From the Hitchin component to opers

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based on joint work with

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Outline

1. Introduction to Higgs bundles and connections
2. Higgs bundles and quantum curves
3. Quantization of Airy function
4. The methamorphosis of quantum curves into opers
5. General theory of Hitchin systems for the Lie group $G = SL_r(\mathbb{C})$
6. Opers

Let C be a smooth projective curve, K_C canonical bundle, E and V be holomorphic rank r vector bundle on C.

Definition

- A holomorphic Higgs bundle is a pair (E, ϕ) where $\phi : E \to E \otimes K_C$ is a $\mathcal{O}_\mathcal{C}$-module homomorphism ie $\phi(sf) = f\phi(s), \forall f \in \mathcal{O}_\mathcal{C}, s \in E$.
- A connection is a pair (V, ∇), $\nabla : V \to V \otimes K_C$ is a $\mathbb{C}$-homomorphism st $\nabla(fs) = df \cdot s + f \cdot \nabla(s), \forall f \in \mathcal{O}_\mathcal{C}, s \in V$.

Example (rank two Higgs filed examples)

$(K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}, \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix})$ and $(\mathcal{O}_C \oplus \mathcal{O}_C, \begin{pmatrix} 0 & P(x)dx \\ P(x)dx & 0 \end{pmatrix})$ for $q = P(x)^2(dx)^2 \in H^0(C, K_C^2)$.

- Locally on C, $\nabla|_U = d + A|_U$ where $A : V \to V \otimes K_C$.
- If $\{f_{\alpha\beta}\}$, $\{g_{\alpha\beta}\}$ are transition functions for E and V respectively then $\phi_\alpha = f_{\alpha\beta}\phi_\beta f_{\alpha\beta}^{-1}$ while $A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - g_{\alpha\beta}^{-1}dg_{\alpha\beta}$.

Let C be a smooth projective curve of arbitrary genus, K_C canonical bundle. Let E a holomorphic rank 2 vector bundle on C.
For $\phi : E \to E \otimes K_C(*)$, the pair (E, ϕ) Higgs bundle of rank 2.

- In [DM '13] ϕ is holomorphic, Hitchin constructed $\Sigma \hookrightarrow T^*C$
- In [DM '14] ϕ is meromorphic

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\ i\ } & Bl\,\overline{T^*C} \\ \big\downarrow & & \big\downarrow \, blow\!-\!up \\ \Sigma & \xrightarrow[\ i\ ]{} & \overline{T^*C} \end{array}$$

Theorem (D-Mulase '13, '14)

For rank 2 Higgs bundle (E, ϕ) and $x \in C$. Locally, we construct a 2nd order differential operator $P(x, \hbar d/dx)$ whose semi-classical limit recovers Σ, s.t. $P(x, \hbar d/dx)\psi(x, \hbar) = 0$.

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  - Gromov-Witten theory $\xrightarrow{T.R.}$ Hitchin Theory

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Airy curve $\equiv$ local spectral curve of a Higgs bundle

- The curve $C = \mathbb{P}^1$, the vector bundle $E = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$
- The meromorphic Higgs field $\phi : E \to E \otimes K_C(*)$ is locally

$$\phi = \begin{pmatrix} 0 & x(dx)^2 \\ 1 & 0 \end{pmatrix}$$

Σ, the spectral curve of ϕ inside $\overline{T^*\mathbb{P}^1} = \mathbb{F}_2$ is locally at the $(0,0)$ chart

$$det(\phi - (ydx)I_2) = (y^2 - x)(dx)^2 = 0$$

with a quintic cusp at (∞, ∞) chart: $u^2 = w^5$

- The spectral curve $\Sigma \to C = \mathbb{P}^1$ double cover.
- Take a resolution of singularity of curve Σ by blowing up $\mathbb{F}_2$. The proper transform $\widetilde{\Sigma}$ of Σ becomes a rational curve.

