# A Vafa-Witten Invariant for projective surfaces.

joint with Yuuji Tanaka.

Begin with a linearised version [VW].

## (Virtual) Euler characteristics of moduli spaces.

Local model (over $\mathbb{R}$ or $\mathbb{C}$)

$$\begin{array}{ccc} & & E \\ & & \downarrow \uparrow s \\ \mathcal{M} & \subset & \mathcal{A} \end{array} \qquad \mathcal{M} = s^{-1}(0)$$

$\mathcal{A}, E$ possibly infinite dimensional, eg $\begin{cases} \mathcal{M} = \mathcal{M}_{asd}(X^4, g) \\ \mathcal{A} = \{\text{connections}\}/\text{gauge} \\ s = F_A^+ \oplus d_{A_0}^*(A - A_0) \end{cases}$

Replace

$$s^{-1}(0) = \boxed{\begin{array}{c} \quad E \\ \quad \downarrow \uparrow s \\ M \subset A \end{array}}$$

by bigger model

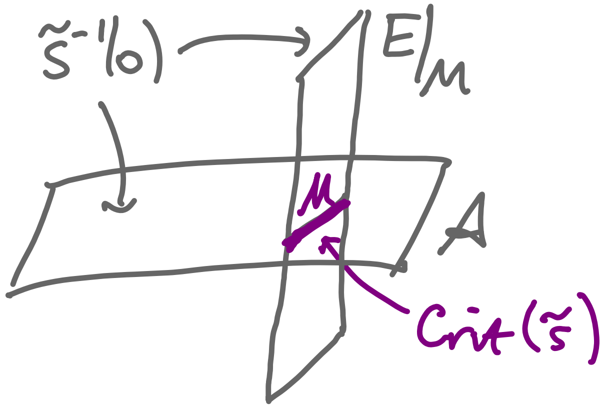
$$(d\tilde{s})^{-1}(0) = \boxed{\begin{array}{c} \qquad T^*_{\tilde{A}} \\ \qquad \downarrow \uparrow d\tilde{s} \\ \tilde{M} \subset \tilde{A} = \mathrm{Tot}(E^*) \end{array}}$$

where $\tilde{s} : \mathrm{Tot}(E^*) \to \mathbb{R}/\mathbb{C}$ is $\langle \,, s \rangle$.

i.e. $\tilde{s}(a, f) = f(s(a))$.

So $\boxed{\tilde{M} = \mathrm{Crit}(\tilde{s})} \supseteq s^{-1}(0) = M$

with equality iff $M$ smooth of correct dimension $vd = \dim A - \mathrm{rk}\, E$



More generally, $\underline{\tilde{M} \text{ is a cone over } M}$
$\underline{\text{with fibre } Ob_M^*}$ $\quad (Ob = \operatorname{coker} ds : T_A|_M \to E|_M)$

<u>Complex case:</u> $\tilde{M} = \operatorname{Spec} \operatorname{Sym}^\bullet(Ob_M) \circlearrowleft \mathbb{C}^*$ fixed locus the zero section $M \subset \tilde{M}$.

If $s = (f_i)_{i=1}^r$ locally then $\tilde{s} = \sum_i y_i f_i$ $\quad$ ($y_i$ local fibre coords on $E^* \to A$)

$$d\tilde{s} = \sum_i f_i \, dy_i + \sum_i y_i \, df_i$$

$$\text{Zeros } (f_i = 0) \cap (\underline{y} \perp df)$$

$s^{-1}(0) = E^*|_M$ $\qquad$ $\underline{y} \in \operatorname{coker}(df)^* = Ob_M^*$

In the good case [ $s$ transverse $\iff M$ smooth of correct dimension vd
$\iff ob_M = 0$ ]

