

# Higgs bundles, branes and applications

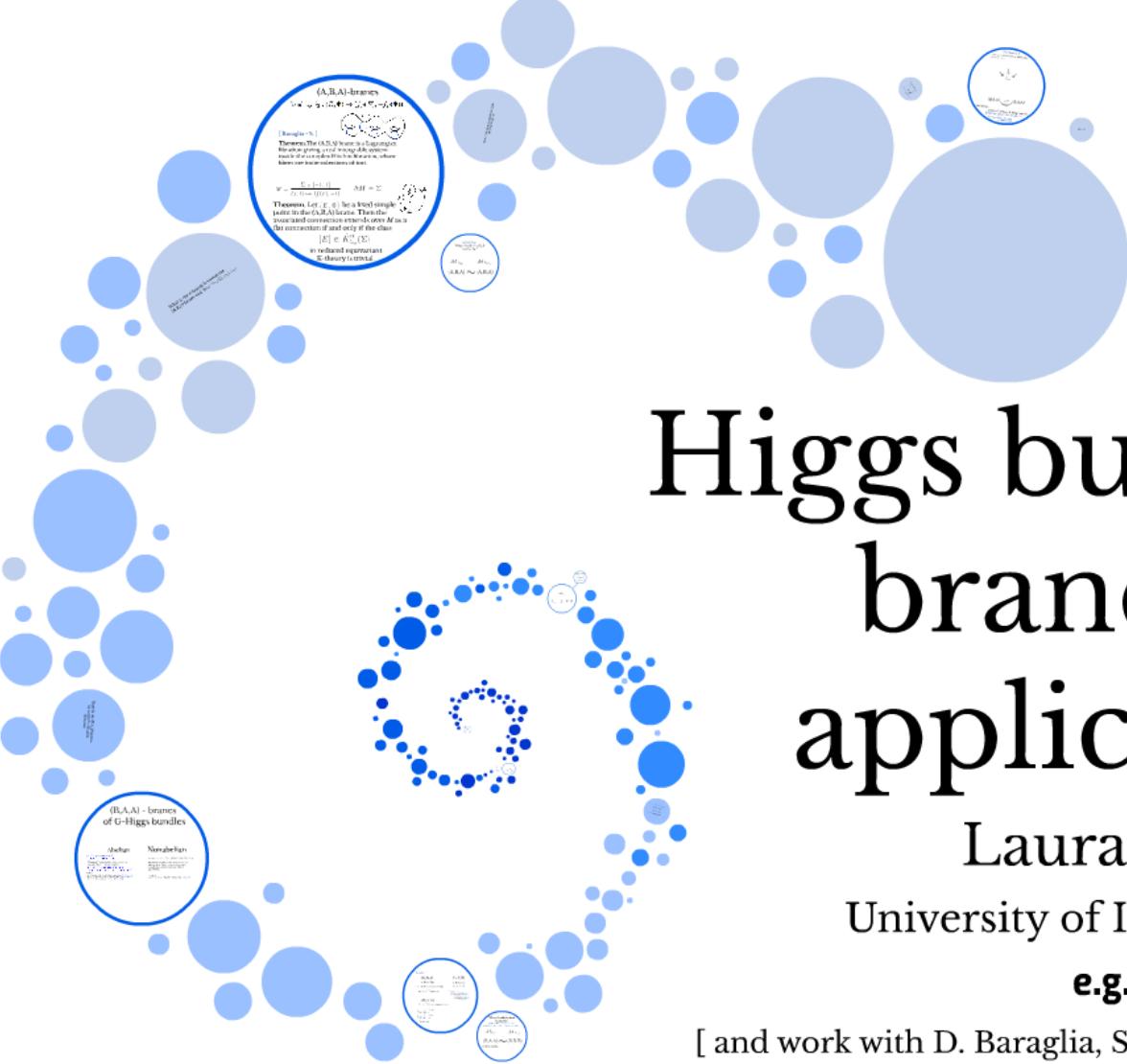
Laura Schaposnik

University of Illinois at Chicago

e.g. arXiv:1603.06691

[ and work with D. Baraglia, S. Bradlow, N. Hitchin ]

String-Math 2016, Collège de France, Paris



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# [Hitchin '87 - THE SELF-DUALITY EQUATIONS ON A RIEMANN SURFACE]

$$\left. \begin{aligned} d''_A \Phi &= 0, \\ F(A) + [\Phi, \Phi^*] &= 0. \end{aligned} \right\}$$

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EQUATIONS ON  
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$$\sum_{\text{Compact Riemann surface, } g \geq 2} K = T^* \Sigma$$