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Geometry of the Airy function

The Airy differential equation = the simplest quantum curve

- $\left(\hbar^2 \frac{d^2}{dx^2} - x\right) A_i(x,\hbar) = 0.$

$$A_i(x,\hbar) := \frac{1}{2\sqrt{\pi}} \exp\left(\sum_{2g-2+n\geq -1}^{\infty} \frac{1}{n!}\hbar^{2g-2+n} F_{g,n}(x)\right)$$

- $F_{g,n}(x)$ satisfy recursion, which can be solved

$$F_{g,n}(x) = \frac{(-1)^n}{2^{2g-2+n}} x^{-\frac{6g-6+3n}{2}} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i - 1)!!$$

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathbb{L}_1)^{d_1} \cdots c_1(\mathbb{L}_n)^{d_n}$$

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Projective coordinate system

Let C be a Riemann surface of genus at least two. Its universal covering is the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} | Imz > 0\}$$

The global coordinate on $\mathbb{H}$ induces by the quotient map $\mathbb{H} \to C$ a particular coordinate system on the Riemann surface C.

Definition (Gunning 1967)

A projective coordinate system on C is a coordinate system on which transition function is given by *Möbius* transformation.

$$C = \bigcup_{\alpha} U_\alpha, \quad z_\alpha \in U_\alpha, \quad z_\alpha = \frac{a_{\alpha\beta} z_\beta + b_{\alpha\beta}}{c_{\alpha\beta} z_\beta + d_{\alpha\beta}}, \quad \begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} \in SL(2, \mathbb{R}).$$

Note $dz_\alpha = \frac{dz_\beta}{(c_{\alpha\beta} z_\beta + d_{\alpha\beta})^2}$ so $K_C^{\frac{1}{2}}$ is given by $\zeta_{\alpha\beta} = \pm(c_{\alpha\beta} z_\beta + d_{\alpha\beta})$.

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Importance of Gunning's definition:

On a projective coordinate system of Σ, the *DM-quantum curve* is globally defined!

Definition (intuitive)

An oper, ∇^{oper}, is a globally defined differential operator.

The quantum curve $P(x, \hbar d/dx)|_{\hbar=1}$ is an oper. Interpret globally

$$P(x, d/dx)\psi = 0 \text{ as } \nabla^{oper}\begin{bmatrix}\psi \\ \psi'\end{bmatrix} = 0$$

Example: The quantum curve of the Higgs field $E = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$, $\phi = \begin{pmatrix} 0 & x(dx)^2 \\ 1 & 0 \end{pmatrix}$ is $\left(\hbar^2 \frac{d^2}{dx^2} - x\right) A_i(x, \hbar) = 0$. It corresponds to a family of opers $\nabla^{\hbar} = d + \frac{1}{\hbar}\begin{pmatrix} 0 & x(dx) \\ dx & 0 \end{pmatrix}$. What is the corresponding vector bundle for this family of connections

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Interpret $\hbar \in \mathbb{C} = Ext^1(K_C^{-\frac{1}{2}}, K_C^{\frac{1}{2}})$.

Theorem (Gunning)

For any $\hbar \in \mathbb{C}$, $\exists!$ rank 2 vector bundle, $V_\hbar$, such that

$$0 \to K_C^{\frac{1}{2}} \to V_\hbar \to K_C^{-\frac{1}{2}} \to 0$$

- proof: $V_\hbar$ is given by transition fn $\{f^\hbar_{\alpha\beta}\}$, $f^\hbar_{\alpha\beta} := \begin{pmatrix} \zeta_{\alpha\beta} & \hbar \cdot \frac{d\zeta_{\alpha\beta}}{dz_\beta} \\ 0 & \zeta^{-1}_{\alpha\beta} \end{pmatrix}$
- $V_0 \cong K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$
- For $\hbar \neq 0$ the *vector bundles* $V_\hbar$ and *their complex structure* are isomorphic. Denote $V := V_\hbar|_{\hbar=1}$.
- Gunning: $H^0(C, K_C^2) \cong \{$moduli space of pairs $(V, \nabla^{oper})\}$
  $0 \leftrightarrow \nabla_{unif}$ = origin
- $(K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}, \begin{bmatrix} 0 & P(x)(dx)^2 \\ 1 & 0 \end{bmatrix}) \overset{DM}{\to} (V_\hbar, d + \frac{1}{\hbar}\begin{bmatrix} 0 & P(x)dx \\ dx & 0 \end{bmatrix})$

