Moduli spaces of curves with non-special divisors

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Outline

1. Moduli of curves with nonspecial divisors and the Krichever map.
2. The case $g = 1$.
3. The case $n = g$.
4. Relation to moduli of A_∞-structures.

Krichever map

Let C be a smooth projective curve, p a point, t a formal parameter at p. Then Laurent series expansion in t defines an embedding

$$H^0(C \setminus \{p\}, \mathcal{O}) \hookrightarrow \mathbb{C}((t)).$$

Let W be the image of this embedding. Then we have identifications

$$W \cap \mathbb{C}[[t]] = H^0(C, \mathcal{O}) = \mathbb{C}, \ \ \mathbb{C}((t))/(W + \mathbb{C}[[t]]) = H^1(C, \mathcal{O}).$$

Thus, we get a point of the Sato Grassmannian SG parametrizing subspaces of $\mathbb{C}((t))$ such that $W \cap \mathbb{C}[[t]] = \mathbb{C}$ and $\mathbb{C}((t))/(W + \mathbb{C}[[t]])$ is finite dimensional. Furthermore, one can recover the data (C, p, t) from W.

Remark. This construction is related to the action of the Virasoro algebra on some natural line bundles over the moduli spaces of curves, studied by Kontsevich, Beilinson-Schechtman, and Arbarello-De Concini-Kac-Procesi.

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Generalized Krichever map

Slight generalization: consider a curve C with n marked points $p_1, \ldots, p_n$ with formal parameters $t_1, \ldots, t_n$. Get an embedding

$$H^0(C \setminus \{p_1, \ldots, p_n\}, \mathcal{O}) \hookrightarrow \mathcal{H} := \bigoplus_{i=1}^{n} \mathbb{C}((t_i)).$$

This is still a point of the appropriate Sato Grassmannian $SG(\mathcal{H})$.

We would like to use this construction to obtain compactifications of the moduli space of curves $\mathcal{M}_{g,n}$. Note that the above construction works for a singular projective curve (reduced and connected), provided $\mathcal{O}(p_1 + \ldots + p_n)$ is ample, i.e., there is at least one marked point on each irreducible component of C.

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Moduli of curves with nonspecial divisors

Consider the moduli stack $\mathcal{U}^{ns}_{g,n}$ of $(C, p_1, \ldots, p_n)$, where C has arithmetic genus g, $p_1, \ldots, p_n$ are smooth and distinct, such that $\mathcal{O}(p_1 + \ldots + p_n)$ is ample and nonspecial, i.e., $H^1(C, \mathcal{O}(p_1 + \ldots + p_n)) = 0$. Consider enhanced spaces

$$\widetilde{\mathcal{U}}^{ns,(\infty)} \xrightarrow{\mathfrak{G}} \widetilde{\mathcal{U}}^{ns}_{g,n} \xrightarrow{(\mathbb{C}^*)^n} \mathcal{U}^{ns}_{g,n}$$

corresponding to choices of formal parameters or nonzero tangent vectors at each marked point. Here $\mathfrak{G}$ is the group of formal changes

$$t_i \mapsto t_i + c_{2,i}t_i^2 + c_{3,i}t_i^3 + \ldots$$

Note that we necessarily have $n \geq g$.
Example. For $g = 0$ the restriction on marked points is that $\mathcal{O}(p_1 + \ldots + p_n)$ is ample. The moduli stack $\mathcal{U}^{ns}_{0,n}$ is related to Boggi-Kontsevich compactification of $\mathcal{M}_{0,n}$

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There is a natural map from $\widetilde{\mathcal{U}}^{ns}_{g,n}$ to the Grassmannian $G(n-g,n)$ defined as follows. The short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(p_1 + \ldots + p_n) \to \bigoplus_{i=1}^{n} T_{p_i} C \to 0$$

gives rise to an exact sequence

$$\bigoplus_{i=1}^{n} T_{p_i} C \to H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C(p_1 + \ldots + p_n)) = 0,$$

The kernel of the first arrow is an $(n-g)$-dimensional subspace in $\mathbb{C}^n$.

For each subset of indices $S \subset [1,n]$ with $|S| = g$, the preimage of the corresponding standard cell in $G(n-g,n)$ is the open subset $\mathcal{U}(S) \subset \widetilde{\mathcal{U}}^{ns}_{g,n}$ consisting of $(C, p_1 + \ldots + p_n)$ such that $H^1(C, \mathcal{O}(\sum_{i \in S} p_i)) = 0$.

