# Chern character of the Verlinde bundle

Dimitri Zvonkine
(Jussieu Mathematical Institute, CNRS, Paris)

String-Math, Collège de France, 30 June 2016

# Chern character of the Verlinde bundle

Alina Marian, Dragos Oprea, Rahul Pandharipande, Aaron Pixton,

Dimitri Zvonkine

String-Math, Collège de France, 30 June 2016

# Chern character of the Verlinde bundle

$$E_{g,n}(\rho_1, \ldots, \rho_n)$$

$$\downarrow$$

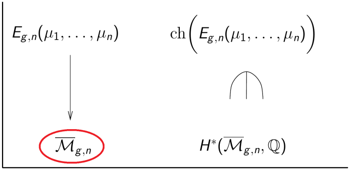
$$\overline{\mathcal{M}}_{g,n}$$

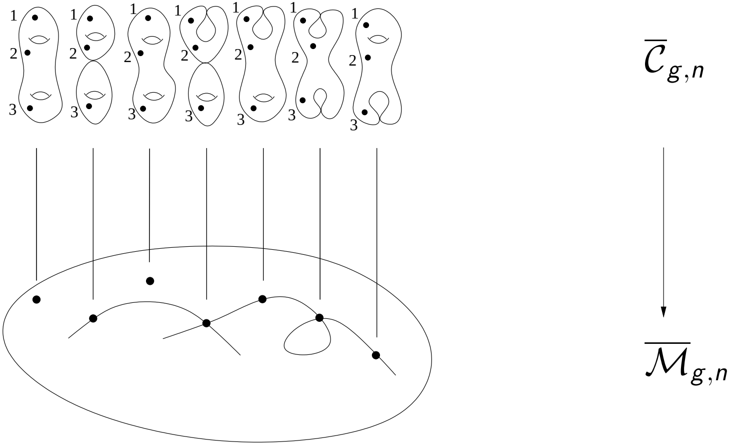
# Chern character of the Verlinde bundle

$$
\begin{array}{ccc}
E_{g,n}(\mu_1, \ldots, \mu_n) & \qquad & \mathrm{ch}\Big(E_{g,n}(\mu_1, \ldots, \mu_n)\Big) \\
\Big\downarrow & & \rotatebox{90}{$\in$} \\
\overline{\mathcal{M}}_{g,n} & & H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})
\end{array}
$$

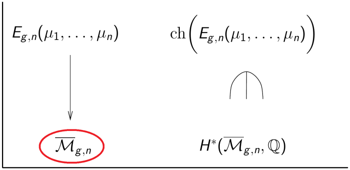

BUCKLE UP!