DM-quantum curve $\hbar\nabla^\hbar$ is an $\hbar-$deformation family

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DM-quantum curve $\hbar\nabla^\hbar$ is an $\hbar -$ deformation family

- E, V be a holomorphic vector bundles of rank r of degree 0
- $\phi : E \to E \otimes K_C$ a traceless holomorphic Higgs field.
- $\nabla : V \to V \otimes K_C$ an irreducible holomorphic connection.

$\mathcal{M}_{\text{Dol}} :=$ {moduli space of rank r stable Higgs bundles (E, ϕ)}

$\downarrow$ [Hitchin-Simpson]

$\mathcal{M}_{\text{deR}} :=$ {moduli space of rank r irreducible connections (V, ∇)}

$\downarrow$ [Riemann-Hilbert]

$$\mathcal{M}_B := Hom(\pi_1(C), G) /\!\!/ G, G = SL_r(\mathbb{C})$$

Let η tautological 1-form on T^*C. The Spectral curve of ϕ is

$$det(\phi - \eta \cdot I_r) = 0 \subset T^*C$$

- $\mathcal{M}_{\text{Dol}}$ is a fibration of abelian varieties, by the Hitchin map

$$\begin{array}{ccc} \mathcal{M}_{\text{Dol}} & \ni & (E, \Phi) \\ \downarrow H & & \\ B := \oplus_{i=2}^{r} H^0(C, K_C^i) & \ni & ((-1)^i tr(\wedge^i \phi)) \end{array}$$

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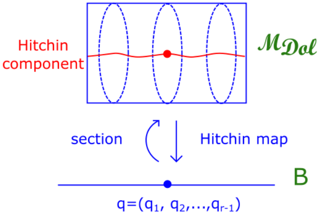
Hitchin component in rank two

- Fix a spin structure for C, a line bundle $\mathcal{L}$ for which $\mathcal{L}^{\otimes 2} \cong K_C$.
- Let $\zeta_{\alpha\beta}$ be the transition functions for the line bundle $\mathcal{L}$.

Hitchin map is just $(E, \phi) \xrightarrow{H} det(\phi)$. Let $q \in B = H^0(C, K_C^2)$.

The Hitchin section is $s(q) = (E_0 := K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}, \phi(q) := \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix})$



Hitchin component (principal $sl_2(\mathbb{C})$)

Let $q := (q_1, \ldots, q_{r-1}) \in B = \oplus_{i=1}^{r-1} H^0(C, K_C^{i+1})$. Denote $p_i := i(r-i)$.

- $X_+ := \begin{bmatrix} 0 & \sqrt{p_1} & 0 & \ldots & 0 \\ 0 & 0 & \sqrt{p_2} & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \sqrt{p_{n-1}} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, X_- := X_+^t,$
- $H := \begin{bmatrix} r-1 & 0 & \ldots & 0 & 0 \\ 0 & r-3 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & -(r-3) & 0 \\ 0 & 0 & \ldots & 0 & -(r-1) \end{bmatrix}.$
- Let $E_0 := K_C^{\frac{r-1}{2}} \oplus K_C^{\frac{r-1}{2}-1} \oplus \ldots \oplus K_C^{-\frac{r-1}{2}}$ with transition functions $\{\zeta_{\alpha\beta}^H = exp(H \cdot log \zeta_{\alpha\beta})\}$.