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Moduli of curves with nonspecial divisors

Theorem. The Krichever map defines a locally closed embedding $\widetilde{\mathcal{U}}^{ns,(\infty)} \hookrightarrow SG(\mathcal{H})$.
Its image is the closed subset of the locus $SG^{ns}(\mathcal{H})$ consisting of W such that $\mathcal{H} = W + \mathcal{H}_{\geq -1}$. The image consists of W such that $W \cdot W \subset W$.
The action of $\mathfrak{G}$ on $SG^{ns}(\mathcal{H})$ is free, and the quotient has an open covering by infinite-dimensional affine spaces.
The moduli space $\widetilde{\mathcal{U}}^{ns}_{g,n}$ is a scheme of finite type, affine over the Grassmannian $G(n-g,n)$.
There is a natural $(\mathbb{C}^*)^n$-action on $\widetilde{\mathcal{U}}^{ns}_{g,n}$ (rescaling the tangent vectors at the marked points), compatible with the standard $(\mathbb{C}^*)^n$-action on $G(n-g,n)$.
The invariant subscheme of the diagonal $\mathbb{C}^* \subset (\mathbb{C}^*)^n$ in $\widetilde{\mathcal{U}}^{ns}_{g,n}$ is a section of a map to $G(n-g,n)$. $\mathbb{C}^*$-action has positive weights.
Remark. Taking GIT quotients of $\widetilde{\mathcal{U}}^{ns}_{g,n}$ one gets birational projective models of $M_{g,n}$.

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Moduli of curves with nonspecial divisors, $g = n = 1$

Example. $g = n = 1$. The algebra $\mathcal{O}(C \setminus p)$ is generated by x and y such that

$$x = \frac{1}{t^2} + \dots, \quad y = \frac{1}{t^3} + \dots$$

Using the ambiguity $x \mapsto x + a$, $y \mapsto y + bx + c$, we can choose x and y uniquely so that

$$y^2 = x^3 + px + q.$$

Thus, $\widetilde{\mathcal{U}}_{1,1}^{ns} = \mathbb{A}^2$.
$\mathbb{C}^*$ acts with the weights $(2, 3)$.
The unique $\mathbb{C}^*$-invariant point, $p = q = 0$, corresponds to the cuspidal cubic.

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Example. $g = 0$, $n \geq 3$. The algebra $\mathcal{O}(C \setminus \{p_1, \ldots, p_n\})$ is generated by $x_1, \ldots, x_n$, where $x_i \in H^0(C, \mathcal{O}(p_i))$, and $x_i = \frac{1}{t_i} + \ldots$. The defining relations are

$$x_i x_j = \alpha_{ij} x_j + \alpha_{ji} x_i + c_{ij}, \quad \text{for } i \neq j, \text{ with}$$

$$(\star)\ c_{ij} = \alpha_{ik}\alpha_{jk} - \alpha_{ij}\alpha_{jk} - \alpha_{ji}\alpha_{ik}.$$

Normalization: $\alpha_{i,i+1} = 0$. Then x_i are unique.

The relation $(\star)$ gives the defining equations of $\widetilde{\mathcal{U}}^{ns}_{0,n}$, same as the miniversal deformation of the coordinate cross in $\mathbb{C}^n$.

Remark. The only singularities that a reduced curve of arithmetic genus 0 can have are rational m-fold points.

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$(\mathbb{C}^*)^n$-action: $(\lambda^{-1})^*\alpha_{ij} = \lambda_i\alpha_{ij}$.
For each character $\chi(\lambda) = \lambda_1^{a_1} \dots \lambda_n^{a_n}$ of $(\mathbb{C}^*)^n$ can consider the GIT-quotient $\widetilde{\mathcal{U}}_{0,n}^{ns} /\!\!/_{\chi} (\mathbb{C}^*)^n$.
If $a_1 > 0, \dots, a_n > 0$ then stable (=semistable) points correspond to $(C, p_1, \dots, p_n)$ such that each irreducible component has ≥ 3 special points. This is exactly Boggi-Kontsevich moduli space.

We get a realization of this space by explicit equations in $(\mathbb{P}^{n-3})^n$. Namely, in equation $(\star)$ we should view α_{ij} as homogeneous coordinates on the ith copy of $\mathbb{P}^{n-3}$.

Remark. For $n > 5$ it is not known whether the scheme $\widetilde{\mathcal{U}}_{0,n}^{ns}$ (or its GIT-quotient) is normal or even reduced.

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Case $g = 1$: fundamental decomposition

Proposition. A curve $(C, p_1, \ldots, p_n)$ is in $\widetilde{\mathcal{U}}_{1,n}^{ns}$ if and only if it has a fundamental decomposition

$$C = E \cup R_1 \cup \ldots \cup R_r,$$

where R_i are connected tails of arithmetic genus 0 attached to E transversally at distinct points, and E is of one of the following types:

- smooth elliptic curve;
- cycle of projective lines (standard m-gon);
- elliptic m-fold curve.

Furthermore, there should be at least one marked point on every irreducible component.

Elliptic m-fold curves.