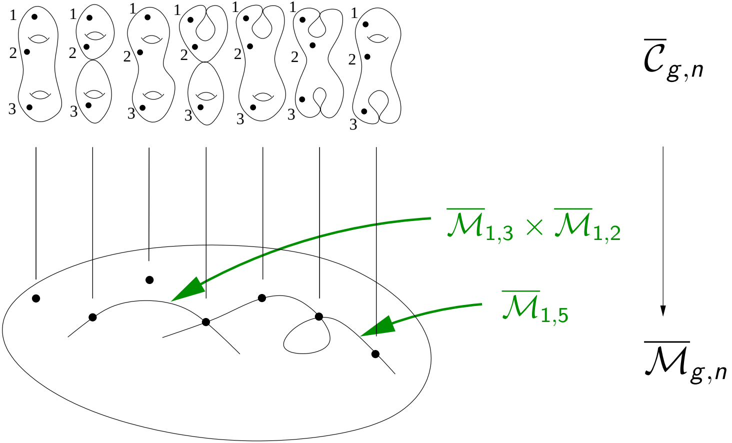
# Moduli space



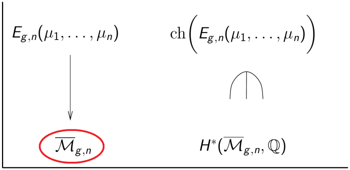



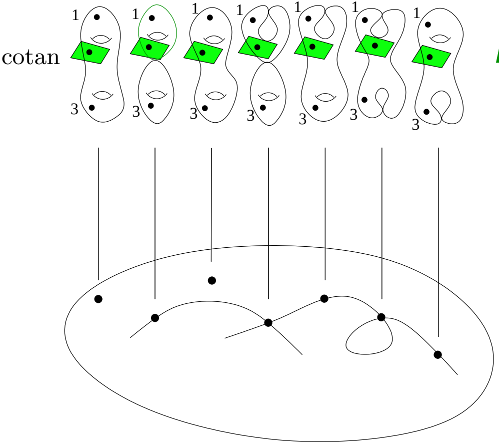
# Boundary strata



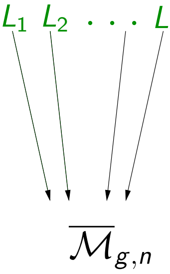



# Cotangent line bundles

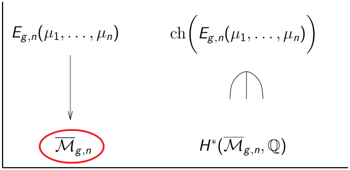


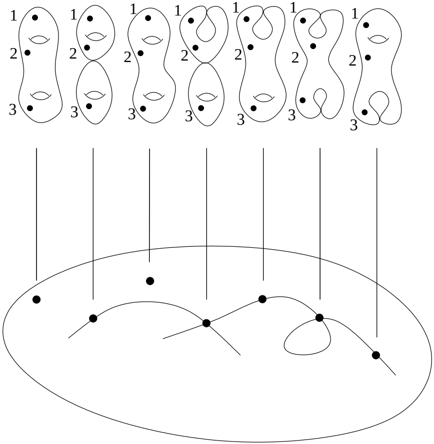




$$L_2 = s_2^*\omega_{\mathrm{rel}}$$



# The Hodge bundle

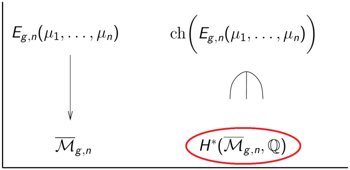



$$E_p = H^0(C_p, \omega_{\mathrm{rel}})$$

$$E$$
$$\downarrow$$
$$\overline{\mathcal{M}}_{g,n}$$

# Cohomology classes: $\psi$, $\lambda$, boundary

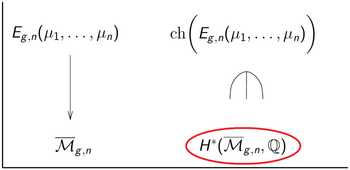



- $\psi_i := c_1(L_i)$ $\qquad$ $\psi_1,\ldots,\psi_n \in H^2(\overline{\mathcal{M}}_{g,n})$
  $(L_i)_p =$ cotangent line to $C_p$ at $i$th marking

- $\lambda := c_1(E)$ $\qquad$ $\lambda \in H^2(\overline{\mathcal{M}}_{g,n})$
  $E_p = \{$holomorphic 1-forms on $C_p\}$

- Boundary strata

# Cohomology classes: $\psi$, $\lambda$, boundary




- $\psi_i := c_1(L_i)$ $\qquad$ $\psi_1, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n})$
  $(L_i)_p =$ cotangent line to $C_p$ at $i$th marking

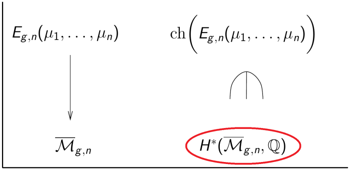
- $\lambda := c_1(E)$ $\qquad$ $\lambda \in H^2(\overline{\mathcal{M}}_{g,n})$
  $E_p = \{$holomorphic 1-forms on $C_p\}$

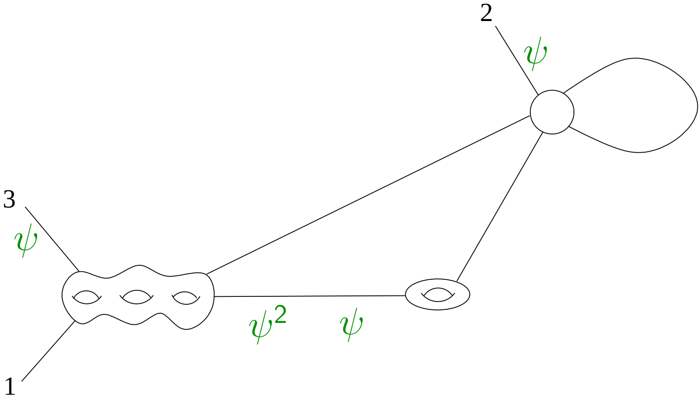
- **Example.** Restrict $E_{g,n}(\mu_1, \ldots, \mu_n)$ to $\mathcal{M}_{g,n}$. Then

$$\frac{c_1\left(E_{g,n}(\mu_1, \ldots, \mu_n)\right)}{\mathrm{rk}\left(E_{g,n}(\mu_1, \ldots, \mu_n)\right)} = \sum_{i=1}^{n} w(\mu_i)\psi_i - \frac{c}{2}\lambda.$$

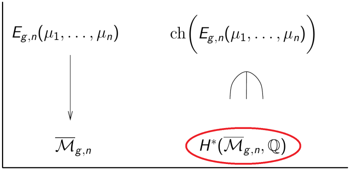
Coefficients $w$, $c$ found by [Marian, Oprea, Pandharipande].

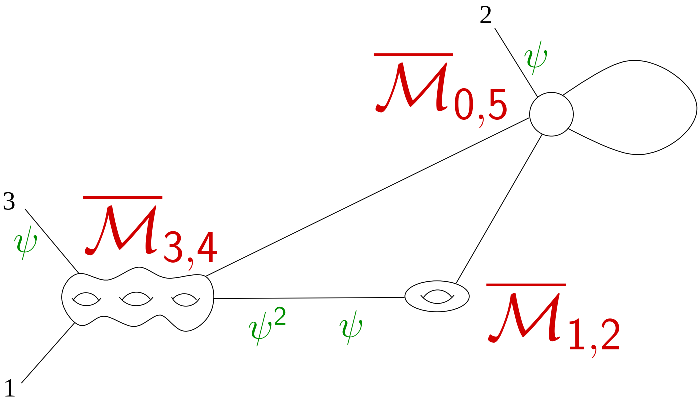
# Combining boundary with $\psi$-classes



