Correlation Functions in Superconformal Field Theories

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with Gerchkovitz, Ishtiaque, Karasik, Komargodski, Pufu; arXiv:1602.0597

Introduction

- In this talk we tackle a long-standing problem in supersymmetric QFT

  Solve for the chiral ring of $d = 4$ $\mathcal{N} = 2$ superconformal field theories

  - Exact solution for simplest non-holomorphic local observables
  - Algorithm to compute these observables as a function of the couplings, which span the conformal manifold $\mathcal{M}$ of the SCFT

  - Non-perturbative solution of QFT correlators in $d > 2$ at finite N

Chiral Operators

- Supersymmetric theories admit chiral operators

$$[\overline{Q}, \mathcal{O}] = 0$$

The "holomorphy" revolution largely unraveled the vacuum structure of supersymmetric quantum field theories (1990-2000's):

- Moduli Space of Vacua: supersymmetric ground states $|\Psi\rangle$
- Holomorphic observables: one-point functions of chiral operators

$$\langle\Psi|\mathcal{O}|\Psi\rangle$$

Two milestones of this era are the Seiberg-Witten solution of 4d $\mathcal{N} = 2$ gauge theories and Nekrasov-Okounkov's ab-initio microscopic instanton counting derivation

Extremal Correlators

- Our goal is to study the following non-holomorphic correlators in 4d $\mathcal{N}=2$ superconformal field theories

$$\left\langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)\ldots\mathcal{O}_{I_n}(x_n)\overline{\mathcal{O}}_{\bar{J}}(y)\right\rangle_{\{\tau^i,\bar{\tau}^{\bar{i}}\}}$$

where

$$[\overline{Q}^a_{\dot{\alpha}},\mathcal{O}_I]=0 \qquad\qquad [Q^a_\alpha,\overline{\mathcal{O}}_{\bar{J}}]=0$$

$\{\tau^i,\bar{\tau}^{\bar{i}}\}$ are coordinates in the space of couplings (space of CFTs) $\mathcal{M}$

- Extremal correlators do not preserve any Poincaré supersymmetries
- Unitarity in the Hilbert space of the superconformal field theory implies

$$\Delta[\mathcal{O}_I]=\frac{1}{2}R_I \qquad\qquad R_I : U(1)_R \text{ charge}$$

- $\mathcal{O}_I$ are sections of a holomorphic vector bundle over $\mathcal{M}$

Chiral Ring

- Chiral primary operators define the so-called chiral ring

$$\mathcal{O}_I(x)\mathcal{O}_J(0) = \sum_K C_{IJ}^K \mathcal{O}_K(0)$$

- Multiplication in the ring is the OPE of CFT

- Chiral ring is freely generated. Choose a basis that

$$\mathcal{O}_I(x)\mathcal{O}_J(0) = \mathcal{O}_I\mathcal{O}_J(0)$$

- Chiral ring data and extremal correlators captured by

$$\left\langle \mathcal{O}_I(x)\,\overline{\mathcal{O}}_{\bar{J}}(0)\right\rangle_{\{\tau^i,\bar{\tau}^{\bar{i}}\}} = \frac{G_{I\bar{J}}(\tau^i,\bar{\tau}^{\bar{i}})}{|x|^{2\Delta_I}}\delta_{\Delta_I\Delta_{\bar{J}}}$$

$G_{I\bar{J}}(\tau^i,\bar{\tau}^{\bar{i}})$ is a Hermitean metric on the chiral ring vector bundle over $\mathcal{M}$

Zamolodchikov Metric

- $G_{i\bar{j}}(\tau^i, \bar{\tau}^{\bar{i}})$ for $\Delta = 2$ determines the Zamolodchikov metric
  - Measures distance in the space of CFTs $\mathcal{M}$
    - In $d = 2$, $\mathcal{M}$ is the space of vacua of string theory
    - $G_{i\bar{j}}$ in $\mathcal{N} = (2, 2)$ SCFTs is the metric on Calabi-Yau moduli space
    - In $d = 4$, $\mathcal{M}$ is the space of vacua in AdS_5 quantum gravity (AdS/CFT)
    - $G_{i\bar{j}}$ in $\mathcal{N} = 2$ SCFTs related to quantization of Teichmuller space
  - Generalizes conformal anomalies. Lead to novel Weyl-cohomology classes

$$T^\mu_\mu \supset G_{ij}\, \nabla_\mu \tau^i \nabla^\mu \tau^j$$

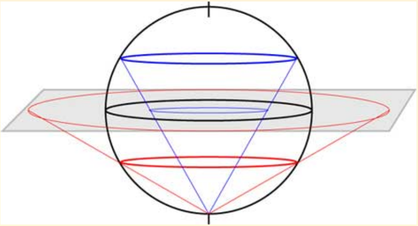
$$T^\mu_\mu \supset G_{ij} \left(\hat{\Box}\tau^i \hat{\Box}\tau^j - 2 \left(R^{\mu\nu} - \frac{1}{3} R g^{\mu\nu} \right) \nabla_\mu \tau^i \nabla_\nu \tau^j \right)$$