- $m = 1$: cuspidal cubic;
- $m = 2$: union of two projective lines glued in a tacnode;
- $m \geq 3$: union of m generic lines through a point in $\mathbb{P}^{m-1}$.

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Case $g = 1$: GIT stability conditions

Fix a rational character $\chi = (a_1, \ldots, a_n)$ of $(\mathbb{C}^*)^n$. Let $(C, p_1, \ldots, p_n)$ be in our moduli space, with fundamental decomposition $C = E \cup R_1 \cup \ldots \cup R_r$. For a marked point p_i lying on an irreducible component $C' \simeq \mathbb{P}^1$, we denote by $N(p_i)$ the number of special points on C'. Define $I_0 \subset I \subset [1, n]$ and $J \subset [1, n]$ by

- $J = \{j \mid p_j \notin E, N(p_j) \geq 3\}$;
- $I = \{i \mid p_i \in E\}$;
- $I_0 = \emptyset$ if E is at most nodal; otherwise, $I_0 = \{i \in I \mid N(p_i) \leq 2\}$.

Theorem. $(C, p_1, \ldots, p_n)$ is $\pi^*\mathcal{O}(1) \otimes \chi$-semistable if and only if

- $a_i \geq 0$ for all i; $a_i = 0$ for $i \notin I \cup J$;
- $\sum_{i \in I_0} a_i \leq 1$;
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All these GIT quotients are projective.

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Example 1. All $a_i > 1$. Then stability (=semistability) means that there are ≥ 3 special points on the normalization of every rational component of C.
Equivalently, there exists a birational map $f : \widetilde{C} \to C$, where $(\widetilde{C}, \widetilde{p}_1, \ldots, \widetilde{p}_n)$ is Deligne-Mumford stable of genus 1, f contracts the unmarked components in $\widetilde{C}$.
Example 2. $a_i = a$, $a \in (\frac{1}{n}, \frac{1}{n-2})$. Then stability means that $C = E$ (equivalently, C is Gorenstein with trivial ω_C, equivalently, $H^1(C, \mathcal{O}(p_i)) = 0$ for each i), and $(C, p_1, \ldots, p_n)$ has no infinitesimal symmetries.

The moduli spaces in both Examples were first constructed by David Smyth. The moduli space in Example 2 was studied in [P-Lekili]. We showed that it is a normal Gorenstein projective scheme, given by explicit equations.

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Case $g = 1$: Strongly non-special curves

Consider $(C, p_1, \ldots, p_n)$ with $H^1(C, \mathcal{O}(p_i)) = 0$ for each i, as in Example 2. Fix a nonzero global section $\omega \in H^0(C, \omega_C)$. Assume $n \geq 3$. For $i \neq j$, there is $h_{ij} \in H^0(C, \mathcal{O}(p_i + p_j))$, unique up to adding a constant, such that $\operatorname{Res}_{p_i} h_{ij}\omega) = 1$, $\operatorname{Res}_{p_j} h_{ij}\omega) = -1$.

Normalize h_{12} and h_{13} by $h_{12}(p_3) = 0$, $h_{13}(p_2) = 0$. Then the algebra $\mathcal{O}(C \setminus \{p_1, \ldots, p_n\})$ is generated by $x_2 = h_{12}, \ldots, x_n = h_{1n}$, with defining relations

$$x_i x_j = x_2 x_3 + c_{ij} x_j + c_{ji} x_i + d_{ij},$$

$$x_2 x_3^2 = x_2^2 x_3 + a x_2 x_3 + b x_2 + c x_3 + d,$$

where $c_{ij} = h_{1i}(x_j)$.

The (normalized) coefficients become (weighted) projective coordinates on the moduli space. For $n \geq 5$ all coordinates are expressed in terms of c_{ij} and a (which have weight 1). One can write explicitly defining equations between them.

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Case $n = g$

Set $D = p_1 + \ldots + p_g$. Then for each i there exist $x_i \in H^0(C, \mathcal{O}(D + p_i))$, $y_i \in H^0(C, \mathcal{O}(D + 2p_i))$ such that

$$x_i = \frac{1}{t_i^2} + \ldots, \quad y_i = \frac{1}{t_i^3} + \ldots$$

at p_i. The algebra $\mathcal{O}(C \setminus D)$ is generated by $(x_1, \ldots, x_g, y_1, \ldots, y_g)$.

Using the ambiguity $x_i \mapsto x + a_i$, $y_i \mapsto y_i + b_i x_i + c_i$, we can choose x_i and y_i uniquely so that

$$y_i^2 - x_i^3 \in H^0(C, \mathcal{O}(3D)), \quad x_i(y_i^2 - x_i^3) \in H^0(C, \mathcal{O}(4D)).$$

Note that $(x_i^n, y_i x_i^n)$ is a basis of $\mathcal{O}(C \setminus D)$.

