its solutions sheaf coincides with the period sheaf of the CY family $\mathcal{Y}$.*
- **Hyperplane Conjecture** (Hosono, Lian, Yau): *Let $[Y^\vee]$ be the canonical class of the mirror toric manifold $X^\vee$. Then the period sheaf of the CY family $\mathcal{Y}$ is precisely the sheaf generated by the functions $\langle B \cup [Y^\vee], \alpha\rangle$, $\alpha \in H_*(X^\vee)$.*
- The hyperplane conjecture has been recently confirmed by Lian and M. Zhu for $X = \mathbb{P}^n$.

36. **Generalizations**

- A few more isolated examples on the RH problem have been constructed 1996-2012.
- For toric hypersurfaces, the GKZ system gives us a candidate PDE system for $\mathbf{\Pi}(E, \Omega)$ that is incomplete. There is a conjecture (the Hyperplane Conjecture) that says how to make the GKZ system complete. More on this later.
- The completeness problem was not decided, even for the quintics in $\mathbb{P}^4$ until 2012.
- Likewise, nothing was known for hypersurfaces in a general flag variety before that.

37. **The Symmetry-Constraints approach – Tautological Systems**

General Idea: First, use *symmetry* of X, and *geometric constraints* of the family $\mathcal{Y}$ to construct a candidate PDE system τ, such that à priori

$$\mathit{sol}(\tau) \supset \mathbf{\Pi}.$$

Then, decide if τ is complete, i.e. if '$\subset$' also holds. Otherwise, describe the obstruction.

38. Set-up

- **Data:**
  G: complex algebraic group
  X: a smooth projective G-variety

- **Notations:**
  $L = K_X^{-1}$
  $V^* := H^0(X, L)$
  $f : V^* \times X \to L, \;\; (b, x) \mapsto b(x)$ universal section of L
  $\pi : \mathcal{Y} \to B$ the universal CY family in X
  $E^* = R^{n-1}\pi_*\mathbb{C}$ local system associated to $\mathcal{Y}$
  $U = V^* \times X \backslash \{f = 0\}$ universal complement

- Recall the Weyl algebra D_{V^*}:=the algebra of polynomial differential operators on V^*. Then

$$D_{V^*} = \mathbb{C}[a_0, .., a_m, \frac{\partial}{\partial a_0}, ..., \frac{\partial}{\partial a_m}]$$

  where $(a_0, .., a_m)$ are linear coordinates on V^*.

39. Set-up

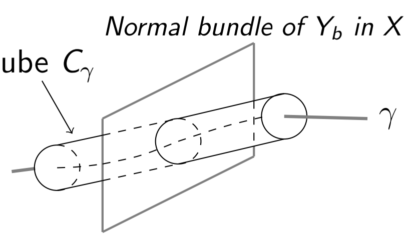
- **Theorem:** [Lian-Yau '12] There is a canonical section of E^* of the form

$$\Omega = Res \frac{\omega}{f}$$

where ω is a certain G-invariant n-form on a $\mathbb{C}^*$-principal bundle on X, and Res is the Leray-Poincaré residue map.

- By the residue formula, periods of $\mathcal{Y}$ (with respect to Ω) are:

$$\int_\gamma \Omega = \int_{C_\gamma} \frac{\omega}{f}.$$



40. Tautological systems

- **Observations:**
  - 1. f^{-1} is a section defined on U, and $\mathcal{O}_U f^{-1}$ is a (cyclic) $D_{V^*}-$module generated by f^{-1}.
  - 2. If $P \in D_{V^*}$ annihilating f^{-1}, then P annihilates the periods:

$$P \cdot \int_{C_\gamma} \frac{\omega}{f} = 0 \quad \text{(Geometric constraints)} .$$

Let $J_X \subset D_{V^*}$ be the annihilating ideal of f^{-1}.

  - 3. Since $G \curvearrowright X$, for any $\xi \in \mathfrak{g} \equiv Lie(G)$, the vector field Z_ξ induced on V^* also annihilates the periods:

$$Z_\xi \cdot \int_{C_\gamma} \frac{\omega}{f} = 0 \quad \text{(Symmetry)} .$$

Let $Z_{\mathfrak{g}}$ be the space of vector fields induced on V^* by $G \curvearrowright X$.