$\tilde{M} = M$ <u>but</u> with $\underline{ob_{\tilde{M}} = T^*_{\tilde{M}} = T^*_M}$

so there's a natural invariant [vw]

$$e(T^*_M) = \underline{\underline{\pm e(M)}}$$

Applying this to ASD eq=s on $(X^4, g, P)$ $\begin{pmatrix} A = \mathcal{A}_P \\ E = \Omega^+(g_P) \\ S = F_A^+ \end{pmatrix}$
gives a <u>linearised</u> version of:

<u>Vafa-Witten Equations</u> on $(X^4, g, P)$

$$\begin{array}{l} \mathcal{A}_P \times \Omega^+(g_P) \times \Omega^0(g_P) \longrightarrow \Omega^+(g_P) \times \Omega^1(g_P) \times \Omega^0(g_P) \\ \ \text{\rotatebox{90}{$\cong$}} \\ \Omega^1(g_P) \end{array}$$

$$(d_A, B, \Gamma) \longmapsto \left(F_A^+ + [B.B] + [B,\Gamma],\ d_A\Gamma + d_A^* B\right)$$

$$, d_{A_0}^*(A - A_0)$$

[VW] proved a vanishing theorem in good cases (positive curvature, or negative canonical bundle when $X$ Kähler)

$$\Rightarrow \widetilde{\mathcal{M}} = \mathcal{M}_{asd} \qquad B = 0 = \Gamma$$

In this case there's a natural invariant $\pm e(\mathcal{M}_{asd})$.

[VW] show generating series <u>modular</u> in many examples. "S-duality"

When vanishing result does not hold, $\nexists$ defn of $VW(X,P)$.

## Kähler/projective case.

From now on $(X,g)$ will be a projective surface $(S,\omega)$

Equations become

$$F_A^{0,2} = 0$$

$\phi \in \Omega^{0,2}(\mathfrak{g}_P \otimes \mathbb{C})$

$$F_A^{1,1} \wedge \omega + [\phi, \bar{\phi}] = c \cdot \mathrm{id} \cdot \omega^2$$

$$\bar{\partial}_A \phi = 0$$

Can see the 2nd equation as a moment map for gauge group action. Solve by Kempf-Ness / stability.

$\Rightarrow$ Get a Hitchin-Kobayashi correspondence

[Alvarez-Consul, Garcia-Prada] [Tanaka]

$$\frac{\text{Solutions equivalent to}}{\text{modulo unitary gauge}} \quad \frac{\text{(poly) stable Higgs pairs}}{\text{modulo complex gauge}}$$

$$\boxed{\begin{array}{ll} (E, \phi) & \phi \in \mathrm{Hom}(E, E \otimes K_S) \\ \text{s.t. } \mu(F) < \mu(E) & \forall\, \phi\text{-invariant subsheaves } F \subset E \end{array}}$$

$$\left( \mu(E) = \int_S c_1(E).\omega \,/\, \mathrm{rk}\, E \right)$$

This has <u>linearised</u> the problem at the expense of enlarging $\mathcal{M}_{asd}$ to <u>stack</u> $\mathcal{M}_S$ of <u>all</u> bundles on $S$.
sheaves

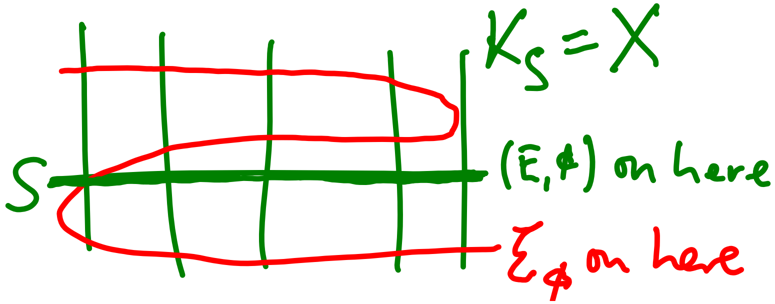
$$\begin{array}{ll} \widetilde{\mathcal{M}} = \{(E,\phi) \text{ stable}\} & \\ \downarrow \qquad\qquad \downarrow & \underline{\text{fibre}}\ \{\phi \in \mathrm{Hom}(E, E \otimes K_S)\} \\ \mathcal{M}_S = \{E\} & \text{is } \underset{H^2(\mathrm{End}\,E)^*}{\mathrm{Ext}^2(E,E)^*} = \underline{\underline{Ob^*_{\mathcal{M}_S,E}}} \\ & \text{by Serre duality} \end{array}$$

