QUANTUM SPECTRAL CURVE

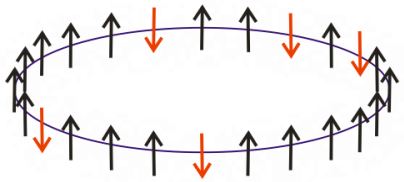
Nikolay Gromov

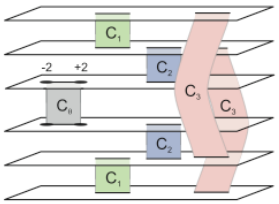
Based on

N. G., V. Kazakov, S. Leurent, D. Volin 1305.1939 (PRL), 1405.4857 (JHEP)
M. Alfimov, N. G., V. Kazakov 1408.1042 (JHEP)
N. G., F. Levkovich-Maslyuk 1601.05679
N.G., F. Levkovich-Maslyuk, G. Sizov 1507.04010 (PRL)
N.G., F. Levkovich-Maslyuk, G. Sizov 1504.06640 (JHEP)
M. Alfimov, N.G., G. Sizov to appear

Paris, 2016

Integrability in gauge theory





Origins of YM integrability: Lipatov's BFKL Hamiltonian

Perturbative integrability in N=4 SYM

Classical integrability of string σ-model on AdS5×S5, quasiclassics

1993

Lipatov
Faddeev,Korchemsky

2002

Minahan,Zarembo,
Beisert,Kristijanssen,Staudacher

2003-2008

Bena,Polchinski,Roiban
Kazakov,Marshakov, Minahan,
Zarembo, Frolov, Tseytlin
Schafer-Nameki
Beisert,Kazakov,Sakai,Zarembo
NG,Vieira

N=4 SYM

The "simplest" generalization of QCD:

$$S = \frac{1}{4g_{YM}^2}\int d^4x \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu} + \ldots)$$

Plus extra scalar fields $\Phi_1, \ldots, \Phi_6$ and fermions

Parameters: $\lambda = g_{YM}^2 N_c$ and $N_c = \infty$

Symmetries:

Lorentz: $so(3,1)$

Conformal: $so(4,2)$ $so(6)$ - rotation of the scalars

$su(2,2)$ $su(4)$

Super (graded) Lie algebra:

$su(2,2|4)$

New type of the Integrable structure

Baxter equation for the spectrum of SU(2) XXX spin chain

$$T(u)Q(u) + (u+i/2)^L Q(u-i) + (u-i/2)^L Q(u+i) = 0$$

Two solutions: polynomial $Q_1 \sim u^S$ and $Q_2 \sim u^{L-S}$

Satisfy Wronskian relation:

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

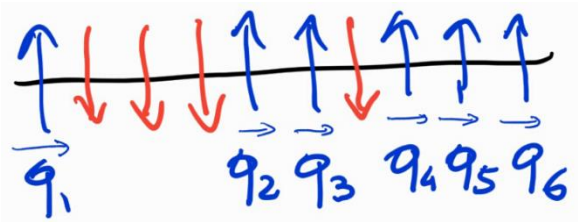
Q_1 - polynomial

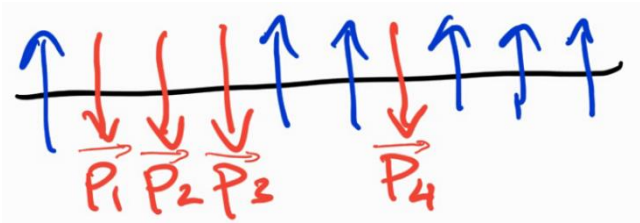
Q_2 - polynomial

By itself has finite set of solutions!

Gives the spectrum of the theory:

$$E = \partial_u \log Q_1|_{u=i/2}$$





$$Q_1 = \prod_{i=1}^{S}(u-u_i) \ , \quad u_i = \frac{1}{2}\cot q_i$$

$$Q_2 = \prod_{i=1}^{L-S}(u-v_i) \ , \quad v_i = \frac{1}{2}\cot p_i$$

New type of the Integrable structure

Whereas the Baxter is already quite complicated for SU(3)

The QQ-formulation remains simple:

$$\begin{vmatrix} Q_1(u+i) & Q_2(u+i) & Q_3(u+i) \\ Q_1(u) & Q_2(u) & Q_3(u) \\ Q_1(u-i) & Q_2(u-i) & Q_3(u-i) \end{vmatrix} = u^L$$

Q_1 - polynomial

Q_2 - polynomial

Q_3 - polynomial

All conserved charges are encoded into:

$$T(u) = \begin{vmatrix} Q_1(u+2i) & Q_2(u+2i) & Q_3(u+2i) \\ Q_1(u) & Q_2(u) & Q_3(u) \\ Q_1(u-i) & Q_2(u-i) & Q_3(u-i) \end{vmatrix}$$

New type of the Integrable structure

Two main ingredients:

- QQ-relations

$$sl(2) \to psu(2,2|4)$$

$$(Q_1,\ Q_2) \quad \longrightarrow \quad (\underbrace{\mathbf{P}_1,\ \mathbf{P}_2,\ \mathbf{P}_3,\ \mathbf{P}_4}_{S^5}|\underbrace{\mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3,\ \mathbf{Q}_4}_{AdS_5})$$

- Analyticity

Q_1 - polynomial

Q_2 - polynomial

What are the analytic properties of the Q's at finite coupling?

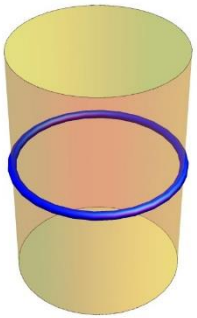
Motivation from classics

[Bena, Polchinski, Roiban]

String world-sheet action:

$$S = g \int \mathrm{str}(J^{(2)} \wedge *J^{(2)} - J^{(1)} \wedge J^{(3)})$$

Flat connection $\mathcal{A}(u) = J^{(0)} + \frac{u}{\sqrt{u^2-4g^2}} J^{(2)} - \frac{2g}{\sqrt{u^2-4g^2}} * J^{(2)} + \dots$



$$\Omega(u,\tau) = \mathrm{Pexp} \oint \mathcal{A}_\sigma d\sigma$$

Eigenvalues of the monodromy matrix:

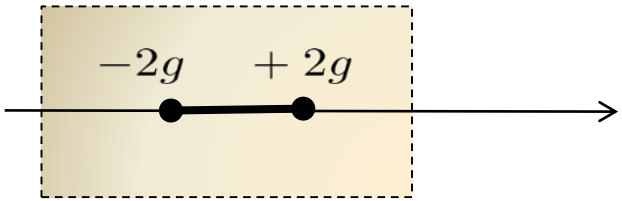
$$(\underbrace{e^{ip_1}, e^{ip_2}, e^{ip_3}, e^{ip_4}}_{S^5} | \underbrace{e^{iq_1}, e^{iq_2}, e^{iq_3}, e^{iq_4}}_{AdS_5})$$

Giving (WKB approximation):

$$\mathbf{P}_a \sim \exp\left(-\int^u p_a(v)dv\right) \quad , \quad \mathbf{Q}_i \sim \exp\left(-\int^u q_i(v)dv\right) \quad ,$$

Analytic properties:

[Dorey, Vicedo] $\oint p(u)du = \mathbb{Z}$

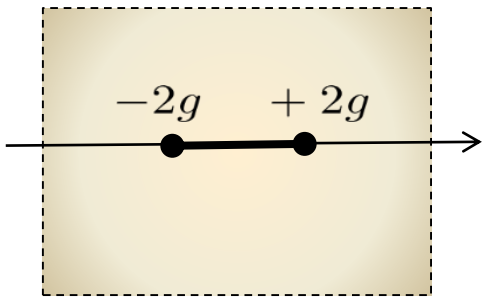


New type of the Integrable structure

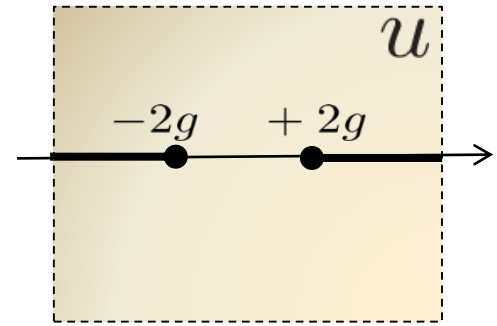
[N.G., Kazakov, Leuren, Volin]

Simplest analyticity assumptions:

$\mathbf{P}_a$



$\mathbf{Q}_a$



$$\mathbf{P}_a \simeq u^{\mathrm{R-charge}} \quad , \quad u \to \infty$$

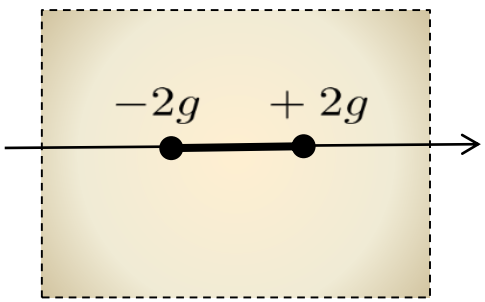
Charges in S^5 are integer

$$\mathbf{Q}_a \sim u^{\mathrm{conformal\ charge}}$$

Charges in AdS_5 contain anom.dimension

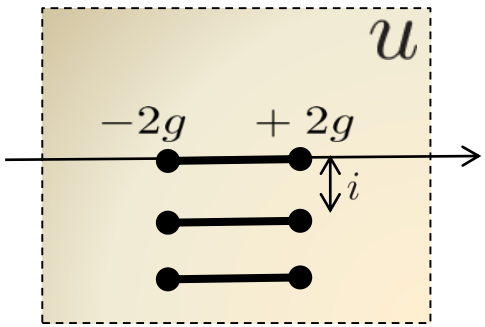
New type of the Integrable structure

$\mathbf{P}_a$

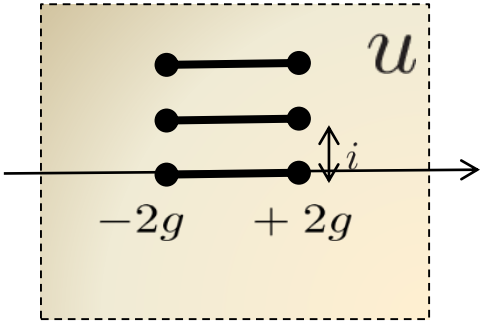


QQ-relations

$\mathbf{Q}_a$

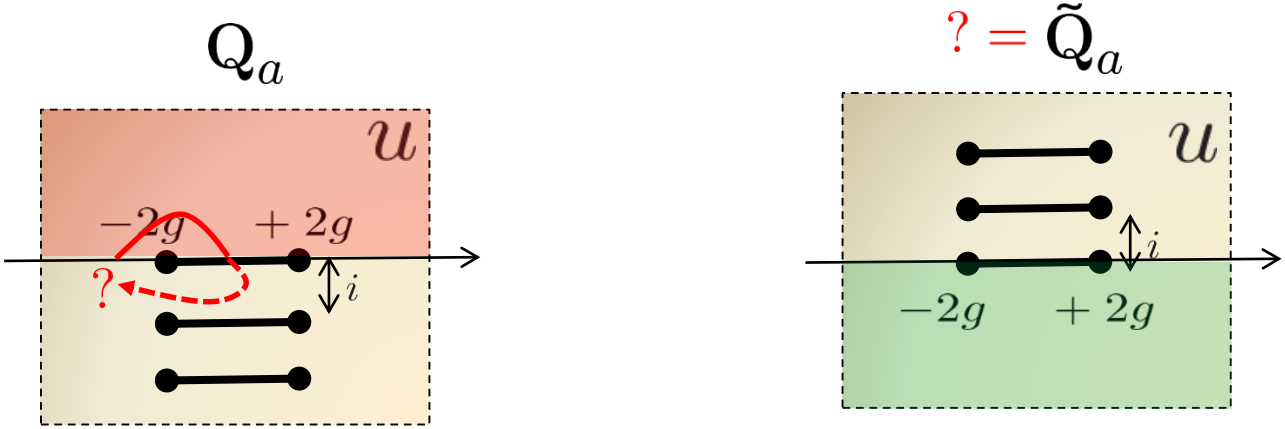


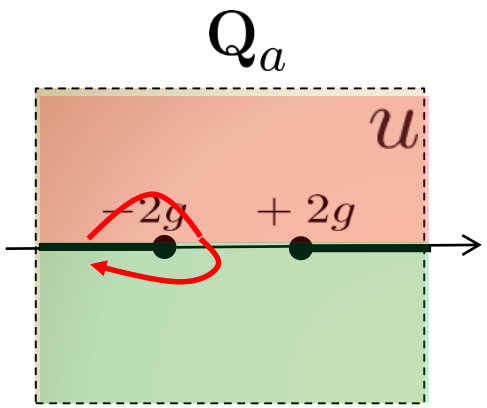
Or another possibility:



New type of the Integrable structure

[N.G., Kazakov, Leuren, Volin]





Deduce gluing condition:

[N.G., Sizov, Levkovich-Maslyuk]

$$\tilde{\mathbf{Q}}_a = \bar{\mathbf{Q}}_b$$

[N.G., Sizov, Levkovich-Maslyuk]

String - Math

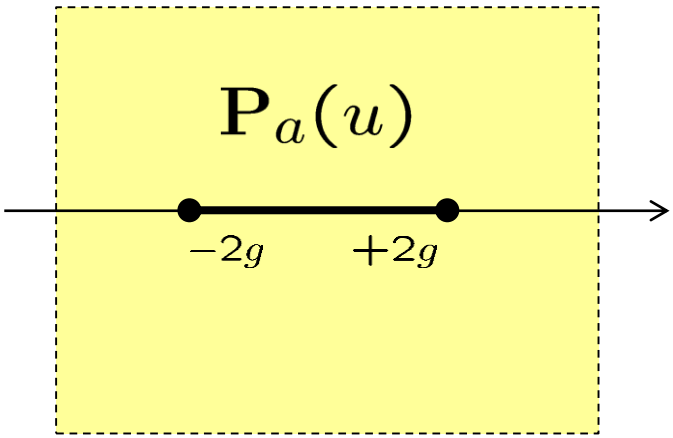
Spectrum of string = Standard QQ-relations + $\tilde{Q}_a = \bar{Q}_b$

Numerical Solution of the Spectral Problem

QSC and P-functions

Gromov, Kazakov, Leurent,Volin 2013,14

For any state all Q-functions can be built using $\mathbf{P}_a, \mathbf{P}^a$



S^5 charges

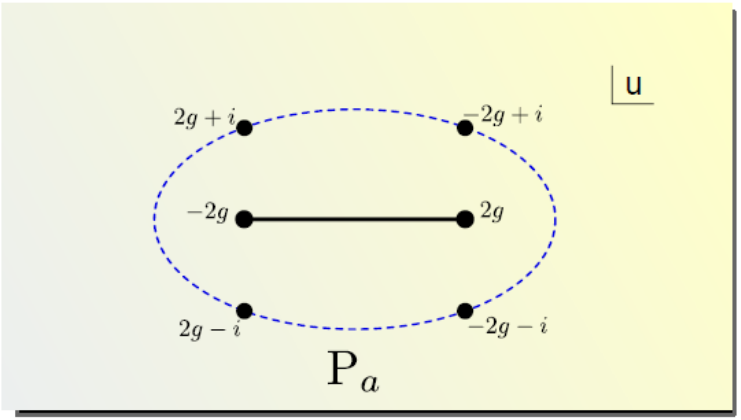
$$\mathbf{P}_a(u) \sim 1/u^{M_a}, \quad u \to \infty$$

$$\mathbf{P}_a(u) = \sum_{n=M_a}^{\infty} \frac{c_{a,n}}{x^n}$$

Marboe,Volin 2014

$$x + \frac{1}{x} = \frac{u}{g} \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

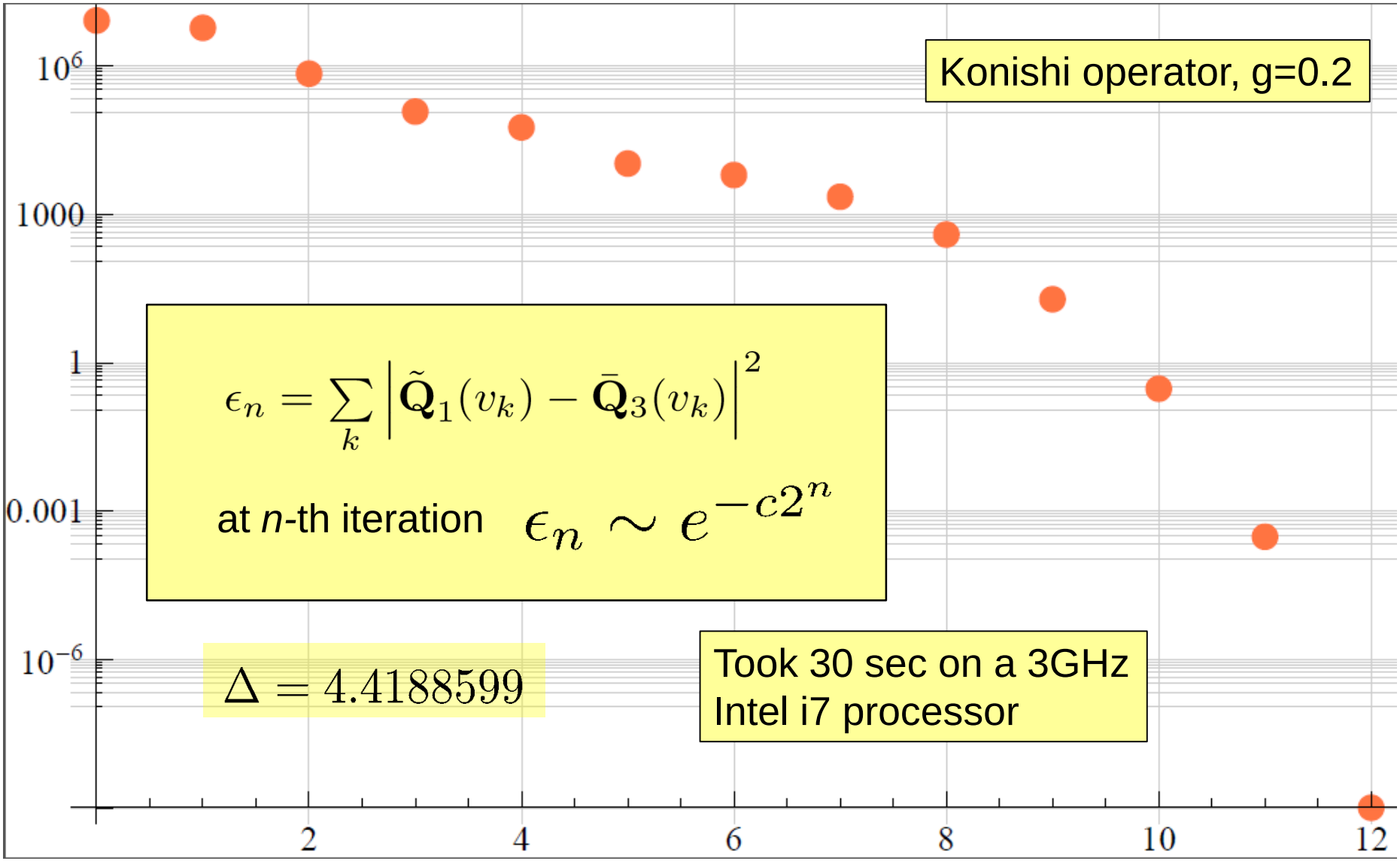
$$a, b = 1, \ldots, 4$$



$c_{a,n}$ are the main parameters in our numerics

$$\tilde{\mathbf{P}}_a(u) = \sum_{n=M_a}^{\infty} c_{a,n} x^n$$

Convergence



$$\epsilon_n = \sum_k \left| \tilde{\mathbf{Q}}_1(v_k) - \bar{\mathbf{Q}}_3(v_k) \right|^2$$