Determining The Exact Chiral Ring

- Chiral ring data and extremal correlators captured by

$$\left\langle \mathcal{O}_I(\infty)\,\overline{\mathcal{O}}_{\bar{J}}(0)\right\rangle_{\mathbb{R}^4} = G_{I\bar{J}}(\tau^i,\bar{\tau}^{\bar{i}})\,\delta_{\Delta_I \Delta_{\bar{J}}}$$

- Strategy: conformally map the SCFT from $\mathbb{R}^4$ to S^4



  - Observable is IR finite
  - $\mathcal{O}_I$ are primary operators, mapping to S^4 should be trivial, right?

Ambiguities And Conformal Anomalies

- Mapping correlators in $\mathbb{R}^4$ to S^4 is afflicted by:
  - Ambiguities
  - New conformal anomalies $\Longrightarrow$ Operator mixing
- Generating function of correlators of primary operators in a CFT in X

$$Z[X] = \left\langle \exp \int d^D x \sqrt{g} \sum_I \tau^{(I)}(x) \mathcal{O}_I(x) \right\rangle_X$$

$\tau^{(I)}(x)$: background sources for primary operators

- In QFT, ambiguities captured by local functionals, e.g.

$$\int d^4 x \sqrt{g(x)}\, \tau^{(2)}(x) R(x) f(\tau^i(x), \bar{\tau}^{\bar{i}}(x))$$

$\tau^{(2)}$: source for operator of dimension 2

R: Ricci scalar in X

Ambiguities

- In $\mathbb{R}^4$, local terms do not lead to separated point ambiguities

$$\langle \mathcal{O}_{I_1}(x_1) \dots \mathcal{O}_{I_n}(x_n) \rangle_{\mathbb{R}^4} = \frac{\delta^n}{\delta\tau^{(I_1)}(x_1) \dots \delta\tau^{(I_n)}(x_n)} \ln Z[\mathbb{R}^4]$$

- In curved space, these local terms lead to separated point ambiguities, e.g

$$\langle \mathcal{O}_{\Delta=2}(x) \rangle = R(x) f(\tau^i, \bar{\tau}^{\bar{i}})$$

  - This can be interpreted as operator mixing between $\mathcal{O}_{\Delta=2}$ and $\mathbb{1}$ on S^4
  - In general, operators with $\Delta \in \mathbb{N}$ mix with lower dimensional operators

$$\mathcal{O}_\Delta \to \alpha_1(\tau^i, \bar{\tau}^{\bar{i}})\, R\, \mathcal{O}_{\Delta-2} + \alpha_2(\tau^i, \bar{\tau}^{\bar{i}})\, R^2\, \mathcal{O}_{\Delta-4} + \dots$$

  - In generic CFTs this mixing can be removed by a choice of local counterterms

Conformal Anomalies

How about in $\mathcal{N}=2$ superconformal field theories?

- There do not exist $\mathcal{N}=2$ supersymmetric counterterms which undo mixing

$$\alpha_k(\tau^i,\bar\tau^{\bar i}) \to \alpha_k(\tau^i,\bar\tau^{\bar i}) + \mathcal{F}_k(\tau^i) + \overline{\mathcal{F}}_k(\bar\tau^{\bar i})$$

- Supersymmetry relates mixing counterterm to a cohomologically non-trivial Weyl anomaly

Mixing counterterm of exactly marginal operators with $\mathbb{1}$ is related by supersymmetry to conformal anomaly

J.G, Komargodski, Hsin, Schwimmer, Seiberg, Theisen

$$\begin{aligned}\delta_\Sigma \ln Z &= \int d^4x d^8\theta E(\Sigma+\overline{\Sigma})K(\tau^i,\bar\tau^{\bar i}) \\ &\supset \int d^4x\sqrt{g}\,\delta\sigma\left[G_{i\bar j}\left(\hat\Box\tau^i\hat\Box\bar\tau^{\bar j} - 2\left(R^{\mu\nu}-\frac{1}{3}Rg^{\mu\nu}\right)\nabla_\mu\tau^i\nabla_\nu\bar\tau^{\bar j}\right)\right] \\ &+\ldots\end{aligned}$$