[In the language of derived algebraic geometry, $\widetilde{\mathcal{M}}$ is an open subset of the $(-1)$-shifted cotangent bundle $T^*[-1]\mathcal{M}_S$ of the stack $\mathcal{M}_S$]

Next apply the <u>spectral construction</u>

Higgs pairs on $S$ $\xleftrightarrow[\text{of categories}]{\text{Equivalence}}$ Torsion sheaves $\mathcal{E}_\phi$
$(E,\phi)$ on $X = K_S$ (finite over $S$)

[On any fibre $K_S|_s$, $s \in S$, put eigenspaces of $\phi_s : E_s \to E_s \otimes K_S|_s$ over corresponding eigenvalues in $K_S|_s$]



<u>Upshot</u> is that

Solutions of (VW) on $S$ $\longleftrightarrow$ Stable torsion sheaves on $X = K_S$

(rank $r$) (finite length $r$ over $S$)

(Possibly singular: $E$ sheaf)

Under this correspondence, $SU(r)$ solutions of (VW) correspond to sheaves $\mathcal{E}_\phi$ on $X = K_S \xrightarrow{\pi} S$ such that (weighted) centre of mass of support of $\mathcal{E}_\phi$ on each fibre $K_S|_s$ is <u>zero</u>, and <u>$\det \pi_* \mathcal{E}_\phi$ fixed</u>.

$X = K_S$ is a Calabi–Yau 3-fold

$\Rightarrow$ Moduli space $\widetilde{\mathcal{M}}$ of compactly supported torsion sheaves has a <u>perfect obstruction theory</u>

| | | |
|---|---|---|
| deformations | $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ | Serre dual |
| obstructions | $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ | |
| higher obstructions | $\mathrm{Ext}^{\geq 3}(\mathcal{E},\mathcal{E})_0 = 0$ | |

Noncompact, but has $\mathbb{C}^*$ action scaling fibres of $X = K_S \downarrow S$

Equivalently, scaling Higgs field $(E,\phi) \xmapsto{\lambda \in \mathbb{C}^*} (E,\lambda\phi)$

$\mathbb{C}^*$-fixed locus $\widetilde{M}^{\mathbb{C}^*}$ is compact so can define invt by virtual localisation.

<u>Preliminary definition:</u>

$$VW_{u(r)}(S, c_1, c_2) = \int_{[\widetilde{M}^{\mathbb{C}^*}]^{vir}} \frac{1}{e(N^{vir})} \in \mathbb{Z}$$

chosen s.t.
semistability = stability

**But** this is <u>zero</u> if $H^{0,2}(S) \neq 0$ ($\Rightarrow$ trivial $H^2(\mathcal{O}_S)$ summand of $ob_{\widetilde{M}}$)

or $H^{0,1}(S) \neq 0$ ($\Rightarrow$ trivial $H^1(K_S)$ summand of $N^{vir}$)

**Fix** Use $SU(r)$ theory instead:

- $(E, \phi)$ : $\det E$ fixed $(= \mathcal{O}_S)$
  $\operatorname{tr} \phi = 0$ $\quad (\phi \in \operatorname{Hom}(E, E \otimes K_S)_0)$

or, equivalently,

- $\mathcal{E}$ on $X$ : centre of mass of (support of) $\mathcal{E}$ on each fibre $K_S|_s$ <u>zero</u>,
  $\det(\pi_* \mathcal{E}) = \det E$ fixed.

Need perfect obstruction theory for this smaller moduli space $\widetilde{\mathcal{M}}_{SU(r)}$.