at *n*-th iteration $\epsilon_n \sim e^{-c2^n}$

$$\Delta = 4.4188599$$

Took 30 sec on a 3GHz Intel i7 processor

Konishi anomalous dimension

NG, F Levkovich-Maslyuk,Sizov 2015

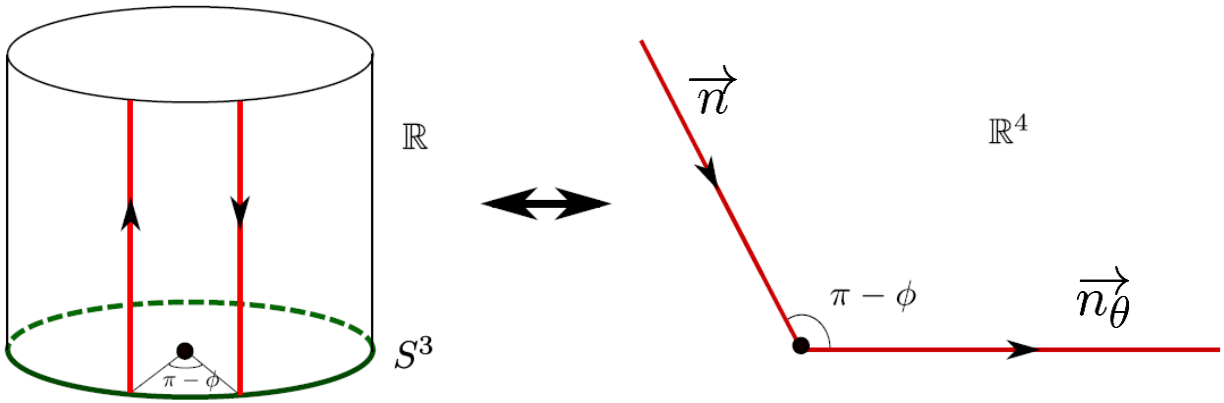
$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_{S=2}(\lambda)$	$\frac{\sqrt{\lambda}}{4\pi}$	$\Delta_{S=2}(\lambda)$
0.1	4.115506377945221056840042671851	0.2	4.418859880802350962250362876243
0.3	4.826948662284842304671283425271	0.4	5.271565182595898008221528540034
0.5	5.712723424787739030626966875973	0.6	6.133862814488691819595425762346
0.7	6.531606077852440195886557953690	0.8	6.907504206024567515828872789717
0.9	7.264169587439112774839639853 9	1	7.604070717047388483428655 5
1.1	7.929294264156845163218626 4	1.2	8.241563441147703542676050
1.3	8.542302872295066744863 42	1.4	8.832699939316309049451 4
1.5	9.11375404891588560886	1.6	9.386314656368554140399
1.7	9.651110426530137814 71	1.8	9.908771708559350878 9
1.9	10.159848013161547364 1	2	10.404821743440506112 7
2.1	10.644119095161757597 2	2.2	10.878118797537726796
2.3	11.107159189584305149	2.4	11.331544000504529107
2.5	11.551547111042160297	2.6	11.76741650605722239
2.7	11.979377579520677 41	2.8	12.18763591669137588
2.9	12.392379650914951 9	3	12.593781471798856 5
3.1	12.792000345714489 8	3.2	12.987182997398639 2
3.3	13.179465191962905 5	3.4	13.36897284920814 4
3.5	13.55582301629291 4	3.6	13.74012472015796 6
3.7	13.92197971739147 4	3.8	14.10148315622714 9
3.9	14.27872416294376 3	4	14.45378636296056
4.1	14.62674834530641	4.2	14.79768407780976
4.3	14.96666327925592	4.4	15.13375175384302
4.5	15.29901169250472	4.6	15.4625019450274
4.7	15.6242782663505	4.8	15.7843935399844
4.9	15.942897981092	5	16.099839321454

Hegedus, Konczer 2016 up to 7, C++

Quark – anti-quark potential

Cusped Wilson line in N=4 SYM

$$W = \mathrm{Tr}\, \mathcal{P} \exp \int dt \left[iA \cdot \dot{x} + \vec{\Phi} \cdot \vec{n}\, |\dot{x}| \right]$$



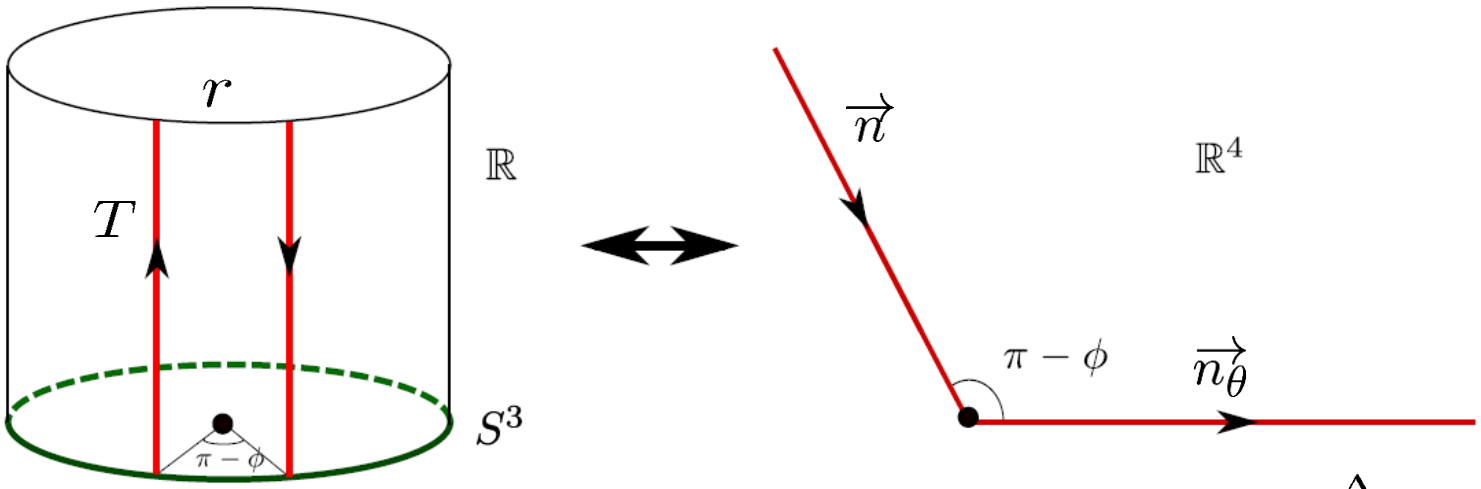
generalized
$q\bar{q}$ potential

$$\langle W \rangle \sim \left(\frac{\Lambda_{IR}}{\Lambda_{UV}} \right)^{\Delta}$$

Parameters:

- Cusp angle ϕ
- Angle θ between the couplings to scalars on two rays
- ‘t Hooft coupling λ

Flat space limit

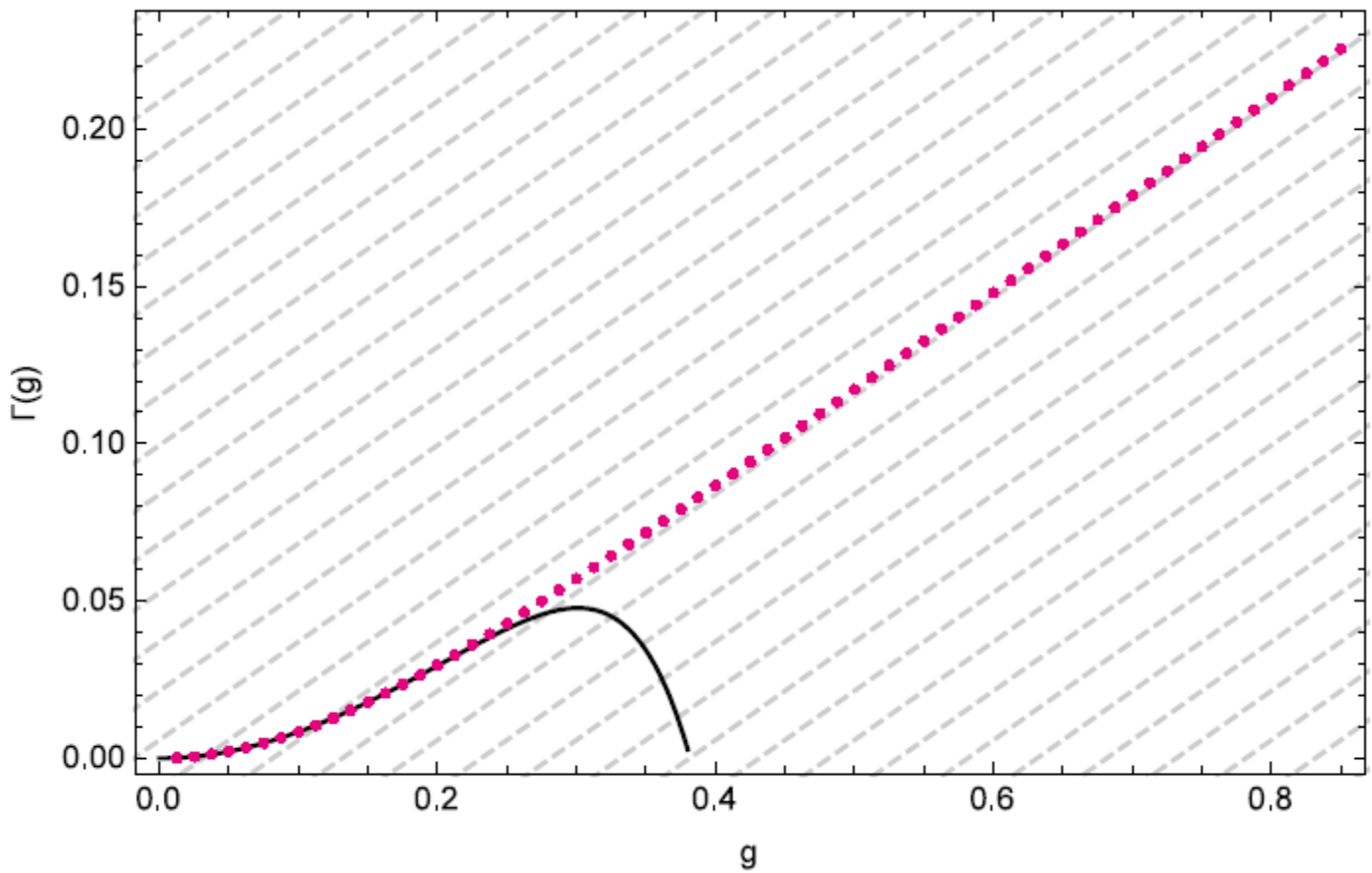


$$\langle W \rangle \sim e^{\frac{T\Omega}{r}}$$

$$\langle W \rangle \sim \left(\frac{\Lambda_{IR}}{\Lambda_{UV}} \right)^{\Delta}$$