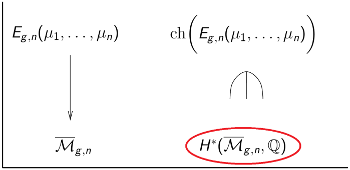



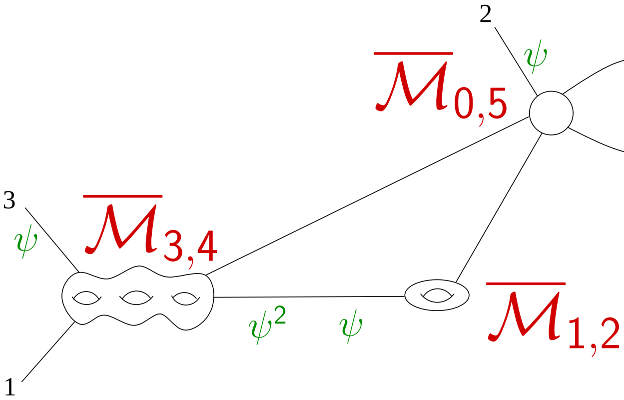
# Combining boundary with $\psi$-classes



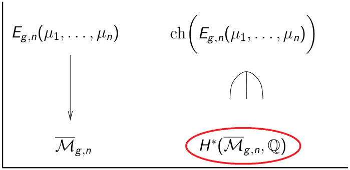
# Combining boundary with $\psi$-classes

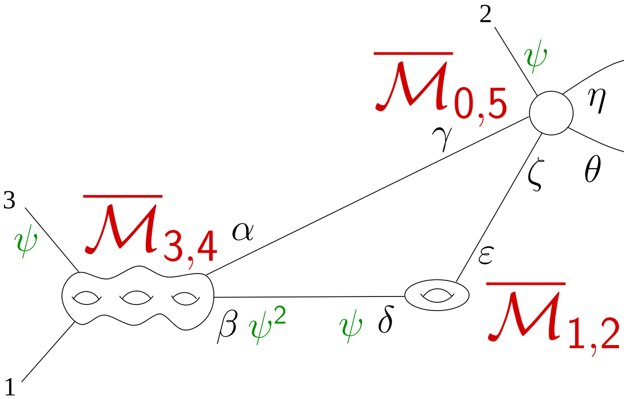



$$j : \overline{\mathcal{M}}_{3,4} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{6,3}$$

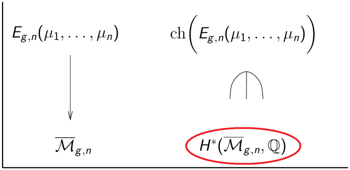
# Combining boundary with $\psi$-classes

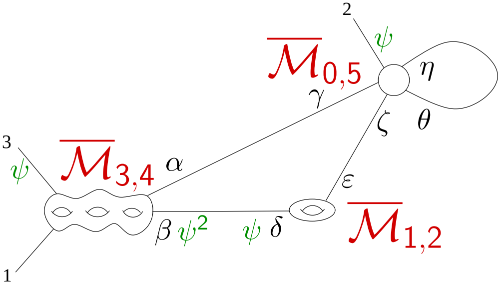



$$j : \overline{\mathcal{M}}_{3,4} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{6,3}$$

# Combining boundary with $\psi$-classes

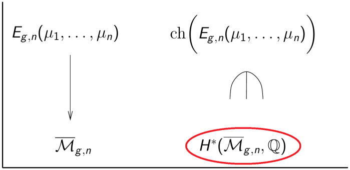


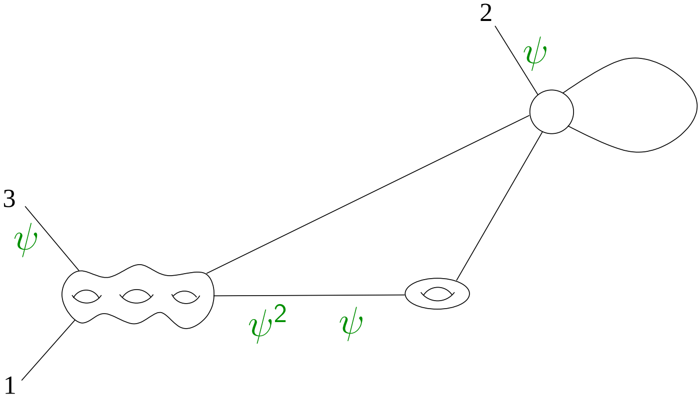




$$j : \overline{\mathcal{M}}_{3,4} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{6,3}$$

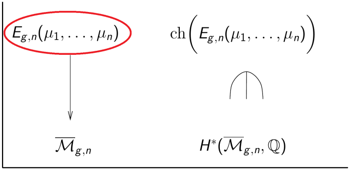
$$\frac{1}{\mathrm{Aut}(\Gamma)}\; j_* \left( \psi_3 \psi_\beta^2 \;\cdot\; \psi_\delta \;\cdot\; \psi_2 \right)$$