At the level of deformations/obstructions

$$\cdots \longrightarrow \mathrm{Hom}(E, E\otimes K_S) \longrightarrow \mathrm{Ext}^1(\mathcal{E}_\phi, \mathcal{E}_\phi) \xrightarrow{\pi_*} \mathcal{E}xt^1(E,E) \longrightarrow \cdots$$

replace by $\mathrm{Hom}(E, E\otimes K_S)_0$

[already equal to $\mathrm{Ext}^1(\mathcal{E}_\phi, \mathcal{E}_\phi)_0$]

replace by $\mathrm{Ext}^1(E,E)_0$

Using either

1. Derived algebraic geometry (results of Toën-Vezzosi-Vaquié) : $T^*[-1]\mathcal{M}_S^2$ quasi-smooth ; $\tilde{\mathcal{M}}_{su(r)}$ open subset where pair is Higgs stable.

or

2. Illusie's full cotangent complex and an analysis of Atiyah classes on $X$ and $S$ to relate deformation theories of $\mathcal{E}_\phi$ and $(E,\phi)$

We can get a symmetric perfect obstruction theory on $\tilde{\mathcal{M}}_{su(r)}$

**Defn:**

$$VW_{SU(\underline{r})}(S, \underline{L, c_2}) = \int_{\left[\left(\widetilde{\mathcal{M}}^{L}_{SU(r)}(S)\right)^{\mathbb{C}^*}\right]^{vir}} \frac{1}{e(N^{vir})} \in \mathbb{Z}$$

s.t. semistability = stability

$\det E \cong L$
$\operatorname{tr} \phi = 0$

Reduces to preliminary definition when $H^{0,1}(S) = 0 = H^{0,2}(S)$.

<u>Deformation invariant</u> under defs of $(S, L)$.

But $\exists$ other (non-deformation invariant) ways to $\mathbb{C}^*$-localise.

__Kai localisation.__ Let $N$ be a compact, complex moduli space.

If it carries a __symmetric__ perfect obstruction theory

$N$ locally critical locus of a holomorphic function

then it carries a constructible function

$$\chi^B : N \to \mathbb{Z} \quad \text{s.t.} \quad \int_{[N]^{vir}} 1 = e(N, \chi^B)$$

$$= \sum_{i \in \mathbb{Z}} i \cdot e(\chi^B = i)$$

multiplicity function [Behrend]

For $N = \widetilde{\mathcal{M}}$ (noncompact) take RHS as another __definition__ of the invariant

$\chi^B$ is $\mathbb{C}^*$-invariant and $e(\mathbb{C}^*\text{-orbit}) = 0$ except for fixed points.

$\Rightarrow$ Get Kai-localisation

$$e(\tilde{M}, \chi^B_{\tilde{M}}) = \boxed{e\left(\tilde{M}^{\mathbb{C}^*}, \chi^B_{\tilde{M}}\big|_{\tilde{M}^{\mathbb{C}^*}}\right)}$$

In general <u>different</u> from virtual localisation.

## Kiem-Li cosection localisation.

$\mathbb{C}^* \curvearrowright \tilde{\mathcal{M}}$ gives a vector field $v$ on $\tilde{\mathcal{M}}$

so $Ob_{\tilde{\mathcal{M}}} = T^*_{\tilde{\mathcal{M}}} \xrightarrow{v} \mathcal{O}_{\tilde{\mathcal{M}}}$ "cosection of obstruction sheaf"

[KL]: localise to zero locus of $v$, ie $\tilde{\mathcal{M}}^{\mathbb{C}^*}$ here to give a class $[\tilde{\mathcal{M}}]^{loc} \in H_0(\tilde{\mathcal{M}}^{\mathbb{C}^*})$ and so another localised invariant $\boxed{\int_{[\tilde{\mathcal{M}}]^{loc}} 1}$

In older work with <u>Yunfeng Jiang</u> we worked out the relationship between these localisations.