41. **Tautological systems**

- **Definition:** Let $\tau \equiv \tau(X, G)$ be the cyclic D_{V^*}

$$\boxed{\tau = D_{V^*} / \left(J_X + D_{V^*} Z_{\mathfrak{g}}\right).}$$

  This D_{V^*}-module is called a tautological system .

- **Theorem:** [Lian-Song-Yau '11, Lian-Yau '12] Every period integral for the family $\mathcal{Y}$ is an analytic solution to τ.

- This is a consequence of observations 1-3 above.

42. **Explicit algebraic description of τ**

- Since L is very ample, we have a canonical 'tautological' embedding

$$X \hookrightarrow \mathbb{P}V \equiv \mathbb{P}^m$$

  where we identify $\mathbb{C}[V] \equiv \mathbb{C}[a_0^\vee, ..., a_m^\vee]$. Let $I_X \subset \mathbb{C}[V]$ be the vanishing ideal of $X \subset \mathbb{P}^m$.

- Define the Fourier transform of I_X:

$$\tilde{I}_X = \{P(\frac{\partial}{\partial a_0}, .., \frac{\partial}{\partial a_m}) \mid P(a_0^\vee, .., a_m^\vee) \in I_X\} \subset \mathbb{C}[\frac{\partial}{\partial a}].$$

- **Theorem:** (Annihilator of f^{-1})

$$J_X = D_{V^*}\tilde{I}_X + D_{V^*}\left(\sum a_i \frac{\partial}{\partial a_i} + 1\right).$$

43. Basic properties and examples of τ

- τ makes sense if we replace I_X by any G-invariant ideal $I \subset \mathbb{C}[V]$, and twist the vector fields $Z_\mathfrak{g}$ and the Euler vector field by a Lie algebra character. The more general set-up is needed to deal with families of Fano and general type varieties.

- τ is a **regular holonomic** D-module if X has only finite number of G orbits. That is to say, any formal power series solution is analytic; the sheaf of analytic solutions has finite rank at each point.

- τ has **solution rank** which is bounded above by the degree of the tautological embedding $X \hookrightarrow \mathbb{P}V$ if the ring $\mathbb{C}[V]/I_X$ is Cohen-Macaulay.

- **Example:** $X = \mathbb{P}^n$. The annihilator ideal J_X is generated by the differential operators:

$$\frac{\partial}{\partial a_\mu}\frac{\partial}{\partial a_\nu} - \frac{\partial}{\partial a_{\mu'}}\frac{\partial}{\partial a_{\nu'}} \quad (\mu + \nu = \mu' + \nu')$$

$$\sum_\mu a_\mu \frac{\partial}{\partial a_\mu} + 1$$

44. Special cases of τ

- $\tau(X, G)$ specializes to a **GKZ system** [GKZ '90] if X is a toric variety and $G = T$ is the dense torus.

- $\tau(X, G \times G)$ specializes to a **Kapranov system** [Kapranov '97] if X a certain *wonderful compactification* of a reductive algebraic group G, i.e. $X = \overline{\mathbb{P}\rho(G)}$ is the Zariski closure in $\mathbb{P}(\text{End } V)$ under a given G-action $G \hookrightarrow \text{End } V$.

- $\tau(X, G)$ has the following explicit description if X is the projective homogeneous space a semisimple group G: its 'constraints' operators are precisely given by

$$(C_{\mathfrak{g}} - \lambda) \sum_{ij} \mathbb{C} \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j}$$

  $C_{\mathfrak{g}}$ =quadratic Casimir operator of $\mathfrak{g}$ acting on 2nd order differential operators; λ is certain distinguished eigenvalue of $C_{\mathfrak{g}}$.
  This follows from an old result of Kostant.