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Case $n = g \geq 2$: equations of the universal curve

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\begin{aligned}
x_i x_j &= \alpha_{ji} y_i + \alpha_{ij} y_j + \gamma_{ji} x_i + \gamma_{ij} x_j + \textstyle\sum_{k \neq i,j} c_{ij}^k x_k + a_{ij}, \\
x_i y_j &= \alpha_{ij} x_j^2 + \beta_{ji} y_i + \gamma_{ij} y_j + r_{ji} x_i + \delta_{ij} x_j + \textstyle\sum_{k \neq i,j} e_{ij}^k x_k + b_{ij}, \\
y_i y_j &= \beta_{ji} x_i^2 + \beta_{ij} x_j^2 + \varepsilon_{ji} y_i + \varepsilon_{ij} y_j + \psi_{ji} x_i + \psi_{ij} x_j + \textstyle\sum_{k \neq i,j} l_{ij}^k x_k + u_{ij}, \\
y_i^2 &= x_i^3 + p_i x_i + \textstyle\sum_{j \neq i} g_i^j y_j + \textstyle\sum_{j \neq i} k_i^j x_j + q_i,
\end{aligned}
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where i and j are distinct.

Using Gröbner basis technique find:

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\begin{aligned}
c_{ij}^k &= \alpha_{ik}\alpha_{jk}, \\
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Case $n = g \geq 2$: GIT quotients

GIT-stability condition depends on a character $\chi(\lambda) = (a_1, \ldots, a_n)$ of $(\mathbb{C}^*)^n$.
All GIT-quotients are projective, empty unless $(a_1, \ldots, a_n)$ belongs to the cone generated by $(2e_i - e_j)$.
Wall structure in $\mathbb{R}^n$: the codim-1 walls are cones spanned by subsets of $(2e_i - e_j, 3e_i - e_j, e_i)$.

Main chamber $\mathbf{C}_0$: $a_1 > 0, \ldots, a_n > 0$.
For $\chi \in \mathbf{C}_0$ every $(C, p_1, \ldots, p_g)$ with smooth C is χ-stable.

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Moduli of A_∞-structures

Recall that an A_∞-algebra is a graded vector space A with operations $m_n : A^{\otimes n} \to A$ of degree $2 - n$, for $n \geq 1$, satisfying A_∞-identities $\sum_{i+j=n}[m_i, m_j] = 0$ (where $[\cdot, \cdot]$ is the Gerstenhaber bracket).

For a given finite-dimensional associative algebra A can consider all A_∞-structures $(m_\bullet)$ on A with $m_1 = 0$ and m_2 the given product on A. These are parametrized by an infinite-dimensional affine scheme $\mathcal{A}_\infty(A)$. There is a natural action of an infinite-dimensional unipotent group $\mathfrak{G}$ of gauge equivalences on $\mathcal{A}_\infty(A)$. We consider the moduli space $\mathcal{M}_\infty(A) = \mathcal{A}_\infty(A)/\mathfrak{G}$.

Theorem. Assume $HH^1(A)_j = 0$ for $j < 0$. Then the action of $\mathfrak{G}$ on $\mathcal{A}_\infty(A)$ admits a section, so that $\mathcal{M}_\infty(A)$ is an affine scheme. If in addition $HH^2(A)_{<0}$ is finite-dimensional then $\mathcal{M}_\infty(A)$ is of finite type.

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Moduli of curves and moduli of A_∞-structures

Given a curve $(C, p_1, \ldots, p_g)$ such that $H^1(\mathcal{O}(p_1 + \ldots + p_g)) = 0$ consider the algebra $E = \mathrm{Ext}^*(G, G)$, where

$$G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \ldots \oplus \mathcal{O}_{p_g}.$$

Generators: $A_i \in \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_{p_i})$, $B_i \in \mathrm{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C)$. Note that the classes $B_1A_1, \ldots, B_gA_g$ form a basis of $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = H^1(C, \mathcal{O})$.

A choice of nonzero tangent vectors at $p_1, \ldots, p_n$ gives an isomorphism of E with the fixed algebra E_g. By homological perturbation, for each curve $(C, p_1, \ldots, p_n)$ there is a canonical gauge equivalence class of A_∞-structures on E_g.

Theorem. This defines an isomorphism $\widetilde{\mathcal{U}}^{ns}_{g,g} \simeq \mathcal{M}_\infty(E_g)$.

Remark. There is a similar interpretation of $\widetilde{\mathcal{U}}^{ns}_{g,n}$ for $n > g$ as moduli of A_∞-structures. In this case m_2 is also allowed to vary.

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References

arXiv:1312.4636 (case $n = g$)

[P-Lekili] arXiv:1408.0611 (case $g = 1$, strongly non-special curves)

arXiv:1511.03797 (general case)

arXiv:1603.01238 (case $g = 1$, GIT stabilities)