**Theorem.** 1. Kai = Kiem–Li $= \pm e(\tilde{\mathcal{M}}^{\mathbb{C}^*})$

2. $$\int_{[\tilde{\mathcal{M}}^{\mathbb{C}^*}]^{vir}} \frac{1}{e(N^{vir})} = \int_{[\tilde{\mathcal{M}}^{\mathbb{C}^*}]^{vir}} \pm c_{top}\left(T^{vir}_{\tilde{\mathcal{M}}^{\mathbb{C}^*}}\right)$$

$$\boxed{\begin{aligned} \pm &= (-1)^{vd\, \tilde{\mathcal{M}}^{\mathbb{C}^*}} \\ &= (-1)^{vd(\mathcal{M}_{asd})} \end{aligned}}$$

And in general $1 \neq 2$.

So 1. gives a different VW invt. (Not deformation invt?)

Modular?

**Computations** We are calculating **both** putative VW invts.

$\tilde{\mathcal{M}}^{\mathbb{C}^*}$ has 2 types of component

$\phi = 0$ 1. $M_{asd}$ – heavily studied in $K_S \le 0$ case
– harder for $K_S \ge 0$ – singularities

$\phi$ nilpotent 2. Various nested Hilbert schemes of curves/points on $S$.

$\mathbb{C}^*$-weight spaces

<u>Eg</u> $r=2$ $\quad E = E_0 \oplus E_{-1} \quad \phi: E_0 \to E_{-1} \otimes K_S$

$L_0 \otimes \mathcal{I}_{Z_0} \quad L_1 \otimes \mathcal{I}_{Z_1}$

$$\deg L_1 \underset{\text{stability}}{\leq} \deg L_0 \underset{\exists\,\phi \neq 0}{\leq} \deg L_1 + \deg K_S$$

$\Rightarrow$ no solutions when $K_S < 0$

<u>Eg</u> $S$ = quintic surface $\subseteq \mathbb{P}^3$, $L = K_S = \mathcal{O}_S(1)$ $\Rightarrow$ $L_0 = \mathcal{O}_S(1)$, $L_1 = \mathcal{O}_S$ $\quad \phi: \mathcal{I}_{Z_0} \longrightarrow \mathcal{I}_{Z_1}$

Get component $S^{[n_1, n_0]} = \{ Z_1 \subseteq Z_0 \subseteq S : \underset{n_0+n_1 = c_2(E)}{\text{length}(Z_i) = n_i} \}$

Eg when $Z_1 = \phi$ get $S^{[n]}$ $\left[ \begin{array}{l} n = \text{length}(Z_0) \\ \quad = c_2(E) \end{array} \right]$

Find $\mathcal{Ob} = K_S^{[n]} \to S^{[n]}$

<u>Cosection</u> localise via section of $K_S$ with zeros a canonical divisor $C \subseteq S$

Get $\pm[C^{[n]}] \subset S^{[n]}$

So these contributions to VW come from integrals over $\mathrm{Sym}^n C$, $C \subset S$

canonical divisor

Can compute individually, but not yet sum over all $n$.

Modular?

The alternative localization gives

$$\pm e\left(S^{[n_1,n_0]}\right)$$

nested Hilb $\{Z_1 \subseteq Z_0 \subseteq S\}$
$n_1 \leq n_0$

$\rightsquigarrow$ Generating series of skew partitions

$$\sum_{n_1 \leq n_0} \pm e\left(S^{[n_0,n_1]}\right) q^{n_0+n_1} =$$

$$\pm(1-q)^{e(S)}\left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^{2e(S)}$$

Modular!

Perhaps both definitions of VW give modular forms?

In compact $CY^3$ case (counting torsion sheaves supported on surfaces in $X$)

either definition gives local contributions which add to <u>same</u> inv't (also expected to be modular).

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Similar trick $\rightsquigarrow$ Kapustin-Witten invariant for projective surfaces. [Uses Borisov-Joyce's $DT^4$ inv't.]