45. **A Numerical Example:**

$X = \mathbb{P}^2$
$G = PSL_3$
$L = K_X^{-1} = O(3)$
$V^* = Sym^3\ \mathbb{C}^3$

$X \hookrightarrow \mathbb{P}V$ is the Segre embedding, $[z_0, z_1, z_2] \mapsto [z_0^3, z_0^2 z_1, z_0^2 z_2, .., z_2^3]$.
I_X=the quadratic ideal generated by the Veronese binomials

$$\zeta_\alpha \zeta_\beta - \zeta_{\alpha'} \zeta_{\beta'}, \quad \alpha + \beta = \alpha' + \beta', \quad \text{where } \zeta_\alpha = z^\alpha$$

D_{V^*}= the Weyl algebra $\mathbb{C}[a_0, ..., a_9, \frac{\partial}{\partial a_0}, .., \frac{\partial}{\partial a_9}]$.

Then the constraints operators of $\tau(X, G)$ in this case are therefore the Fourier transforms

$$\frac{\partial}{\partial a_\alpha} \frac{\partial}{\partial a_\beta} - \frac{\partial}{\partial a_{\alpha'}} \frac{\partial}{\partial a_{\beta'}}$$

of the Veronese binomials.

46. **Tautological systems**

Generalizations: We can replace the vanishing ideal of contraints I_X of $\tau(X, G)$ by any other G-invariant ideal I in the polynomial ring $\mathbb{C}[V^*]$. This generalization arise in many important geometric settings:

- Period integrals of general type manifolds Y with K_Y very ample;
- Period integrals of Fano manifolds with K_Y^{-1} very ample;
- Differential equations for period mappings.

For now, let's restrict to period integrals of CY manifolds for simplicity. Non-CY will be discussed separately later.

47. **General bound on solutions**

Theorem: [Lian-Song-Yau '11] Suppose X has only finite number of G orbits. Then the tautological system $\tau = \tau(X, G)$ is regular holonomic, i.e. restriction of τ to any smooth curve in V^* has only regular singularity. Moreover, the solution rank is bounded above by the degree of the tautological map $X \to \mathbb{P}V$ if the ring $\mathbb{C}[X]$ is Cohen-Macaulay.

Therefore any formal power series solution is analytic; the sheaf of solutions is a locally constant sheaf of rank $\leq \deg X$ on some open $V^*_{gen} \subset V^*$.

Proof is based partly on result of Brylinski on Fourier transforms, and on argument of Kapranov and Hotta on G-equivariant D-modules.

48. Examples of special functions – old and new

- For X a projective toric variety, and G is taken to be the dense torus $T \subset X$, then $\tau(X, T)$ is a GKZ system. Its solutions are special functions given by the cohomology valued B-series we discussed earlier.
- For X a projective toric variety, and $G = \operatorname{Aut} X$, then $\tau(X, G)$ is the extended GKZ system introduced by Hosono-Lian-Yau '95. Its solutions are special functions given by a monodromy invariant component of the B-series.
- For X a wonderful variety (which is a finite-orbit algebraic compactification of a reductive group G), then $\tau(X, G \times G)$ specializes to the Kapranov systems. Its solutions are special functions which can be given by generalized Euler integrals as shown by Kapranov.
- For X a generalized flag variety of a semisimple group G, $\tau(X, G)$ is a new quadratic system which can be described in terms of the Casimir operator of Kostant. Its solutions form a new class of special functions defined on the irreducible representation space $V^* = H^0(X, K_X^{-1})$.
- Work in progress: construct Euler integral representations of the new special functions.

49. Solution rank of τ – special case

Consider the family of CY hypersurfaces Y_b in X, and put

$$D \equiv D_{V^*}, \; V^* \equiv H^0(X, K_X^{-1}), \; \tau \equiv \tau(X, G).$$

Theorem(Bloch,Huang,Lian,Srinivas,Yau): Let X^n a projective homogeneous G-space such that for $r > 0$, $\mathfrak{g} \otimes \Gamma(X, K_X^{-r}) \twoheadrightarrow \Gamma(X, TX \otimes K_X^{-r})$. Then the solution space of τ at any point $b \in V^*$ is canonically isomorphic to $H_n(X - Y_b)$, where the isomorphism is given by the integral of a canonically defined meromorphic form Ω/f on X, over cycles in $H_n(X - Y_b)$, where Ω is certain canonical form on a principal bundle over X, and f is the universal section of L, that will be explained later.