# Combining boundary with $\psi$-classes

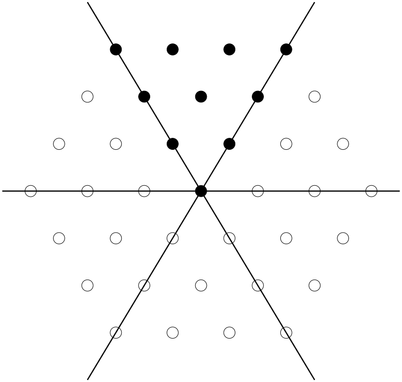


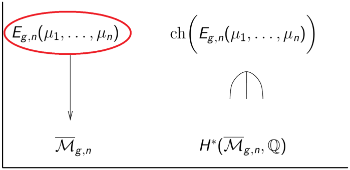
# Representations of $\mathfrak{g}$



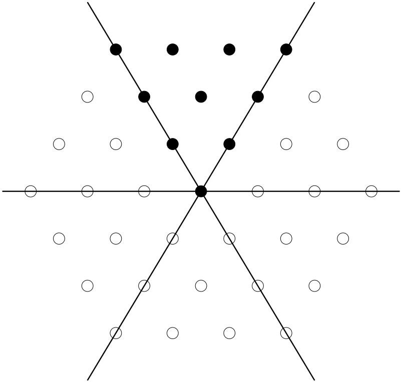

$\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$.



# Representations of $\mathfrak{g}$

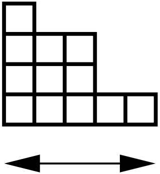



$\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$.



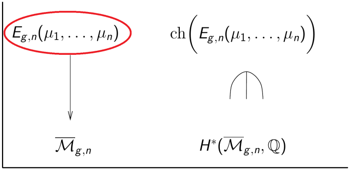
**Example:** $\mathfrak{g} = sl_N$.

Black dots =

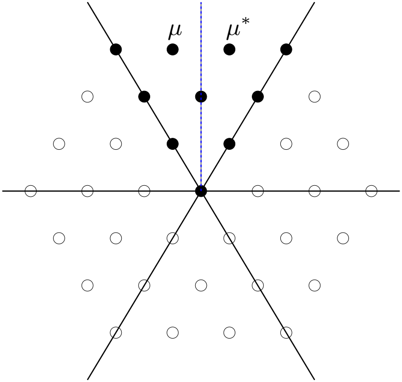


$N - 1$

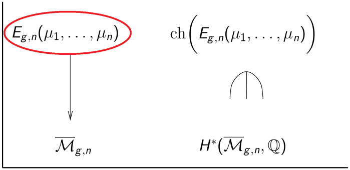
# Conjugate representations



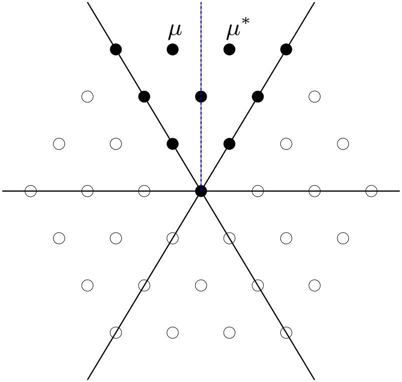

$\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$.



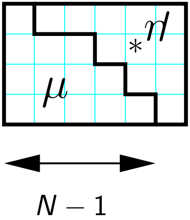
# Conjugate representations



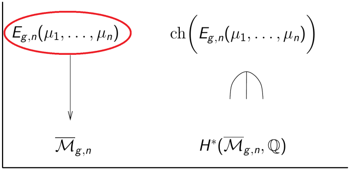

$\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$.



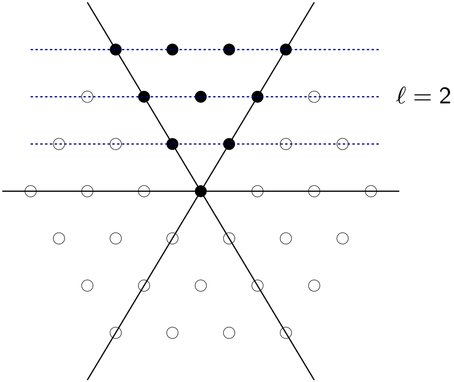

**Example:** $\mathfrak{g} = sl_N$.



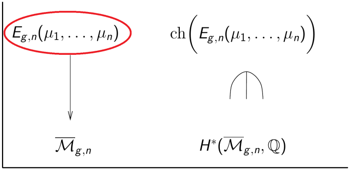
# Choosing a level



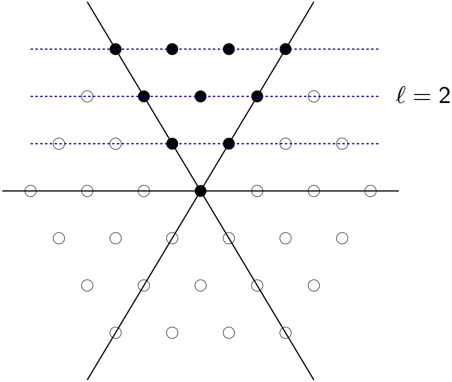

$\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$.
$\ell \geq 0$ = an integer (level).