In the singular limit $\phi \to \pi$ we get the flat space potential

$$\Delta = -\frac{\Omega(\lambda,\theta)}{\pi - \phi}$$



String theory prediction [Drukker, Forini 11]

$$\Gamma_{\text{cusp}}\left(\phi=\frac{\pi}{4},\theta=\frac{4\pi}{10},g\right)\simeq 0.3122892\,g-0.0410591+\frac{0.00073853}{g}+\mathcal{O}\left(\frac{1}{g^2}\right)$$

$$\Gamma_{\text{cusp}}^{\text{classical}}\simeq 0.3122881g.$$

Weak coupling: results

NG, F Levkovich-Maslyuk 2016

We have computed the first 7 orders of the expansion

$$
\begin{aligned}
\frac{\Omega}{4\pi} = {} & \hat{g}^2 + \\
& \hat{g}^4 \left[16L - 8\right] + \\
& \hat{g}^6 \left[128L^2 + L\left(64 + \frac{64\pi^2 T}{3}\right) - 112 - \frac{8\pi^2}{3} + 72T\zeta_3\right] + \\
& \hat{g}^8 \left[\frac{2048L^3}{3} + \frac{1024}{3}\pi^2 L^2 T + 2048L^2 + LT\left(768\zeta_3 + \frac{2176\pi^2}{3}\right) + \left(-768 - \frac{640\pi^2}{3}\right)L \right. \\
& \left. + T^2\left(128\pi^2\zeta_3 - 760\zeta_5\right) + T\left(384\zeta_3 - 640\pi^2 + \frac{32\pi^4}{9}\right) + \frac{1664\zeta_3}{3} + \frac{1216\pi^2}{9} - 1280\right] + \dots
\end{aligned}
$$

$$
\hat{g} \equiv g\cos\frac{\theta}{2} \quad , \quad T \equiv \frac{1}{\cos^2\frac{\theta}{2}} \quad , \quad L \equiv \log\sqrt{8e^{\gamma}\pi\hat{g}^2}
$$

Pefect match with known results
(first 3 orders and partial data at higher loops)

Ericksson, Semenoff, Szabo, Zarembo 2000; Pineda 2007; Drukker, Forini 2011; Stahlhofen 2012; Correa, Henn, Maldacen, Sever 2012; Bykov, Zarembo 2012; Prausa, Steinhauser 2013;

+ new simple formula for subleading logs to all orders

$$
\frac{\Omega}{4\pi} = \sum_{n=0}^{\infty} \hat{g}^{2n+2}\frac{16^n L^n}{n!} \times \left(1 + \frac{3n^2 - 5n}{4L} + \pi^2 T\frac{n^2 - n}{12L} + \mathcal{O}(1/L^2)\right)
$$

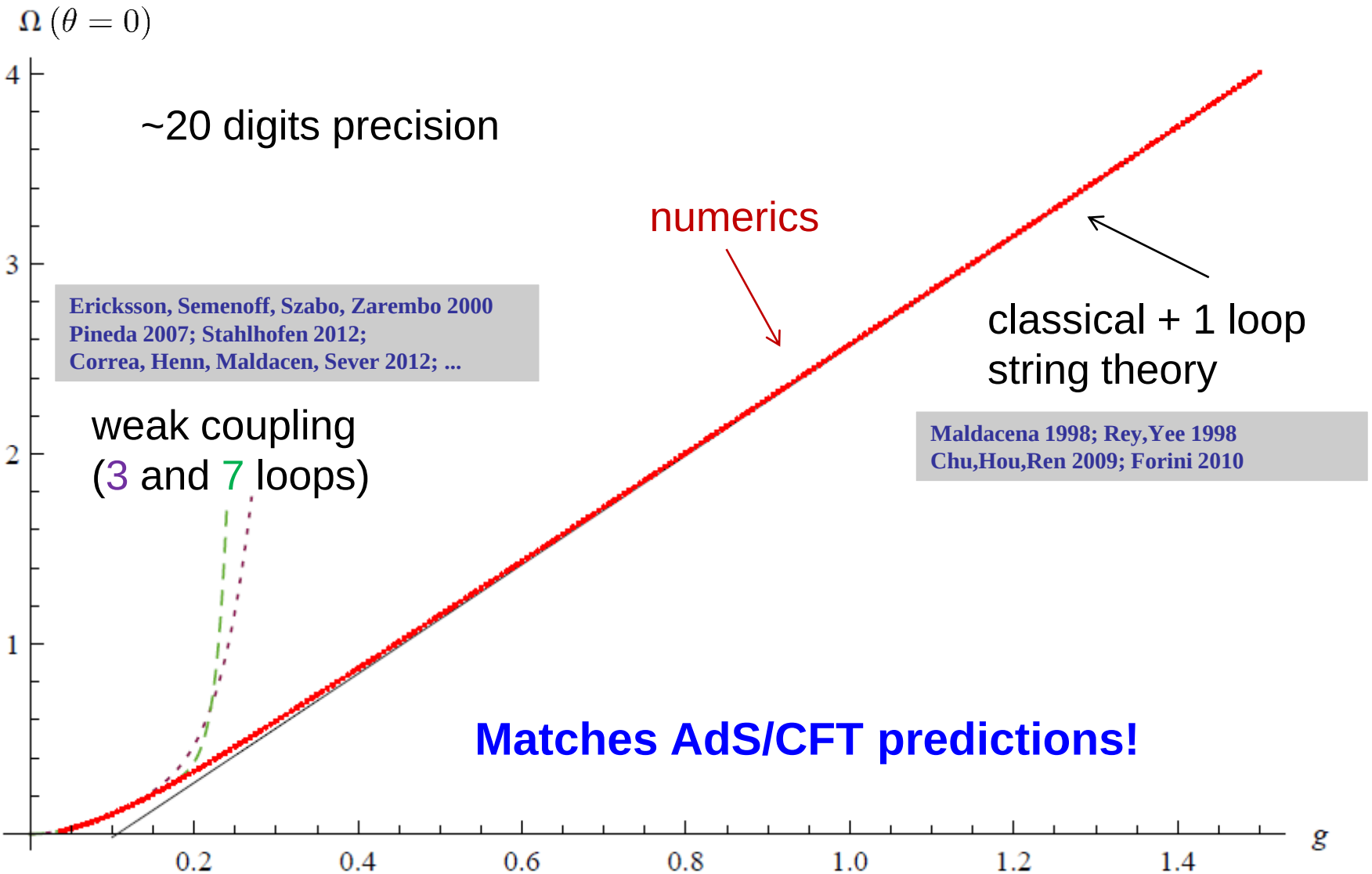
7-loop result. The term of order $\hat{g}^{14}$ in $\frac{\Omega}{4\pi}$ is given by