Hitchin section

For any $q \in B$, the Higgs pair $(E_0, \phi(q) := X_- + \sum_{i=1}^{r-1} q_i X_+^i) \in \mathcal{M}_{Dol}$

A stable holomorphic Higgs bundle (E, ϕ) corresponds to (D, ϕ, h)

- h is a *hermitian metric*, D an *h-unitary connection*

$$D = D^{(1,0)} + D^{(0,1)}$$

- F_D is the curvature, $2F_D = [D, D]$
- $\phi^{\dagger_h}$ is the adjoint of ϕ with respect to h

satisfying Hitchin's equations, a nonlinear system of PDE

$$F_D + [\phi, \phi^{\dagger_h}] = 0 \quad (1)$$

$$\bar{\partial}_D \phi = 0. \quad (2)$$

This is equivalent to the flatness of a family of connections, $\zeta \in \mathbb{C}^*$

$$D(\zeta) := \frac{1}{\zeta}\phi + D + \zeta\phi^{\dagger_h}$$

$$(E, \phi) \xrightarrow{NAH} (V, \nabla := D(1)^{(1,0)}), V = (E, D(1)^{(0,1)})$$

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Gaiotto's conjecture

Take $(E_0, \phi(q))$ on the Hitchin section and twist by $R \in \mathbb{R}_+$,
$(E_0, R\phi(q)) \in \mathcal{M}_{Dol}$

$$F_D + R^2[\phi, \phi^{\dagger_h}] = 0, \tag{3}$$
$$\bar{\partial}_D \phi = 0. \tag{4}$$

where $R \in \mathbb{R}$. The solution to Hitchin's equations corresponds to family of flat connections on a topologically trivial bundle, $\zeta \in \mathbb{C}^*$ and $R > 0$.

$$D(\zeta, R) := \zeta^{-1} R\phi + D + \zeta R\phi^{\dagger}.$$

D. Gaiotto predicted in 2014

For a Higgs bundle on a Hitchin section $\lim_{R,\zeta\to 0, \frac{\zeta}{R}=\hbar} D(\zeta, R)$ exists and is an oper.

We recall: in rank 2 oper is (V, ∇) with $0 \to K_C^{\frac{1}{2}} \to V \to K_C^{-\frac{1}{2}} \to 0$.

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$$\bar{\partial}_D \phi = 0. \quad (4)$$

where $R \in \mathbb{R}$. The solution to Hitchin's equations corresponds to family of flat connections on a topologically trivial bundle, $\zeta \in \mathbb{C}^*$ and $R > 0$.

$$D(\zeta, R) := \zeta^{-1} R\phi + D + \zeta R\phi^{\dagger}.$$

D. Gaiotto predicted in 2014

For a Higgs bundle on a Hitchin section $\lim_{R,\zeta\to 0, \frac{\zeta}{R}=\hbar} D(\zeta, R)$ exists and is an oper.

We recall: in rank 2 oper is (V, ∇) with $0 \to K_C^{\frac{1}{2}} \to V \to K_C^{-\frac{1}{2}} \to 0$.

Definition (Beilinson-Drinfeld, 1993)

Let V be a holomorphic vector bundle of rank r. An $SL_r(\mathbb{C})$-oper is a pair (V, ∇) with a filtration st

1. $0 = F_r \hookrightarrow F_{r-1} \hookrightarrow \ldots \hookrightarrow F_0 = V$
2. $\nabla|_{F_{i+1}} : F_{i+1} \to F_i \otimes K_C$ Griffiths transversality.
3. $F_{i+1}/F_{i+2} \cong F_i/F_{i+1} \otimes K_C$ an $\mathcal{O}_C$ linear isomorphism.

The following hold:

- There is a unique filtration and vector bundle $V_\hbar$ with $F_{r-1} = K_C^{\frac{r-1}{2}}$.
- $V_\hbar$ is given by $\{f^\hbar_{\alpha\beta}\}$ where $f^\hbar_{\alpha\beta} = exp(H \cdot log\zeta_{\alpha\beta})exp(\hbar\frac{dlog\zeta_{\alpha\beta}}{dz_\beta} \cdot X_+)$
- $V_0 = K_C^{\frac{r-1}{2}} \oplus \ldots \oplus K_C^{-\frac{r-1}{2}}$, since $f^{\hbar=0}_{\alpha\beta} = exp(H \cdot log\zeta_{\alpha\beta}) = \zeta^H_{\alpha\beta}$.
- For $\hbar \neq 0$ the vector bundles $V_\hbar$ and their complex structures are isomorphic. Let $V_1 := V_\hbar|_{\hbar=1}$.
- Let $\nabla^\hbar_\alpha|_{U_\alpha} := d + \frac{1}{\hbar}\phi(q)|_{U_\alpha}$. Then $(V_\hbar, \nabla^\hbar)$ is a family of opers.