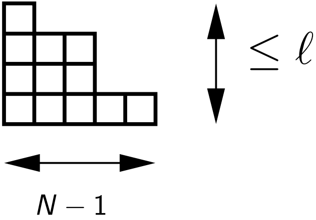
# Choosing a level



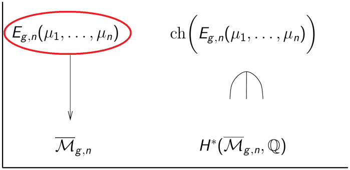

$\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$.
$\ell \geq 0$ = an integer (level).



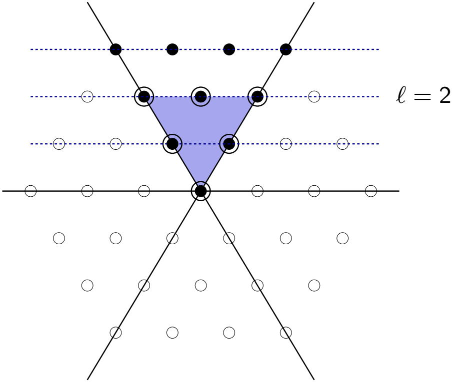

**Example:** $\mathfrak{g} = sl_N$.



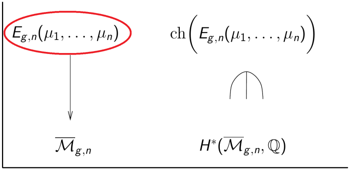
# Choosing a level

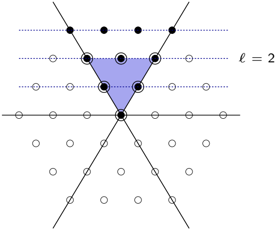



Indices $\mu_i$ for $1 \leq i \leq n$ belong to this set:



# Choosing a level




These are the $\mu$'s for which:

- the $\mathfrak{g}$-module $V_\mu$ extends to a representation of the central extension of $\mathfrak{g}((z))$ on which the central element acts by $\ell$;
- the symplectic form $\omega$ on the conjugacy class $C_\mu$ of $\exp(\mu)$ satisfies $\ell \cdot \omega \in H^2(C_\mu, \mathbb{Z})$.

# The moduli space

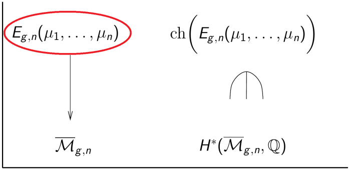
$$E_{g,n}(\mu_1, \ldots, \mu_n) \qquad \mathrm{ch}\Big(E_{g,n}(\mu_1, \ldots, \mu_n)\Big)$$

$$\downarrow \qquad\qquad\qquad\qquad \bigcap$$

$$\overline{\mathcal{M}}_{g,n} \qquad H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

| $G = U(1)$ | $G =$ compact form of $\mathfrak{g}$ |
|---|---|

# The moduli space



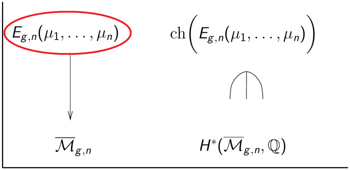

$$G = U(1)$$

- $C$ = Riemann surface

$$G = \text{compact form of } \mathfrak{g}$$

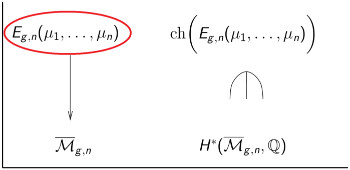
- $(C, x_1, \ldots, x_n)$ = Riemann surface with $n$ marked points

# The moduli space




| $G = U(1)$ | $G =$ compact form of $\mathfrak{g}$ |
|---|---|
| • $C =$ Riemann surface | • $(C, x_1, \ldots, x_n) =$ Riemann surface with $n$ marked points |
| • $\mathrm{Jac}(C) =$ flat connections on $C \times U(1)$ | • $\mathcal{M} =$ flat connections on $(C \backslash \{x_1, \ldots, x_n\}) \times G$ with monodromy $\exp(\mu_i)$ at $x_i$ |

# The line bundle $\mathcal{L}$



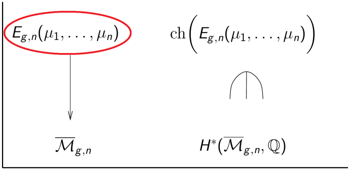

$G = U(1)$

- $C$ = Riemann surface
- $\mathrm{Jac}(C)$ = flat connections on $C \times U(1)$
- $\mathcal{L} \to \mathrm{Jac(C)}$ the line bundle with $c_1(\mathcal{L}) = [\Theta]$

$G$ = compact form of $\mathfrak{g}$

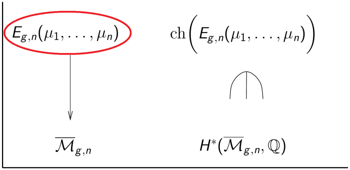
- $(C, x_1, \ldots, x_n)$ = Riemann surface with $n$ marked points
- $\mathcal{M}$ = flat connections on $(C\backslash\{x_1,\ldots,x_n\})\times G$ with monodromy $\exp(\mu_i)$ at $x_i$
- $\mathcal{L} \to \mathcal{M}$ a natural line bundle