- Operator mixing on S^4 is physical in $\mathcal{N}=2$ theories

Summary

- Sphere background induces non-trivial operator mixing in $\mathcal{N} = 2$ theories
- Disentangle operator mixing on S^4 to extract correlators in $\mathbb{R}^4$
- Perform a Gram-Schmidt diagonalization of S^4
  - Operators only mix with lower dimensional operators
  - Diagonalize sequentially for $\Delta = 0, 1, 2, \ldots$

$\Longrightarrow$

Need to learn how to compute

$$\left\langle \mathcal{O}_I(N) \overline{\mathcal{O}}_{\bar{I}}(S) \right\rangle_{S^4}$$

Deformed the $\mathcal{N} = 2$ SCFT on S^4

- Deform $\mathcal{N} = 2$ SCFT with operators in the chiral ring
  - Assign to a chiral operator $\mathcal{O}$ an $\mathcal{N} = 2$ supersymmetric invariant

$$\mathcal{O} \longrightarrow \tau_{\mathcal{O}} \int d^4\theta \, \mathcal{E} \, \mathcal{O} + \text{c.c.}$$

  - Deformation preserves the "massive" $\mathcal{N} = 2$ supersymetry algebra on S^4

$$SU(2,2|2) \to OSp(2|4)$$

  - The deformation induced by $\mathcal{O}$ takes the form

$$\mathcal{L}_{\mathcal{O}}[S^4] = \mathcal{L}_{\mathcal{O}}[\mathbb{R}^4] + O(1/r) + O(1/r^2)$$

Ward Identity

- Consider the partition function of the deformed $\mathcal{N}=2$ SCFT on S^4

$$Z[S^4](\tau^i,\bar{\tau}^{\bar{i}};\tau^A,\bar{\tau}^{\bar{A}})$$

Using the remarkable curved space Ward identity J.G, Ishtiaque

$$\int d^4x\sqrt{g(x)}\int d^4y\sqrt{g(y)}\,\langle\int d^4\theta\,\mathcal{E}\,\mathcal{O}_I(x)\int d^4\bar{\theta}\,\bar{\mathcal{E}}\,\overline{\mathcal{O}}_{\bar{I}}(y)\rangle_{S^4}=\langle\mathcal{O}_I(N)\overline{\mathcal{O}}_{\bar{I}}(S)\rangle_{S^4}$$

$\Longrightarrow$

$$\frac{1}{Z[S^4](\tau^i,\bar{\tau}^{\bar{i}})}\partial_{\tau^I}\partial_{\bar{\tau}^{\bar{I}}}Z[S^4](\tau^i,\bar{\tau}^{\bar{i}};\tau^A,\bar{\tau}^{\bar{A}})\bigg|_{\tau^A=\bar{\tau}^{\bar{A}}=0}=\langle\mathcal{O}_I(N)\overline{\mathcal{O}}_{\bar{I}}(S)\rangle_{S^4}$$

can relate derivatives of $Z[S^4](\tau^i,\bar{\tau}^{\bar{i}};\tau^A,\bar{\tau}^{\bar{A}})$ to chiral primary correlators

Kähler Potential on $\mathcal{M}$

- Consider chiral operators with $\Delta = 2$: Zamolodchikov metric
- Must diagonalize operator mixing of $\Delta = 2$ operators with $\mathbb{1}$ on S^4

$$\langle \mathcal{O}_i(\infty)\, \overline{\mathcal{O}}_{\bar{j}}(0)\rangle_{\mathbb{R}^4} \equiv G_{i\bar{j}}(\tau^i, \bar{\tau}^{\bar{i}}) = \det\left[\frac{1}{Z[S^4]}\begin{pmatrix} Z[S^4] & \partial_{\tau^i} Z[S^4] \\ \partial_{\bar{\tau}^{\bar{j}}} Z[S^4] & \partial_{\tau^i}\partial_{\bar{\tau}^{\bar{j}}} Z[S^4]\end{pmatrix}\right]$$

$$= \partial_{\tau^i}\partial_{\bar{\tau}^{\bar{j}}} \ln Z[S^4]$$

This gives another proof of the relation:

$$\boxed{Z[S^4] = r^{-4a} e^{K(\tau^i, \bar{\tau}^{\bar{i}})}}$$

Gerchkovitz, J.G, Komargodski; J.G, Ishtiaque; J.G, Komargodski, Hsin, Schwimmer, Seiberg, Theisen

- K is the Kähler potential on $\mathcal{M}$: $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$
- $Z[S^4]$ exactly computable by supersymmetric localization Pestun