$$
\begin{aligned}
&\frac{1048576L^6}{45}+\frac{524288}{9}L^5\pi^2T+\frac{6815744L^5}{15}+\frac{262144}{9}L^4\pi^4T^2-65536L^4T\zeta_3+\frac{40632320}{9}L^4\pi^2T\\
&-\frac{15007744}{9}L^4\pi^2+2752512L^4+\frac{131072}{81}L^3\pi^6T^3+65536L^3\pi^2T^2\zeta_3+\frac{655360}{3}L^3T^2\zeta_5\\
&+\frac{12255232}{9}L^3\pi^4T^2-\frac{64159744}{135}L^3\pi^4T-65536L^3T\zeta_3+\frac{13303808}{3}L^3\pi^2T+\frac{3407872L^3\zeta_3}{9}\\
&-\frac{11141120}{9}L^3\pi^2+\frac{15073280L^3}{3}+\frac{2080768}{45}L^2\pi^4T^3\zeta_3-\frac{499712}{3}L^2\pi^2T^3\zeta_5-129024L^2T^3\zeta_7\\
&+32768L^2\pi^6T^3-\frac{2828288}{405}L^2\pi^6T^2-36864L^2T^2\zeta_3^2+\frac{11444224}{3}L^2\pi^2T^2\zeta_3+20480L^2T^2\zeta_5\\
&+\frac{2351104}{3}L^2\pi^4T^2-\frac{7610368}{9}L^2\pi^2T\zeta_3-40960L^2T\zeta_5-\frac{27344896}{45}L^2\pi^4T+1671168L^2T\zeta_3\\
&-3817472L^2\pi^2T+\frac{7221248L^2\pi^4}{45}+2555904L^2\zeta_3+\frac{17096704L^2\pi^2}{9}-\frac{6914048L^2}{3}+\frac{8192}{9}L\pi^6T^4\zeta_3\\
&-\frac{133120}{3}L\pi^4T^4\zeta_5+369152L\pi^2T^4\zeta_7-628992LT^4\zeta_9+\frac{1176832L\pi^8T^3}{42525}+\frac{210944}{3}L\pi^2T^3\zeta_3^2\\
&-71680LT^3\zeta_3\zeta_5+30720LT^3\zeta_{6,2}+\frac{7872512}{15}L\pi^4T^3\zeta_3-1899520L\pi^2T^3\zeta_5+867328LT^3\zeta_7\\
&+\frac{212992}{27}L\pi^6T^3-\frac{1150976}{15}L\pi^4T^2\zeta_3+665600L\pi^2T^2\zeta_5-268800LT^2\zeta_7+\frac{2378752}{405}L\pi^6T^2\\
&+43008LT^2\zeta_3^2+\frac{757760}{3}L\pi^2T^2\zeta_3-1587200LT^2\zeta_5-\frac{14838784}{9}L\pi^4T^2-\frac{2152448L\pi^6T}{2835}\\
&-163840LT\zeta_3^2+\frac{24051712}{9}L\pi^2T\zeta_3+364544LT\zeta_5+\frac{390412288}{405}L\pi^4T+2457600LT\zeta_3\\
&-\frac{39706624}{9}L\pi^2T-\frac{5324800}{9}L\pi^2\zeta_3+\frac{1998848L\zeta_5}{5}-\frac{34199552L\pi^4}{225}+\frac{9797632L\zeta_3}{3}\\
&+\frac{61534208L\pi^2}{81}-\frac{23560192L}{3}-\frac{11264}{105}\pi^6T^5\zeta_5+\frac{73216}{5}\pi^4T^5\zeta_7-285120\pi^2T^5\zeta_9\\
&+1271952T^5\zeta_{11}-\frac{10544\pi^{10}T^4}{93555}+\frac{91136}{9}\pi^4T^4\zeta_3^2-\frac{520832}{3}\pi^2T^4\zeta_3\zeta_5+179424T^4\zeta_5^2\\
&+361088T^4\zeta_3\zeta_7+\frac{16768}{3}\pi^2T^4\zeta_{6,2}-26432T^4\zeta_{8,2}+\frac{65536}{45}\pi^6T^4\zeta_3-63488\pi^4T^4\zeta_5\\
&+401408\pi^2T^4\zeta_7-508032T^4\zeta_9+\frac{5137792\pi^6T^3\zeta_3}{2835}-768T^3\zeta_3^3+30976\pi^4T^3\zeta_5\\
&-\frac{941632}{3}\pi^2T^3\zeta_7+\frac{2211904T^3\zeta_9}{3}-\frac{142816\pi^8T^3}{14175}+\frac{1183232}{3}\pi^2T^3\zeta_3^2-337664T^3\zeta_3\zeta_5 \qquad + \cdots
\end{aligned}
$$

NG, F Levkovich-Maslyuk 2016

Numerical solution

We adapted the efficient algorithm of NG, F Levkovich-Maslyuk, Sizov 2015



Ladders limit

Double scaling limit $\theta \to i\infty, \; g \to 0, \quad \frac{g}{e^{i\theta/2}} = \text{fixed}$

Selects only ladder diagrams

$$\frac{\Omega}{4\pi} = \hat{g}^2 + \hat{g}^4\left[16L - 8\right] + \hat{g}^6\left[128L^2 + L\left(64 + \cancel{\frac{64\pi^2 T}{3}}\right) - 112 - \frac{8\pi^2}{3} + \cancel{72T\zeta_3}\right] + \hat{g}^8\left[\frac{2048L^3}{3} + \cancel{\frac{1024}{3}\pi^2 L^2 T} + 2048L^2 + \cancel{LT\left(768\zeta_3 + \frac{2176\pi^2}{3}\right)} + \left(-768 - \frac{640\pi^2}{3}\right)L + \cancel{T^2\left(128\pi^2\zeta_3 - 760\zeta_5\right)} + \cancel{T\left(384\zeta_3 - 640\pi^2 + \frac{32\pi^4}{9}\right)} + \frac{1664\zeta_3}{3} + \frac{1216\pi^2}{9} - 1280\right] + \ldots$$

$$T \equiv \frac{1}{\cos^2\frac{\theta}{2}} \to 0 \qquad L \equiv \log\sqrt{8e^{\gamma}\pi\hat{g}^2}$$

$$\hat{g} \equiv g\cos\frac{\theta}{2} = \text{fixed}$$

$$\Omega = \Omega(\hat{g})$$

Ladders limit

Double scaling limit $\theta \to i\infty, \; g \to 0, \quad \frac{g}{e^{i\theta/2}} = \text{fixed}$

Bethe-Salpeter techniques reduce sum of ladder diagrams to a Schrodinger problem for the ground state

$$-F''(z) - \frac{4\hat{g}^2}{z^2+1} F(z) = -\frac{\Omega^2}{4} F(z)$$

$\hat{g} \equiv g \cos\frac{\theta}{2} = \text{fixed}$

Ericksson, Semenoff, Szabo, Zarembo 2000
Correa, Henn, Maldacen, Sever 2012

Captures all orders in $\hat{g}$ including all finite-size effects

Can we get it from the QSC ?

Ladders limit in the QSC

Great simplification as 4th order Baxter equation on $\mathbf{Q}_i$ factorizes

$$-2(2\hat{g}^2 - \Omega u + u^2)q_1(u) + u^2 q_1(u-i) + u^2 q_1(u+i) = 0$$

$$q_1(u) \equiv \mathbf{Q}_1(u) e^{\pm \pi u}/\sqrt{u}$$

The Schrodinger equation

$$-F''(z) - \frac{4\hat{g}^2}{z^2+1} F(z) = -\frac{\Omega^2}{4} F(z)$$

maps to this Baxter equation after a Mellin-type transform!

$$q_1(u) = 2u \int_i^{+\infty} \frac{e^{-\frac{\Omega z}{2}}}{z^2+1} \left(\frac{z+i}{z-i}\right)^{iu} F(z) dz$$

(similar to ODE/IM ?)

ABJ Theory?

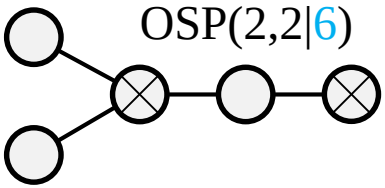
Spectral curve for ABJM

SYM

PSU(2,2|4)

ABJ(M)

OSP(2,2|6)



Quantum Spectral Curve construction is known for ABJM theory

[A. Cavaglia , D. Fioravanti, N. G., R. Tateo]

Important difference is the position of the branch points:

SYM: $\pm 2g(\lambda) = \pm \frac{\sqrt{\lambda}}{2\pi}$

ABJM: $\pm 2h(\lambda) = ?$

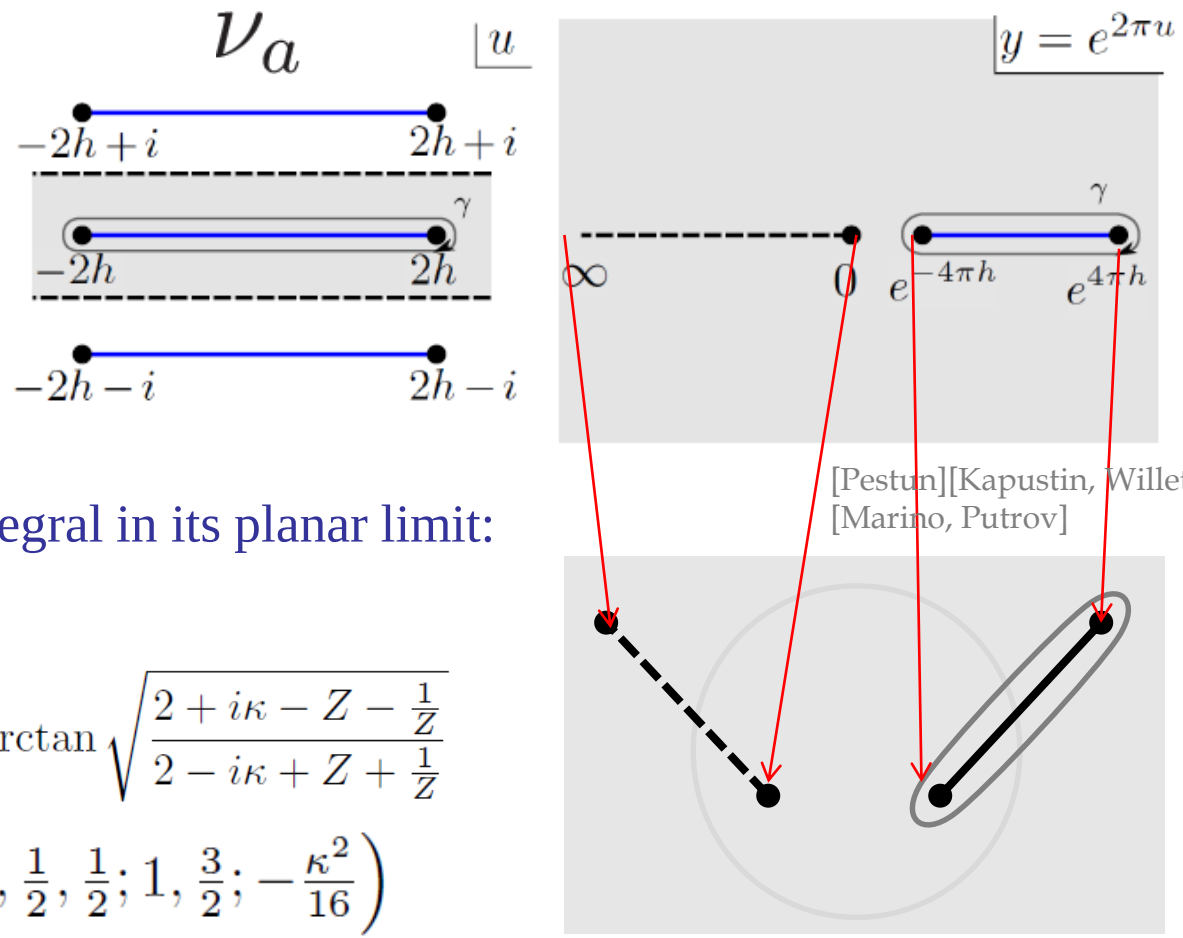
$h(\lambda)$ enters into many important quantities: cusp dimension, magnon dispertion