# The space of sections of $\mathcal{L}^\ell$




| $G = U(1)$ | $G =$ compact form of $\mathfrak{g}$ |
|---|---|
| • $C$ = Riemann surface | • $(C, x_1, \dots, x_n)$ = Riemann surface with $n$ marked points |
| • $\mathrm{Jac}(C)$ = flat connections on $C \times U(1)$ | • $\mathcal{M}$ = flat connections on $(C \backslash \{x_1, \dots, x_n\}) \times G$ with monodromy $\exp(\mu_i)$ at $x_i$ |
| • $\mathcal{L} \to \mathrm{Jac(C)}$ the line bundle with $c_1(\mathcal{L}) = [\Theta]$ | • $\mathcal{L} \to \mathcal{M}$ a natural line bundle |
| • $E_g = H^0(\mathrm{Jac}(C), \mathcal{L}^{\otimes \ell})$ = {$\theta$-functions of weight $\ell$} | • $E_{g,n}(\mu_1, \dots, \mu_n) = H^0(\mathcal{M}, \mathcal{L}^{\otimes \ell})$ = space of conformal blocks |

# The rank



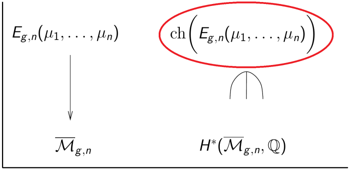

**$G = U(1)$**

- $E_g = H^0(\mathrm{Jac}(C), \mathcal{L}^{\otimes \ell})$ $= \{\theta$-functions of weight $\ell\}$
- $\mathrm{rk}(E_g) = \ell^g$

**$G =$ compact form of $\mathfrak{g}$**

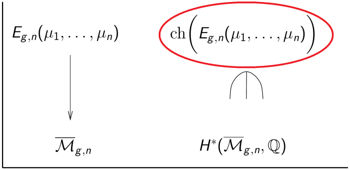
- $E_{g,n}(\mu_1, \ldots, \mu_n) = H^0(\mathcal{M}, \mathcal{L}^{\otimes \ell})$ $=$ space of conformal blocks
- $\mathrm{rk}(E_{g,n}(\mu_1, \ldots, \mu_n))$ given by the Verlinde formula

# Coefficients $c$ and $w(\mu)$



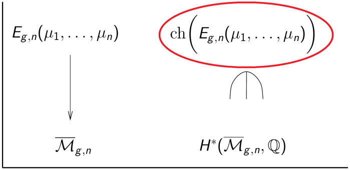

- $(\ ,\ )$ = Killing form; length of longer root $= \sqrt{2}$.

# Coefficients $c$ and $w(\mu)$



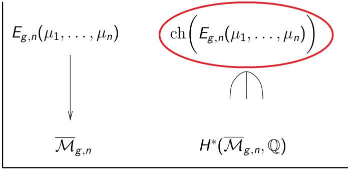

- $( \, , \, )$ = Killing form; length of longer root $= \sqrt{2}$.
- $\rho = \frac{1}{2}(\sum$ positive roots$)$

# Coefficients $c$ and $w(\mu)$



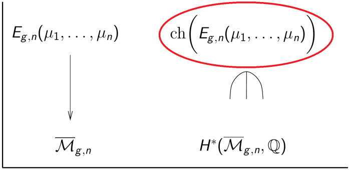

- $(\ ,\ )$ = Killing form; length of longer root $= \sqrt{2}$.
- $\rho = \frac{1}{2}(\sum$ positive roots$)$
- $\check{h} =$ dual Coxeter number of $\mathfrak{g}$
  $= 1 + (\sum$ of coefficients of the highest *short* root
  in the basis of simple roots$)$.

# Coefficients $c$ and $w(\mu)$



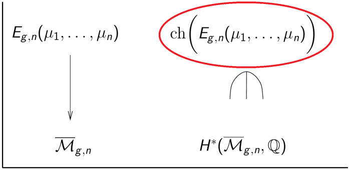

- $(\ ,\ )$ = Killing form; length of longer root $= \sqrt{2}$.
- $\rho = \frac{1}{2}(\sum$ positive roots$)$
- $\check{h}$ = dual Coxeter number of $\mathfrak{g}$
  $= 1 + (\sum$ of coefficients of the highest *short* root
  in the basis of simple roots$)$.
- $c = \dfrac{\ell \dim(\mathfrak{g})}{\check{h} + \ell}$ = conformal anomaly

# Coefficients $c$ and $w(\mu)$




- $(\ ,\ ) =$ Killing form; length of longer root $= \sqrt{2}$.
- $\rho = \frac{1}{2}(\sum$ positive roots$)$
- $\check{h} =$ dual Coxeter number of $\mathfrak{g}$
  $= 1 + (\sum$ of coefficients of the highest *short* root
  in the basis of simple roots).
- $c = \dfrac{\ell \dim(\mathfrak{g})}{\check{h} + \ell} =$ conformal anomaly
- $w(\mu) = \dfrac{(\mu, \mu + 2\rho)}{\check{h} + \ell}.$