Remark:

(1) It was conjectured that the statement holds without the surjectivity assumption. Many examples (e.g. Grassmannians) are known to have this property.

(2) Proof is based on a method of Dimca to interpret the de Rham cohomology of the complement, in terms of Lie algebra homology of certain $\mathfrak{g}$-module. Will return to this point later.

50. Solution rank of τ & the completeness problem

Theorem(Huang,Lian,Zhu): Let X^n a projective homogeneous G-space. Then the solution space of τ at any point $b \in V^*$ is canonically isomorphic to $H_n(X - Y_b)$. The proof uses the Riemann-Hilbert correspondence for algebraic

D-modules. Recall that $\Pi(E, \omega) \subset$ solution sheaf of τ.

When do the two sheaves coincide, i.e. when is τ **complete**?

Corollary: τ is complete iff the primitive cohomology $H^n(X)_{prim} = 0$.

Corollary: For $X = \mathbb{P}^n$, $G = PSL_{n+1}$, the system τ is complete.

Remark: This was conjectured by Hosono et al in the 90's.

51. **Algebraic rank formula**

Introduce notations:

Fix a very ample line bundle L over a projective G-variety X, and put

$$R = \oplus_{j=0}^{\infty} H^0(X, L^j)$$

the coordinate ring of X with respect to the tautological embedding $X \hookrightarrow \mathbb{P}V$, $V := H^0(X, L)^*$.

Let

$$Z^* : \hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C} \to \mathrm{End}\, V^*$$

be the dual representation of V.

Let

$$f = \sum_i a_i a_i^* : V^* \times X \to L$$

be the universal section of L as before.

52. Algebraic rank formula

Observation:

(1) The space $R[V]e^f$ has a natural $D_{V^*} = \mathbb{C}[a, \partial]$-module structure given by:

$$a_i \mapsto a_i, \quad \partial_i \mapsto \partial_i.$$

(2) The operators $Z^*(\hat{\mathfrak{g}})$ commute with D_{V^*}, hence $Z^*(\hat{\mathfrak{g}})R[V]e^f$ is a D_{V^*}-submodule of $R[V]e^f$. So one can form the quotient D_{V^*}-module $R[V]e^f/Z^*(\hat{\mathfrak{g}})R[V]e^f$.

Theorem(BHLSY,HLZ): There is a canonical D-module isomorphism

$$\tau \simeq R[V]e^f/Z^*(\hat{\mathfrak{g}})R[V]e^f, \quad 1 \mapsto e^f.$$

- The appearance of e^f was inspired by Borcherds' work on Feynman measures.
- In the special case of GKZ systems, the Lie algebra coinvariant space on the right side (in slightly different form) also appeared in Adolphson's work.

53. Applications

The theorem has many interesting applications.

- **Corollary**(BHLSY,HLZ): Let X be any finite-orbit G-space. Then the space of solutions of the differential system τ at any point $b \in V^*$ is canonically isomorphic to the dual of the Lie algebra coinvariant space:

$$H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})^*.$$

- **Example:** $G = PSL_n$, and $X = \mathbb{P}^{n-1}$. Then $L = K_X^{-1} = \mathcal{O}(n)$. Let $x_1, ..., x_n$ be the homogeneous coordinates of X. Then for generic $b \in V^*$, the monomials

$$x_1^{k_1} \cdots x_n^{k_n} e^{f_b}, \ \ n | \sum k_i, \ \ 0 \leq k_i \leq n-2$$

form a basis of $H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})$.

54. **Applications**

- **Completeness.** Counting the monomials, i.e. the number of integral solutions to $k_1 + ... + k_n = sn, 0 \leq k_i \leq (n-2), 0 \leq s \leq (n-2)$, we find that generically there are exactly

$$\frac{n-1}{n}((n-1)^{n-1} - (-1)^{n-1})$$

solutions to the tautological system τ for the universal CY family in $\mathbb{P}^{n-1}$ above. This proves τ is complete, because the period sheaf of this family has this rank.