$\Longrightarrow$

Exact metric on $\mathcal{M}$ in many theories, including instanton corrections

- This is the four-dimensional avatar of the two-dimensional counterpart

Jockers,Kumar,Lapan,Morrison,Romo

J.G.,Lee

$$Z_A[S^2] = r^{c/3} e^{-K_{tc}} \qquad\qquad Z_B[S^2] = r^{c/3} e^{-K_c}$$

- K_{tc} : Kähler potential on Kähler moduli space of Calabi-Yau
- K_c : Kähler potential on complex structure moduli space of Calabi-Yau
- $Z_A[S^2]$ and $Z_B[S^2]$ exactly computable by supersymmetric localization

Benini, Cremonesi
Doroud, J.G.,Le Floch,Lee

Doroud, J.G.

$\Longrightarrow$

Computation of novel Gromov-Witten invariants

Deformed Partition Function

- Can generalize Pestun and compute $Z[S^4](\tau^i, \bar{\tau}^{\bar{i}}; \tau^A, \bar{\tau}^{\bar{A}})$ by localization

$$Z[S^4](\tau^i, \bar{\tau}^{\bar{i}}, \tau^A, \bar{\tau}^{\bar{A}}) = \int_{\mathfrak{t}} da \, \left| Z_\Omega(a, \tau^i, \tau^A) \right|^2$$

where

Nekrasov,...

Nakajima

$$Z_\Omega(a, \tau^i, \tau^A) = Z_{\Omega,\text{cl}}(a, \tau^i, \tau^A) \cdot Z_{\Omega,\text{loop}}(a) \cdot Z_{\Omega,\text{inst}}(a, \tau^i, \tau^A)$$

with

$$Z_{\Omega,\text{cl}}(a, \tau, \tau^A) = \exp\left[i\pi\tau \, \text{Tr} a^2 + i \sum_{A=3}^{N} \pi^{A/2} \tau^A \text{Tr} a^A \right]$$

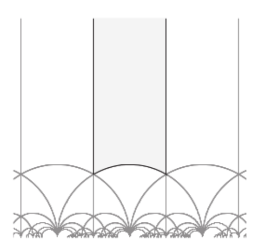
Examples

- Consider $\mathcal{N}=2$ SCFTs with one chiral ring generator, with $\Delta = 2$

$$\mathcal{O}_1 = \mathrm{Tr}\,\phi^2$$

ϕ: complex scalar in $\mathcal{N}=2$ vector multiplet

  - Admits one exactly marginal coupling: $\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$



1. $\mathcal{N}=4$ $SU(2)$ Yang-Mills ($\mathcal{N}=2$ vector & one adjoint hypermultiplet)
2. $\mathcal{N}=2$ $SU(2)$ SQCD ($\mathcal{N}=2$ vector & four fundamental hypermultiplets)

- Chiral ring OPE is

$$\mathcal{O}_n(x)\mathcal{O}_m(0) = \mathcal{O}_{n+m}(0)$$

- Goal is to compute the chiral ring data

$$G_{2n}(\tau,\bar{\tau}) = \left\langle \mathcal{O}_n(\infty)\overline{\mathcal{O}}_n(0)\right\rangle_{\mathbb{R}^4}$$

- By virtue of Ward identity

$$\langle \mathcal{O}_n(N)\overline{\mathcal{O}}_m(S)\rangle_{S^4} = \frac{1}{Z[S^4]}\partial_\tau^n \partial_{\bar{\tau}}^m Z[S^4]$$

- Gram-Schmidt $\Longrightarrow$ correlator written as ratio of determinants

$$G_{2n}(\tau,\bar{\tau}) = \frac{1}{Z[S^4]}\frac{\det_{(k,l)=0,\dots,n}\left(\partial_\tau^k \partial_{\bar{\tau}}^l Z[S^4]\right)}{\det_{(k,l)=0,\dots,n-1}\left(\partial_\tau^k \partial_{\bar{\tau}}^l Z[S^4]\right)}$$

$\Longrightarrow$ Connections with integrability

Toda Chain

- Chiral ring data acted on by integrable differential equations: Toda chain

$$\partial_\tau \partial_{\bar{\tau}} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}} \qquad n = 1, 2, \cdots$$

where

$$q_n = \ln \big(G_{2n} Z[S^4] \big)$$

- Chiral ring data governed by a set of coupled oscillators q_n
- Prescribed dependence of the left-most oscillator $q_0 = \ln \big(Z[S^4] \big)$
- In agreement with 4d $\mathcal{N} = 2$ tt^* equations in the holomorphic gauge