Gaiotto's conjecture holds

Theorem [D, Fredrickson, Kydonakis, Mazzeo, Mulase, Neitzke]

For an arbitrary simple and simply connected Lie group G
$\lim_{R,\zeta\to 0,\frac{\zeta}{R}=\hbar} D(\zeta, R)$ exists and is the oper $(V_\hbar, \nabla^\hbar)$.

for rank 2

The limit oper is $d + \frac{1}{\hbar}\phi(q)$ i.e. DM quantum curve of Topological Recursion.

- [Hitchin, Simpson] Nonabelian-Hodge correspondence

$$\mathcal{M}_{\text{Dol}} \xrightarrow{\text{diffeomorphism}} \mathcal{M}_{\text{deR}}$$

- [Gaiotto's conjecture] holomorphic corresp. between Lagrangians
  Hitchin Component $\xrightarrow{\text{holomorphic}}$ moduli space of opers

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$$\mathcal{M}_{\text{Dol}} \overset{\textit{diffeomorphism}}{\longrightarrow} \mathcal{M}_{\text{deR}}$$

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sketch of the proof in rank two

- The hermitian metric on the vector bundle of the Hitchin component $V_0 = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$ is given by $h = \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}$
- the Chern connection $D = d - \partial log\lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- If $\phi = \begin{bmatrix} 0 & P \\ 1 & 0 \end{bmatrix} dz$ then $\phi^{\dagger_h} = \overline{h \cdot \phi^t \cdot h^{-1}} = \begin{bmatrix} 0 & \lambda^2 \\ \lambda^{-2}\overline{P} & 0 \end{bmatrix} d\bar{z}$
- The flatness condition of $D(\zeta, R)$ gives

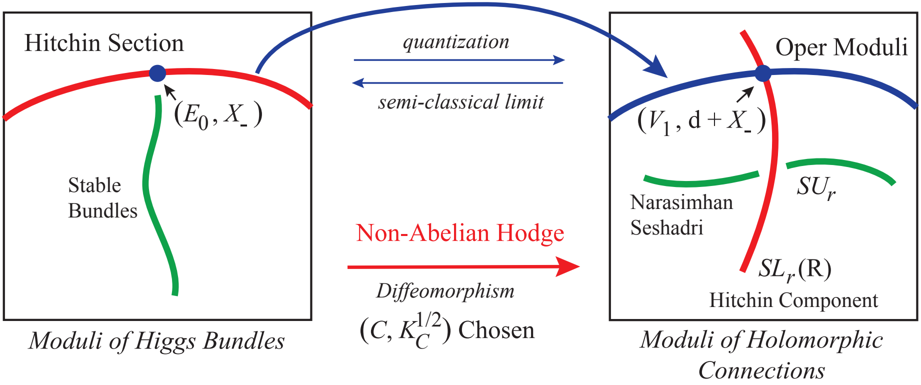
$$\partial\bar{\partial} log\lambda - R^2(\lambda^{-2}P\overline{P} - \lambda^2) = 0$$

- For $P = 0$, ie $\phi = X_-$, solving the harmonicity equation for λ one obtains on $\mathbb{H}$

$$\lambda_0 = \frac{1}{R} \cdot \frac{i}{z - \overline{z}}$$

- In general $\lambda = \lambda_0 \cdot e^{f(R)}$ analysis proves f is real analytic with $f(R) = R^4 + HOT$.

Gaiotto Correspondence = Canonical Biholomorphic Map



Thank you for your attention!

Geometry and Physics

- In Physics: Higgs fields make massless particles massive.

Figure: Peter Higgs, University of Edinburgh and Nigel Hitchin, University of Oxford

- In Mathematics: Higgs fields promote massive interactions between different fields and collaborations.