- **Explicit solutions** Combined with the *Hyperplane Conjecture* recently proved by Lian-M. Zhu (based on the above completeness result), we now have a complete description of all period integrals of this universal CY family – it is given by the B-series near the LCSL $a_0 = \infty$. [Hosono-Lian-Yau '95]

55.**Applications: mirror symmetry**

- **Constructing LCSL degenerations.** Recall that a LCSL degenerate CY Y_{b_∞} corresponds to $b_\infty \in V^*$, where the local monodromy is maximally unipotent, hence there is just one analytic solution at b_∞. By the rank formula, we have
$$\dim H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_{b_\infty}}) = 1.$$
- **Example.** Consider the degenerate CY $b_\infty = x_1 \cdots x_n = 0$. Then one finds that
$$H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_{b_\infty}}) = \mathbb{C}e^{f_{b_\infty}}.$$
This is the famous LCSL degeneration for the CY family in $\mathbb{P}^{n-1}$, where instanton counting can be done by Mirror Symmetry.

56. Applications: constructing LCSL degenerations

- More generally, for X^n any projective homogenous G-variety and $L = K_X^{-1}$, we have

$$H_n(X - Y_b) \simeq Hom_D(\tau, \mathcal{O}_b^{an}) \simeq H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})^*$$

  for any $b \in V^*$. So, we can construct LCSL by either geometric methods (lhs) or representation theoretic methods (rhs): *look for points $b \in V^*$ where either side is 1 dim.*
- **Detecting rank 1 fibers.** We say that a fiber Y_b has rank 1 if $\dim H_n(X - Y_b) = 1$. Thus to look for LCSL CY, we can look for divisor Y_b in X whose complement has a particular homotopy type.
- **Example.** For $b_\infty = x_1 \cdots x_n = 0$ in $\mathbb{P}^{n-1}$, the complement is homotopic to n-torus.

57. Applications: constructing LCSL degenerations

- (Bloch,Huang,Lian,Srinivas,Yau) For the Grassmannian $X = G(k, n)$, we consider the degenerate CY

$$b_\infty = x_{1\cdots k} x_{2\cdots(k+1)} \cdots x_{n1\cdots(k-1)} = 0$$

where the x_I are the Plücker coordinates. We can compute directly the sl_n coinvariants on the module $Re^{f_{b_\infty}}$:

$$H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_{b_\infty}}) = \mathbb{C}e^{f_{b_\infty}}.$$

Or, we can also compute $H_n(X - Y_{b_\infty})$ topologically by induction on the n components of the divisor Y_{b_∞}, starting from

$$x_{1\cdots k} = 0.$$

58. **Applications: constructing LCSL degenerations**

(Huang,Lian,Zhu): Next, we generalize in two ways.

- First, we can "glue" together lower dimensional rank 1 fibers in smaller Grassmannians to yield rank 1 fibers in an arbitrary (type A) partial flag variety.
- Second, we can construct directly a canonical rank 1 fiber in every projective homogenous variety $X = G/P$ as follows.

59. Applications: constructing LCSL degenerations

- There is a natural stratification of the flag variety G/B, called the Richardson stratification: it is given by intersecting the Schubert cells of a Borel subgroup, with that given by the opposite Borel subgroup.
- **Example:** take $X = \mathbb{P}^1$ as a flag variety for SL_2. Then Schubert stratification gives Schubert cells $\{0\}$ and $\mathbb{P}^1 - \{0\}$. The opposite Borel subgroup gives the Schubert cells $\{\infty\}$ and $\mathbb{P}^1 - \{\infty\}$. The Richardson stratification is given by $\{0\}$, $\{\infty\}$, and $\mathbb{P}^1 - \{0, \infty\}$.
- The Richardson stratification induces a similar stratification under the projection $G/B \to G/P$. Then

  $$Y_{b_\infty} := \text{union of closures of codimension 1 strata.}$$

  is Y_{b_∞} an anticanonical divisor.
- Moreover, Y_{b_∞} is a rank 1 fiber of the universal CY family in $X = G/P$. This is a consequence of the solution rank formula, together with the classical BGG multiplicity theorem for Verma modules.
- **Remark:** Taking $X = G(k, n)$ recovers the rank 1 fiber