# Coefficients $c$ and $w(\mu)$

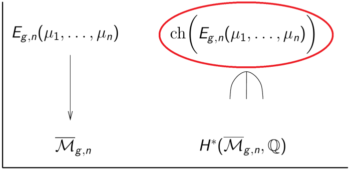



**Example:** $\mathfrak{g} = sl_N$.

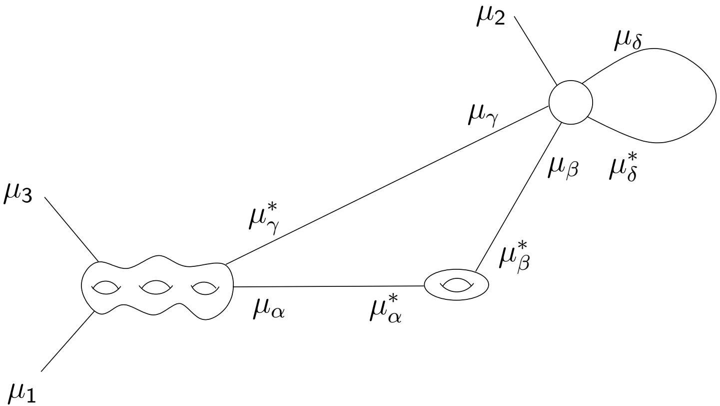
- $\rho = (N-1, N-2, \dots, 1)$
- $\check{h} = N$
- $c = \dfrac{\ell(N^2-1)}{N+\ell}$
- $\mu = (a_1 \geq a_2 \geq \cdots \geq a_{N-1})$

$$w(\mu) = \frac{1}{2(N+\ell)}\left[\sum_{i=1}^{N-1} a_i(a_i + N - 2i + 1) - \frac{1}{N}\left(\sum a_i\right)^2\right]$$

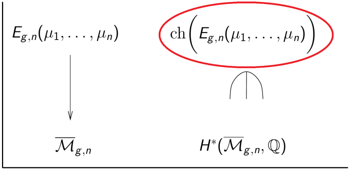
# Graph decorations

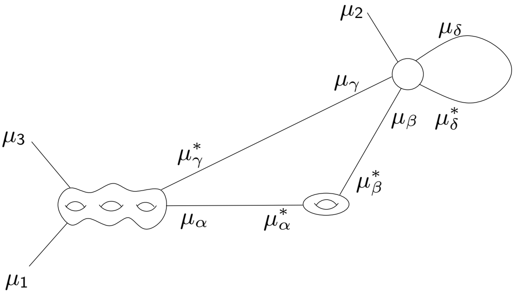



**Definition.** $\Gamma$ = stable graph. A *decoration* assigns a level $\ell$ representation $\mu(h)$ to every half-edge $h$ of $\Gamma$:



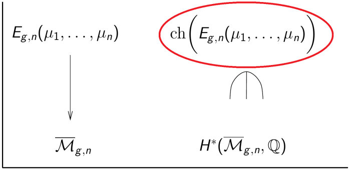
# Contribution of a graph

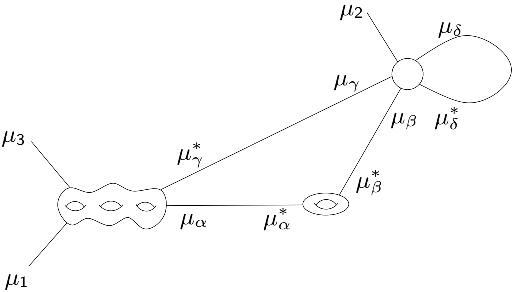



- **Vertex contribution:** $\mathrm{rk}_v$ given by the Verlinde formula.

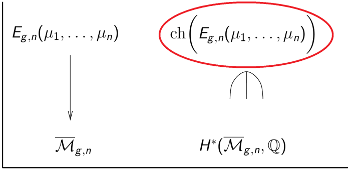
# Contribution of a graph

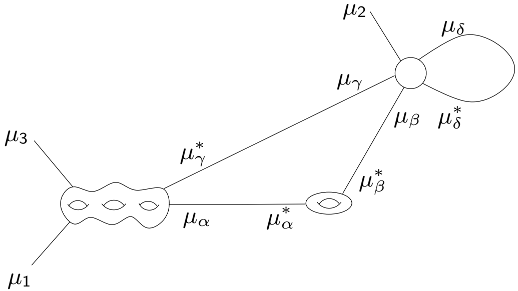



- **Vertex contribution:** $\mathrm{rk}_v$ given by the Verlinde formula.
- **Leg contribution:** $L_i = e^{w(\mu_i)\psi_i}$.

# Contribution of a graph

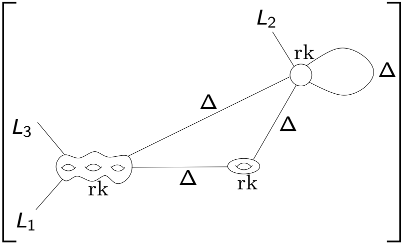



- **Vertex contribution:** $\mathrm{rk}_v$ given by the Verlinde formula.
- **Leg contribution:** $L_i = e^{w(\mu_i)\psi_i}$.
- **Edge contribution:** $\Delta_e = \dfrac{1 - e^{w(\mu(e))(\psi'_e + \psi''_e)}}{\psi'_e + \psi''_e}$.