Finding Interpolation function h

In the near BPS limit we should be able to match with localization

[N.G., Sizov]

Integrability:
Elliptic type integral



[Pestun][Kapustin, Willett, Yaakov]
[Marino, Putrov]

ABJM Matric model integral in its planar limit:

Localization:

$$\langle W_{m=1}^{1/6}\rangle = \int_{\frac{1}{A^+}}^{A^+} \frac{dZ}{2\pi^2 i\lambda} \arctan\sqrt{\frac{2+i\kappa-Z-\frac{1}{Z}}{2-i\kappa+Z+\frac{1}{Z}}}$$

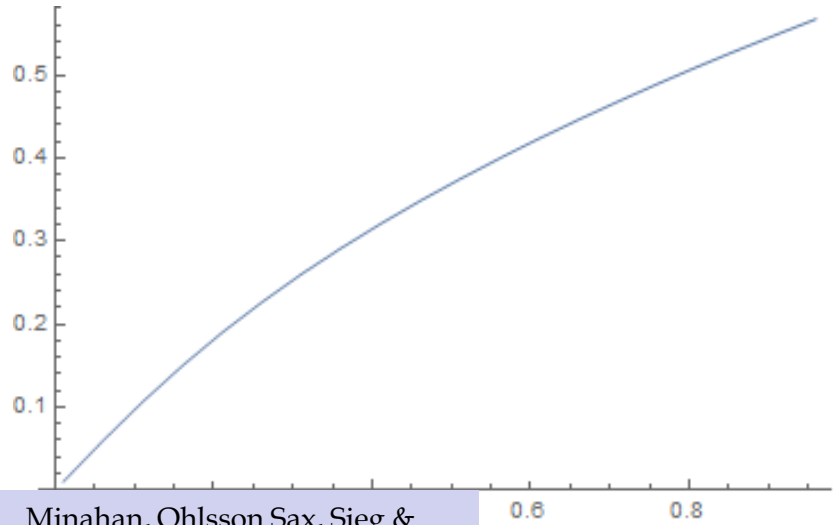
$$\lambda = \frac{\kappa}{8\pi}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2}{16}\right)$$

Comparing cross-ratios of the branch points:

$$\kappa = 4\sinh(2\pi h)$$

Interpolation function h

$$\lambda = \frac{\sinh(2\pi h)}{2\pi} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\sinh^2(2\pi h)\right)$$



Minahan, Ohlsson Sax, Sieg & Leoni, Mauri, Minahan, Ohlsso Sax, Santambrogio, Sieg, Tartagli Mazzucchelli,

McLoughlin, Roiban Tseytlin Abbott, Aniceto, Bombardelli Lopez-Arcos, Nastase

Minahan, Zarembo

$$h(\lambda) = \lambda - \frac{\pi^2\lambda^3}{3} + \frac{5\pi^4\lambda^5}{12} - \frac{893\pi^6\lambda^7}{1260} + \mathcal{O}(\lambda^9),$$

Reproduces ~4 nontrivial coefficients!

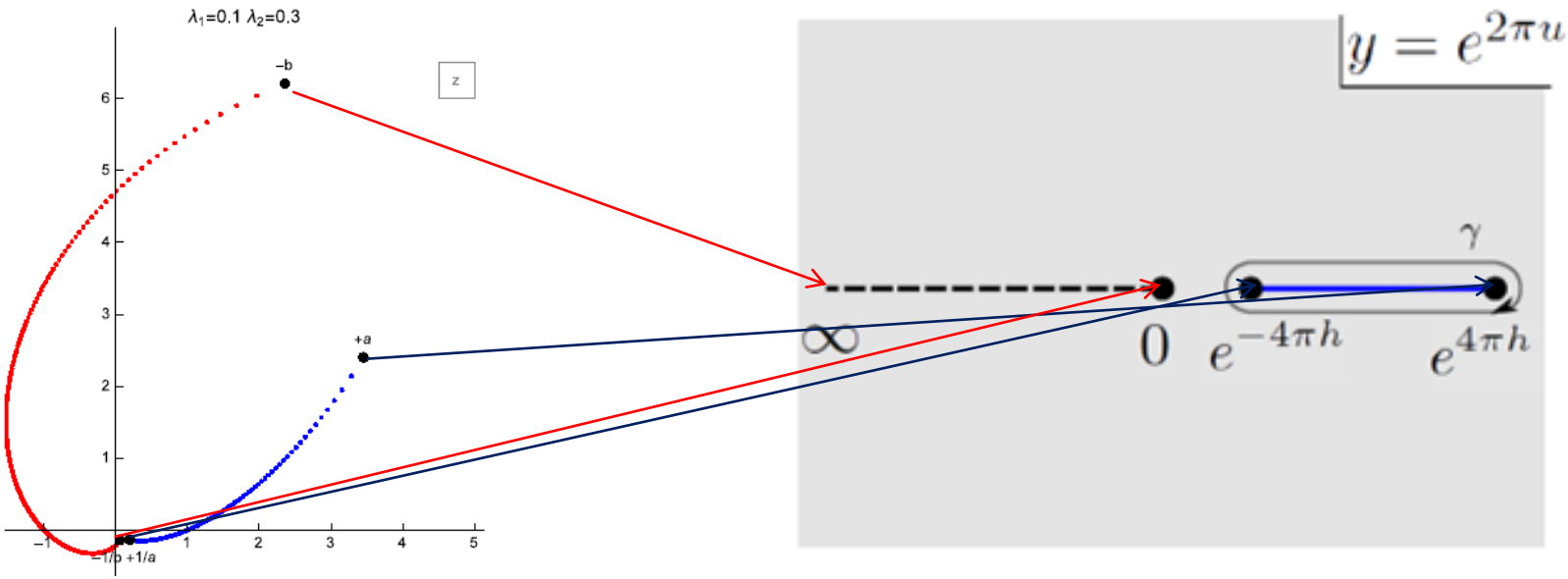
$$h(\lambda) = \sqrt{\frac{\lambda}{2} - \frac{1}{48}} - \frac{\log 2}{2\pi} + \mathcal{O}\left(e^{-\pi\sqrt{8\lambda}}\right),$$

?

Bergman, Hirano
Bianchi, Bianchi, Bres, Forini, Vescovi

Is ABJ theory Integrable?

Integrability structure of ABJM is too rigid. Impossible to deform



[Cavaglia, N.G., Levkovich-Maslyuk]

$$h(\lambda_1, \lambda_2) = \frac{1}{4\pi} \log\left(\frac{ab+1}{a+b}\right)$$

$$\lambda_1 = -\frac{1}{4\pi^2}\oint_a^{1/a} \omega(Z)\frac{dZ}{Z} \quad , \quad \lambda_2 = +\frac{1}{4\pi^2}\oint_{-b}^{-1/b} \omega(Z)\frac{dZ}{Z}$$

$$\omega(Z) = \log\left(\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)}\right)$$

Tests of the conjecture

[Drukker, Marino, Putrov]

1) Reality

$$4e^{2\pi i(B-1/2)} = a + \frac{1}{a} + b + \frac{1}{b}\,, \quad 2\kappa e^{\pi i B} = a + \frac{1}{a} - b - \frac{1}{b} \qquad u = \frac{\kappa^2}{8} - \cos 2\pi B$$

$$h(\lambda_1, \lambda_2) = \frac{1}{4\pi} \log\left(u + \sqrt{u^2 - 1}\right)$$

2) Weak coupling

[Minahan, Ohlsson Sax, Sieg]

[Leoni, Mauri, Minahan, Ohlsson Sax, Santambrogio, Sieg, Tartaglino-Mazzucchelli]

$$h^2(\lambda_1, \lambda_2) = \lambda_1\lambda_2 - \frac{\pi^2}{6}\lambda_1\lambda_2\,(\lambda_1 + \lambda_2)^2 + \frac{\pi^4}{360}\lambda_1\lambda_2\,(\lambda_1 + \lambda_2)^2\,\left(3\lambda_1^2 + 3\lambda_2^2 + 79\,\lambda_1\lambda_2\right)$$

1) Partial weak coupling

[Minahan, Ohlsson Sax, Sieg]

[Bianchi, Lioni]

$$h^2(\lambda_1, \lambda_2) = \frac{\lambda_1}{\pi}\sin(\pi\lambda_2) + \frac{\lambda_1^2}{3}\sin^2\left(\frac{\pi\lambda_2}{2}\right)(1 - 5\cos(\pi\lambda_2))$$

[Honda]

2) Strong coupling

[Aharony, Hashimoto, Hirano, Ouyang]

$$h(\lambda_1, \lambda_2) = \sqrt{\frac{\hat{\lambda}}{2}} - \frac{\log(2)}{2\pi} - \frac{e^{-2\pi\sqrt{2\hat{\lambda}}}\cos(2\pi B)}{\pi}\left(1 + \frac{1}{\pi\, 2\sqrt{2\hat{\lambda}}}\right)$$