  $$Y_{b_\infty} = \{x_{1\cdots k} x_{2\cdots(k+1)} \cdots = 0\}.$$

60. LCSL degeneration for toric hypersurfaces

- (Hosono,Lian,Yau): Consider the case a projective toric manifold X^n. Then
$$Y_{b_\infty} := \text{union of T-invariant divisors in } X$$
is anticanonical in X.
- Once again, we find
$$H_0^{Lie}(\hat{\mathfrak{t}}, Re^{f_{b_\infty}}) = \mathbb{C}e^{f_{b_\infty}}$$
hence Y_{b_∞} is a rank 1 fiber. This is also a LCSL degeneration.

61. Chain integral solutions

- The injective map $H_n(X - Y_*) \to Hom_D(\tau, \mathcal{O}^{an})$ is not surjective in general, (e.g. X being toric) where $Hom_D(\tau, \mathcal{O}^{an})$: the solution space of τ, is in general given as compactly supported middle cohomology of a perverse sheaf.
- **Remark**: This generalizes the famous GKZ formula for toric X, giving the generic rank of the GKZ system τ as the volume of a convex polytope in $\mathbb{R}^n$.
- **Proposition**: More concretely, denote $U_b := X - V(b)$, and $\cup D$ the union of all G-invariant divisors in X (which may be empty), for any relative cycle $C \in H_n(U_b, U_b \cap (\cup D))$, the chain integral $\int_C \frac{\Omega}{f_b}$ is a local analytic solution to τ at b.
- We understand the above *chain integral map* from $H_n(U_b, U_b \cap (\cup D))$, to the space of local holomorphic solutions of τ at b, in several interesting cases.

62. **Chain integral solutions**

- **Theorem**: [Huang,Lian,Yau,Zhu] Let X be a smooth toric variety, and take $G = T$ to be the torus acting on X with the open dense orbit, then the chain integral map is an isomorphism.
- Proof is based on a general geometric formula for τ due to Huang,Lian,Zhu, and some local Weyl algebra computation in the toric case.
- **Remark:** Note that τ in this case reduces to a GKZ system. This theorem gives a canonical geometric construction for all solutions to this GKZ system. For e.g. $X = \mathbb{P}^2$, this was explicitly realized by physicists Avram et al (who call these chain integrals "semi-periods").
- **Theorem**: [Huang,Lian,Yau,Zhu] Suppose $X = G/B$ is a complete flag variety, and take the group in defining τ to be a Borel subgroup B, then the chain integral map is an isomorphism.
- There are generalizations of these results to τ associated to other groups. The chain integral map is not surjective in general, for interesting geometric reasons. On the other hand, there is a direct generalization of these results to the general type case.

63. **Remarks**

- In some cases, there is an interesting interpretation of these chains, in the context of the SYZ conjecture. Namely, some invariant chains appears to be “Lefschetz thimbles”, whose boundary components lie in Lagrangian submanifolds in X. (Cf. Walchers et al).
- In some special cases, Jie Zhou recently found that these chains have another interesting geometric interpretation, near the orbifold points in the moduli space.
- The chain integrals also appears in the problem of counting points in CY defined over finite fields (Candelas, de la Ossa, Rodriguez-Villegas). All solutions to GKZ (which include chain integrals) appear as certain counting generating functions.

64. Topological proof of GKZ rank formula

As another application, one gets a purely topological proof of a famous formula of GKZ for the rank of solution space of τ_{GKZ}.

- **Theorem:** [GKZ '90]

$$\textit{generic solution rank of } \tau_{GKZ} = vol(P_L)$$

  where the right side is the volume of the Newton polytope of the space of sections of the anticanonical line bundle L.

- **Sketch of topological proof:** [Fu,Huang,Lian'16] For generic b,

$$\begin{aligned} &\chi(X - Y_b, D - Y_b) \\ &= \cancel{\chi(X)} - \chi(Y_b) - \cancel{\chi(D)} + \chi(Y_b \cap D) && \text{(std. toric geometry)} \\ &= (-1)^n vol(P_L) && \text{(Riemann-Roch thm.)} \end{aligned}$$

  On the other hand,

$$\begin{aligned} &\chi(X - Y_b, D - Y_b) \\ &= (-1)^n \dim H_n(X - Y_b, D - Y_b) && \text{(quasi-projective Lefschetz hyperplane)} \\ &= (-1)^n \times \textit{ generic solution rank of } \tau_{GKZ} && \text{(chain integral isomorphism)} \end{aligned}$$

  Q.E.D.