# Final answer

$$\mathrm{ch}\left(E_{g,n}(\mu_1,\ldots,\mu_n)\right) = e^{-\frac{c}{2}\lambda} \sum_{\substack{\text{stable graphs}\\ \text{decorations}}}$$




- **Vertex contribution:** $\mathrm{rk}_v$ given by the Verlinde formula.
- **Leg contribution:** $L_i = e^{w(\mu_i)\psi_i}$.
- **Edge contribution:** $\Delta_e = \dfrac{1 - e^{w(\mu(e))(\psi'_e + \psi''_e)}}{\psi'_e + \psi''_e}$.

# What was known

- $\mathrm{ch}\left(E_{0,n}(\mu_1, \ldots, \mu_n)\right)$ for $g = 0$
[Fakhruddin]

# What was known

- $\mathrm{ch}\,(E_{0,n}(\mu_1,\ldots,\mu_n))$ for $g=0$
  [Fakhruddin]
- $c_1\,(E_{g,n}(\mu_1,\ldots,\mu_n)) = \sum w(\mu_i)\psi_i - \frac{c}{2}\lambda$
  [Marian, Oprea, Pandharipande]

# What was known

- $\mathrm{ch}\left(E_{0,n}(\mu_1, \ldots, \mu_n)\right)$ for $g = 0$
  [Fakhruddin]
- $c_1\left(E_{g,n}(\mu_1, \ldots, \mu_n)\right) = \sum w(\mu_i)\psi_i - \frac{c}{2}\lambda$
  [Marian, Oprea, Pandharipande]
- $E_{g,n}(\mu_1, \ldots, \mu_n)$ is projectively flat over $\mathcal{M}_{g,n}$.
  $$\Longrightarrow \quad \mathrm{ch} = \mathrm{rk} \cdot e^{\mathrm{slope}}$$
  [Tsuchiya, Ueno, Yamada]

# What was known

- $\mathrm{ch}\left(E_{0,n}(\mu_1,\ldots,\mu_n)\right)$ for $g=0$
  [Fakhruddin]
- $c_1\left(E_{g,n}(\mu_1,\ldots,\mu_n)\right)=\sum w(\mu_i)\psi_i-\frac{c}{2}\lambda$
  [Marian, Oprea, Pandharipande]
- $E_{g,n}(\mu_1,\ldots,\mu_n)$ is projectively flat over $\mathcal{M}_{g,n}$.
  $$\Longrightarrow \quad \mathrm{ch}=\mathrm{rk}\cdot e^{\mathrm{slope}}$$
  [Tsuchiya, Ueno, Yamada]
- $\mathrm{ch}\left(E_{g,n}(\mu_1,\ldots,\mu_n)\right)$ form a CohFT (factorization rules at the boundary of $\overline{\mathcal{M}}_{g,n}$).
  [Faltings]

# What was known

- $\mathrm{ch}\left(E_{0,n}(\mu_1, \ldots, \mu_n)\right)$ for $g = 0$
  [Fakhruddin]
- $c_1\left(E_{g,n}(\mu_1, \ldots, \mu_n)\right) = \sum w(\mu_i)\psi_i - \frac{c}{2}\lambda$
  [Marian, Oprea, Pandharipande]
- $E_{g,n}(\mu_1, \ldots, \mu_n)$ is projectively flat over $\mathcal{M}_{g,n}$.
  $$\Longrightarrow \quad \mathrm{ch} = \mathrm{rk} \cdot e^{\mathrm{slope}}$$
  [Tsuchiya, Ueno, Yamada]
- $\mathrm{ch}\left(E_{g,n}(\mu_1, \ldots, \mu_n)\right)$ form a CohFT (factorization rules at the boundary of $\overline{\mathcal{M}}_{g,n}$).
  [Faltings]
- This CohFT is semisimple (because the Verlinde algebra is).
  [Verlinde]

# What was known

- $\mathrm{ch}\left(E_{0,n}(\mu_1, \dots, \mu_n)\right)$ for $g = 0$
  [Fakhruddin]
- $c_1\left(E_{g,n}(\mu_1, \dots, \mu_n)\right) = \sum w(\mu_i)\psi_i - \frac{c}{2}\lambda$
  [Marian, Oprea, Pandharipande]
- $E_{g,n}(\mu_1, \dots, \mu_n)$ is projectively flat over $\mathcal{M}_{g,n}$.
  $$\Longrightarrow \quad \mathrm{ch} = \mathrm{rk} \cdot e^{\mathrm{slope}}$$
  [Tsuchiya, Ueno, Yamada]
- $\mathrm{ch}\left(E_{g,n}(\mu_1, \dots, \mu_n)\right)$ form a CohFT (factorization rules at the boundary of $\overline{\mathcal{M}}_{g,n}$).
  [Faltings]
- This CohFT is semisimple (because the Verlinde algebra is).
  [Verlinde]
- Semisimple CohFTs are classified.
  [Givental, Teleman]

# And therefore...

- $\mathrm{ch}\,(E_{g,n}(\mu_1, \ldots, \mu_n))$ is obtained from the Verlinde algebra by an $R$-matrix action.
- $\mathrm{rk} \cdot e^{\mathrm{slope}}$ implies $R = e^{zr}$.
- The knowledge of $c_1$ determines $r$.


obrigado
Dank U
Merci
mahalo
Köszi
спасибо
Grazie
Thank you
mauruuru
Takk
Gracias
Dziękuję
Děkuju
danke
Kiitos