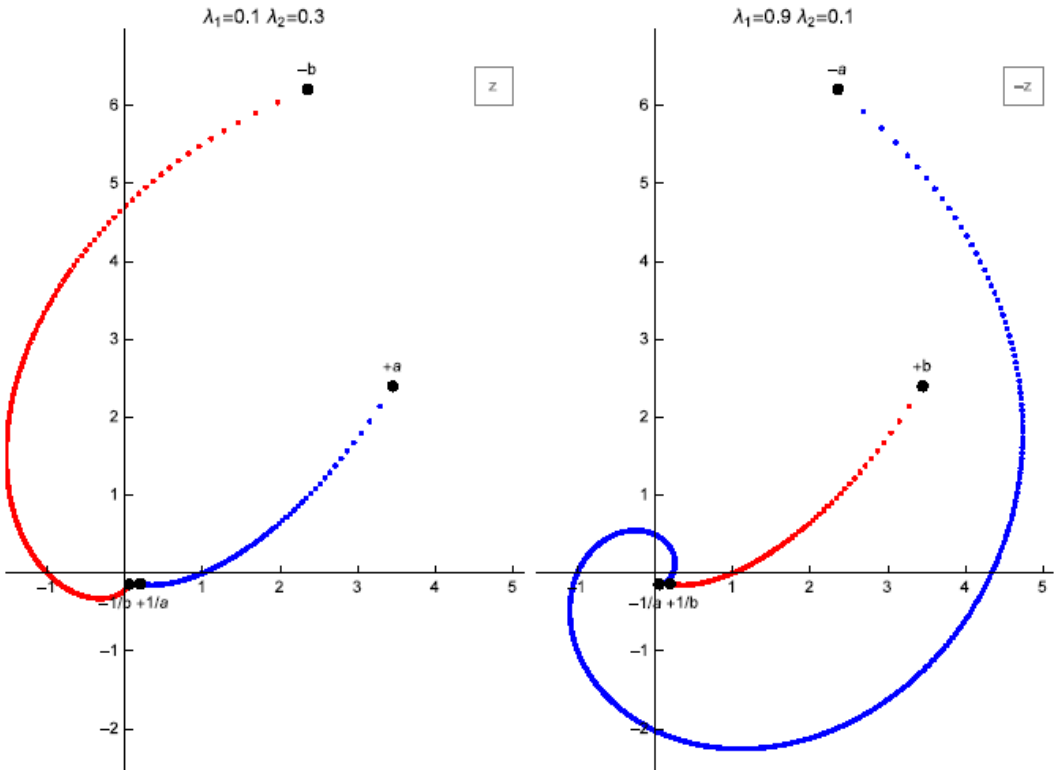
$$\hat{\lambda} \equiv \frac{\lambda_1 + \lambda_2}{2} - \frac{1}{2}\left(B - \frac{1}{2}\right)^2 - \frac{1}{24} = \frac{\lambda_1 + \lambda_2}{2} - \frac{1}{2}(\lambda_1 - \lambda_2)^2 - \frac{1}{24}$$

Seinberg-like symmetry

4) Expected symmetry duality between two gauge group [Aharony, Bergman, Jafferis]

$$U(N_1)_k \times U(N_2)_{-k} \text{ and } U(2N_2 - N_1 + k)_k \times U(N_1)_{-k}$$

$$(\lambda_1, \lambda_2) \to (2\lambda_2 - \lambda_1 + 1, \lambda_1)$$



So the expression is clearly invariant $h(\lambda_1, \lambda_2) = \frac{1}{4\pi} \log\left(\frac{ab+1}{a+b}\right)$

Conclusions

- More general systems with different symmetries?
- Classification of all consistent matching conditions is needed.
- How Qs are related to the wave functions? Applications for 3-point correlators?
- Relation to Integrability found in localization?
- Etc…

BFKL regime

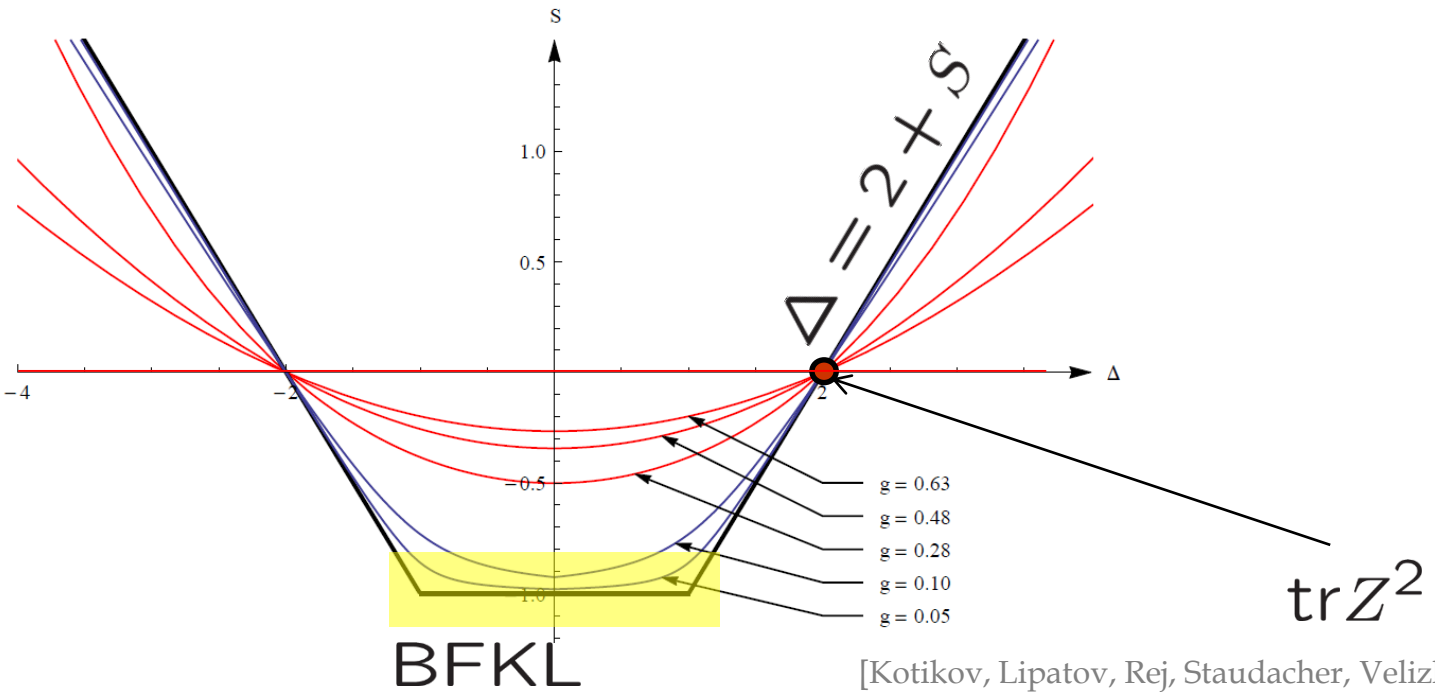
BFKL regime

Important class of single trace operators:

$$\mathrm{tr} D^S Z^2 + \text{permutations}$$

Spectrum for different spins:

[Brower, Polchinski, Strassler, -Itan `06]



[Kotikov, Lipatov, Rej, Staudacher, Velizhanin]

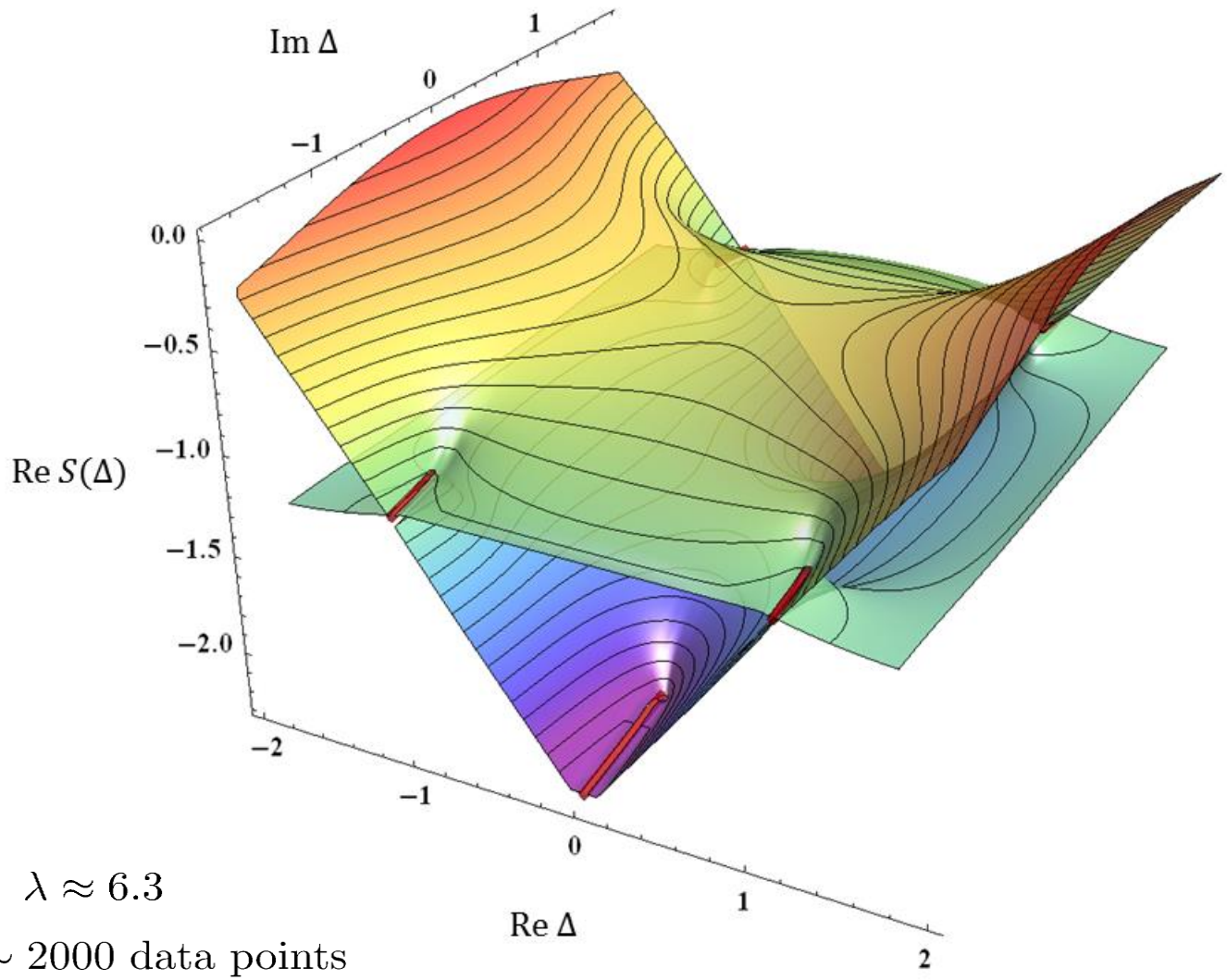
BFKL regime:

$S \to -1 \ , \ g \to 0$ So that: $\dfrac{g^2}{S+1} \simeq 1$ Resumming to all loops terms $\left(\dfrac{g^2}{S+1}\right)^n$

In this regime SYM is undistinguishable from the real QCD

Analytic structure at finite coupling

Gromov,FLM,Sizov 2015



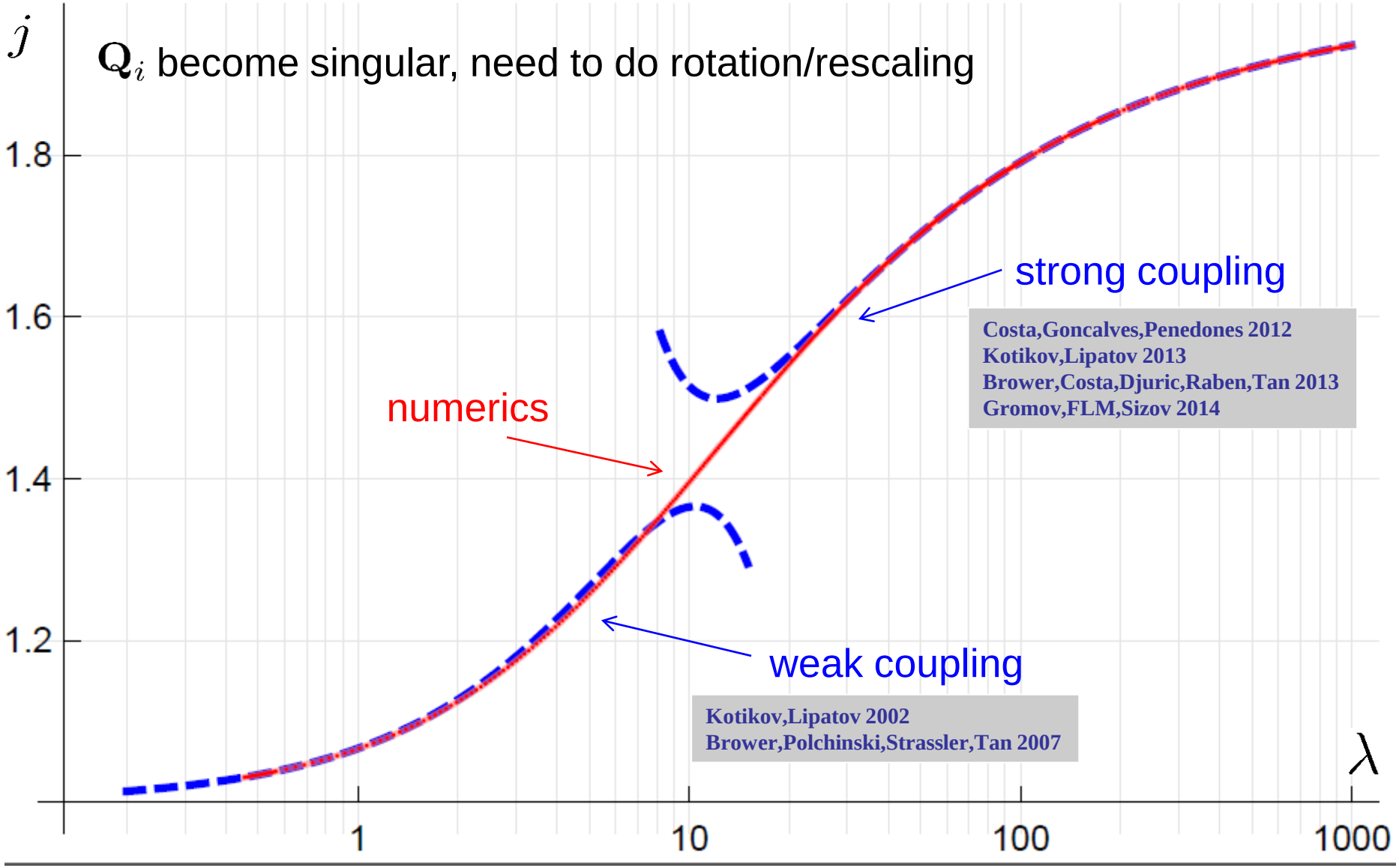
$\lambda \approx 6.3$

~ 2000 data points

BFKL pomeron intercept

$$j = 2 + S(\Delta)|_{\Delta=0}$$

Gromov,FLM,Sizov 2015



BFKL in QCD

- At the LO:

Jaroszewicz, 1982
Lipatov 1986
Kotikov,Lipatov 2002

$$\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma), \quad \Psi(\gamma) = \Gamma'(\gamma)/\Gamma(\gamma)$$

- At NLO:

Kotikov,Lipatov 2002
Kotikov,Lipatov 2000

$$\delta(\gamma) = -\left[\left(\frac{11}{3} - \frac{2n_f}{3N_c}\right)\frac{1}{2}\left(\chi^2(\gamma) - \Psi'(\gamma) + \Psi'(1-\gamma)\right) - \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{N_c}\right)\chi(\gamma\right.$$

$$-6\zeta(3) + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left(3 + \left(1 + \frac{n_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)$$

$$\left. -\Psi''(\gamma) - \Psi''(1-\gamma) - \frac{\pi^3}{\sin(\pi\gamma)} + 4\phi(\gamma)\right] .$$

$$\phi(\gamma) = -\int_0^1 \frac{dx}{1+x}\left(x^{\gamma-1} + x^{-\gamma}\right)\int_x^1 \frac{dt}{t}\ln(1-t)$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\frac{\Psi(n+1+\gamma) - \Psi(1)}{(n+\gamma)^2} + \frac{\Psi(n+2-\gamma) - \Psi(1)}{(n+1-\gamma)^2}\right]$$

BFKL regime

$S \to -1, g \to 0, \frac{g^2}{S+1} = \text{fixed}$ Resums contributions from all loop orders

N=4 SYM should give the highest transcendentality part of the QCD result

$$S = -1 + \sum_{n=1}^{\infty} g^{2n} \left[F_n\left(\frac{\Delta-1}{2}\right) + F_n\left(\frac{-\Delta-1}{2}\right) \right]$$

Costa,Goncalves,Penedones 2012

$$S_a(x) = \sum_{k=1}^{x} \frac{(\text{sign}(a))^k}{k^{|a|}}$$

$$S_{a,b,c,\ldots}(x) = \sum_{k=1}^{x} (\text{sign}(a))^k S_{b,c,\ldots}(k)$$

Leading order $F_1(x) = -4S_1(x)$

Reproduced from QSC in [Alfimov,Gromov,Kazakov 2014]

NLO

$$F_2(x) = 4\left(-\frac{3}{2}\zeta(3) + \pi^2 \log 2 + \frac{\pi^2}{3} S_1(x) + 2S_3(x) + \pi^2 S_{-1}(x) - 4S_{-2,1}(x)\right)$$

NNLO $F_3(x) =$?????

Basis for NNLO

Each term has transcendentality 5

$\Downarrow$

harmonic sums with transcendentality up to 5, and constants:

$$\pi, \log(2), \zeta(3), \zeta(5), \mathrm{Li}_4(1/2), \mathrm{Li}_5(1/2)$$

In total 288 elements

We derive analytic constraints from QSC
for expansion around poles at $\Delta_0 = 1, 3, 5, 7, \ldots$

Our result: BFKL at NNLO

$$S = -1 + \sum_{n=1}^{\infty} g^{2n} \left[F_n\left(\tfrac{\Delta-1}{2}\right) + F_n\left(\tfrac{-\Delta-1}{2}\right)\right]$$

NG,Levkovich-Maslyk,Sizov
Phys.Rev.Lett. 115 (2015)

$$\frac{1}{256} F_3 =$$

$$-\frac{5S_{-5}}{8} - \frac{S_{-4,1}}{2} + \frac{S_1 S_{-3,1}}{2} + \frac{S_{-3,2}}{2} - \frac{5S_2 S_{-2,1}}{4}$$

$$+\frac{S_{-4}S_1}{4} + \frac{S_{-3}S_2}{8} + \frac{3S_{3,-2}}{4} - \frac{3S_{-3,1,1}}{2} - S_1 S_{-2,1,1}$$

$$+S_{2,-2,1} + 3S_{-2,1,1,1} - \frac{3S_{-2}S_3}{4} - \frac{S_5}{8} + \frac{S_{-2}S_1S_2}{4}$$

$$+\pi^2\left[\frac{S_{-2,1}}{8} - \frac{7S_{-3}}{48} - \frac{S_{-2}S_1}{12} + \frac{S_1S_2}{48}\right]$$

$$+\zeta_3\left[-\frac{7S_{-1,1}}{4} + \frac{7S_{-2}}{8} + \frac{7S_{-1}S_1}{4} - \frac{S_2}{16}\right]$$

$$+\left[2\mathrm{Li}_4 - \frac{\pi^2\log^2 2}{12} + \frac{\log^4 2}{12}\right](S_{-1} - S_1) - \pi^4\left[\frac{2S_{-1}}{45} - \frac{S_1}{96}\right]$$

$$+\frac{\log^5 2}{60} - \frac{\pi^2\log^3 2}{36} - \frac{2\pi^4\log 2}{45} - \frac{\pi^2\zeta_3}{24} + \frac{49\zeta_5}{32} - 2\mathrm{Li}_5$$

Found from
iterative solution of QSC

Confirmed recently by an independent calculation by Caron-Huot, Herran