65. Differential zeros of hypergeometric functions

- Based on [HLZ '14], [Chen,Huang,Lian '15] gives a systematic way to construct zeros of derivatives of period integrals, including classical *hypergeometric functions*.

$$\textbf{E.g.} \quad X = \mathbb{P}^n, \boxed{\left(\partial_{a_0}^{n+1} + \frac{(-1)^n}{(n+1)^{n+1}}\right) B_X(a)|_{b_F} = 0, \quad b_F := \sum x_i^{n+1}.}$$

where a_0 is the coefficient of $x_0 \dots x_n$.

- Note that we do not yet know how to explicitly compute the analytic continuation of the function $B_X(a)$ to the Fermat point b_F, the above identity provides nontrivial information. It would be interesting to see the implications to special function theory, even for the case $n = 2$.

- As an example for illustration, for $n = 2$ (i.e. elliptic curves), the above equality follows from the identity

$$\begin{aligned}(\partial_{a_0}^3 + \frac{1}{27})e^f|_{b_F} = \frac{1}{27}(&\partial_3 x_3 \partial_2 x_2 \partial_1 x_1 - \partial_3 x_3 \partial_2 x_2 \\ &- \partial_2 x_2 \partial_1 x_1 - \partial_1 x_1 \partial_3 x_3 + \partial_3 x_3 + \partial_2 x_2 + \partial_1 x_1)e^{f_{b_F}}\end{aligned}$$

Namely, $(\partial_{a_0}^3 + \frac{1}{27})e^f|_{b_F} = 0$ in the Lie algebra coinvariant space $H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})$.

66. **Differential zeros of hypergeometric functions**

- It turns out that, in general, this method can find all differential zeros of solutions to τ, via computations in this Lie algebra homology, that appears in the algebraic rank formula:

- **Theorem** [CHL '16] $\forall$ polynomial differential operator P, and $\forall$ b in the base V^*, $(Ps)(b) = 0$ for all local analytic solutions s of τ at b, iff $Pe^f|_b = 0$ in $H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})$.

- Proof is based on the algebraic rank formula.

67. Concluding remarks

- Tautological systems for families of general type complete intersections Y in X with K_Y very ample, have also been constructed [Lian-Yau '12].
- The solution rank formulas above have also been generalized to include this case. [Huang-Lian-Zhu '13].
- Tautological systems for period integrals of non-holomorphic forms on Y have also be constructed [Chen-Huang-Lian '15]. In this case, we consider period integrals
$$\int_\gamma \Omega$$
where $\Omega \in \Gamma(B, R^{n-1}\pi_*\mathbb{C})$ is an arbitrary section, taking values in $H^{n-1-p,p}(Y)$ for general p, not just $p = 0$. This allows us construct a D-module for the full period mapping, not just its first "Hodge component".

68. Concluding remarks

- Using tautological systems, we have also constructed Picard-Fuch equations for some **non Kähler CY manifolds.**
  Example. Complete intersections in $SU(n)$ (n odd) with respect to a left invariant complex structure [Lian-Yau '11].
- **Example.** Non Kähler CY manifolds studied by Fu-Yau and Goldstein-Prokushkin: T^2 bundle over K3, can be realized as complete intersections in $SU(3)$. Their period integrals are governed by a tautological system.
- Tautological systems provide a new approach to study period integrals for manifolds of general type – **higher dimension analogues** of hyper-elliptic integrals
  $$\int_C \frac{x^k dx}{\sqrt{Q(x)}}.$$
- Therefore, the theory of tautological systems provides a way to **unify and generalize** classical and modern theory special functions, including GKZ hypergeometric functions.
- Thanks to the powerful tools in D-module theory, we now have very good control of their differential systems as well.

THANK YOU.