  Baggio, Niarchos, Papadodimas
- Chiral ring computable from $Z[S^4]$

Integrability

- The chiral ring of any 4d $\mathcal{N}=2$ SCFT is acted on by an integrable system

tt^* equations are integrable

- $\mathcal{N}=4$ super-Yang-Mills for any gauge group get decoupled Toda chains
- Not for $\mathcal{N}=2$ $SU(N)$ SQCD ($2N$ fundamental hypermultiplets)
- Formulae expressing the chiral ring data as ratios of determinants of derivatives of $Z_{\Omega,\mathrm{cl}}(a,\tau,\tau^A)$ generalizes to any $\mathcal{N}=2$ theory

$\Longrightarrow$

Computation of extremal correlators in $\mathcal{N}=2$ theories

1. $\mathcal{N}=4$ super-Yang-Mills with arbitrary G

- Chiral ring is tree-level exact

  Lee, Minwalla, Rangamani, Seiberg

  Baggio, de Boer, Papadodimas

- Zamolodchikov metric is the Poincaré metric in the upper half-plane

2. $\mathcal{N}=2$ $SU(2)$ SQCD

- Zamolodchikov metric around weak coupling has a perturbative expansion

$$G_2(\tau,\bar{\tau})_{\text{pert}} = \frac{6}{(\text{Im}\tau)^2} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(\text{Im}\tau)^4} + \frac{1575\zeta(5)}{4\pi^3}\frac{1}{(\text{Im}\tau)^5} + \dots$$

  reproduces and extends to all orders two-loop computations

  Baggio, Niarchos, Papadodimas

  with instanton corrections

$$G_2(\tau,\bar{\tau})_{\text{1-inst}} = \cos\theta\, e^{-\frac{8\pi^2}{g^2}} \left(\frac{6}{(\text{Im}\tau)^2} + \frac{3}{\pi}\frac{1}{(\text{Im}\tau)^3} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(\text{Im}\tau)^4} + \dots\right)$$

- In $\mathcal{N}=2$ $SU(N)$ SQCD (2N fundamental hypermultiplets)
  - Chiral ring has $N-1$ generators

$$\mathrm{Tr}\,\phi^2, \mathrm{Tr}\,\phi^3, \ldots, \mathrm{Tr}\,\phi^N$$

  - Chiral ring

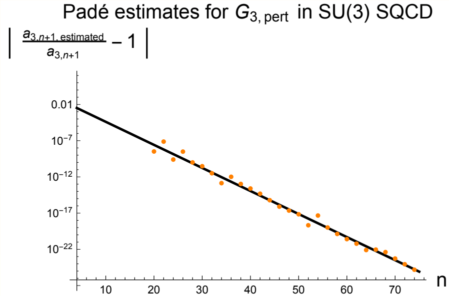
$$\mathcal{O}_{\vec{n}} = \prod_{k=2}^{N} \left(\mathrm{Tr}\,\phi^k\right)^{n_k}$$

  - Degeneracies in the spectrum: e.g in $SU(3)$ $\{\phi_2^3, \phi_3^2\}$
  - Using Gram-Schmidt can compute chiral ring to arbitrary order, e.g:

$$\left\langle \mathrm{Tr}\,\phi^3(\infty)\overline{\mathrm{Tr}\,\phi^3}(0)\right\rangle_{\mathbb{R}^4} = \left(\frac{g^2}{4\pi}\right)^3 \left(40 - \frac{135\,\zeta(3)}{2\pi^4}g^4 + \frac{6275\,\zeta(5)}{48\pi^6}g^6 + \ldots\right)$$

Resurgence

- Use our exact results to probe behaviours and conjectures in QFT, such as
  - Borel summability of the perturbative expansion
  - Convergence of the Pade approximation



Realizes a conjecture made about QCD perturbation theory

Karliner

Conclusions and Open Problems

- Problem of determining chiral ring data essentially solved
- Anomalies and operator mixing on S^4 key to the solution
- Geometrical properties of the conformal manifold $\mathcal{M}$
- Physical consistency conditions also have topological implications for $\mathcal{M}$

J.G, Komargodski, Hsin, Schwimmer, Seiberg, Theisen

$$[K] = 0$$

- Chiral rings of $\mathcal{N} = 2$ come equipped with a integrable system. Toda in simplest cases and of Hitchin-type in general
- Allow to test conjectures and ideas about large order behaviour in QFT
- Ideas can be extended to the chiral ring of 2d $\mathcal{N} = (2, 2)$ theories
- Connections with theory of Riemann surfaces "Quantum Teichmuller Theory" and AGT
- Connections with the conformal bootstrap program
- Open problems include: instantons in $SU(N)$, anomalies, large N, action of S-duality,. . .