A Higgs bundle is a pair  $(E, \Phi)$

$E$  holomorphic vector bundle

  $\Phi : E \rightarrow E \otimes K$  holomorphic map

Higgs field

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Higgs field

$$G_{\mathbb{C}} \subset \mathrm{GL}(n, \mathbb{C})$$

A  $G_{\mathbb{C}}$ -Higgs bundle is  $(E, \Phi) + \text{cond.}$

E.g. For  $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$  take

$$\Lambda^n E \cong \mathcal{O} \text{ and } \mathrm{Tr}(\Phi) = 0$$

 $\mathcal{M}_{G_{\mathbb{C}}}$ 

Moduli space of  $G_{\mathbb{C}}$ -Higgs bundles

?

$$\mathcal{M}_{G_{\mathbb{C}}}$$

Moduli space of  $G_{\mathbb{C}}$ -Higgs bundles



$$\text{Hom}^{red}(\pi_1(\Sigma), G_{\mathbb{C}})/G_{\mathbb{C}},$$

non-abelian Hodge  
correspondence

[ Corlette, Donaldson, Hitchin, Simpson,  
Uhlenbeck, Yau ]

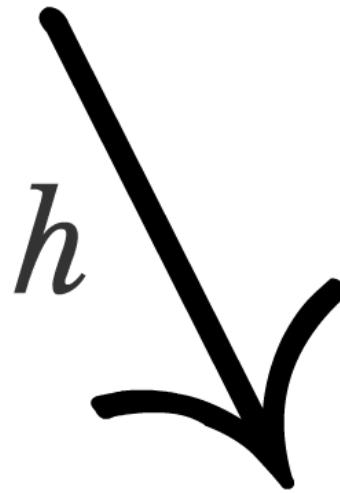
# The Hitchin fibration (abelianization)

Ab. variety

$\cap$   
 $\text{Jac}(S)$



$\mathcal{M}_{G_{\mathbb{C}}}$

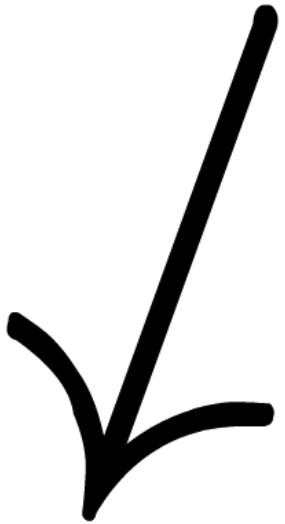


$$S = \{\det(\Phi - \lambda \text{Id}) = 0\} \subset \text{Tot} K$$

$\mathcal{A}_{G_{\mathbb{C}}}$

 Spectral curve

$$\mathcal{M}_{L_{G_{\mathbb{C}}}}$$



$$\mathcal{A}_{G_{\mathbb{C}}} \cong \mathcal{A}_{L_{G_{\mathbb{C}}}}$$

$$G_{\mathbb{C}} \quad {}^L G_{\mathbb{C}}$$

$$\mathrm{GL}(n,\mathbb{C}) \quad \mathrm{GL}(n,\mathbb{C})$$

$$\mathrm{SL}(n,\mathbb{C}) \quad \mathrm{PGL}(n,\mathbb{C})$$

$$\mathrm{Sp}(2n,\mathbb{C}) \quad \mathrm{SO}(2n+1,\mathbb{C})$$

$$\mathrm{SO}(2n,\mathbb{C}) \quad \mathrm{SO}(2n,\mathbb{C})$$

[ Hausel-Thaddeus, Donagi-Pantev,  
Kapustin-Witten, Hitchin... ]

[SYZ]  
Kontsevich's  
homological mirror symmetry

## The Hitchin fibration (abelianization)

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$\mathcal{M}_{^L G_{\mathbb{C}}}$



$$\mathcal{A}_{G_{\mathbb{C}}} \cong \mathcal{A}_{^L G_{\mathbb{C}}}$$

$G_{\mathbb{C}}$      ${}^L G_{\mathbb{C}}$

$\text{GL}(n, \mathbb{C})$      $\text{GL}(n, \mathbb{C})$

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[ Hausel-Thaddeus, Donagi-Pantev,  
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$\mathcal{M}_{L G_{\mathbb{C}}}$

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E.g.  $G_{\mathbb{C}} = \text{GL}(n, \mathbb{C}) \Rightarrow$  generic fibre =  $\text{Jac}(S)$

$G_{\mathbb{C}} = \text{SL}(n, \mathbb{C}) \Rightarrow$  generic fibre =  $\text{Prym}(S, \Sigma)$

${}^L G_{\mathbb{C}} = \text{PGL}(n, \mathbb{C}) \Rightarrow$  generic fibre =  $\text{Prym}(S, \Sigma)^{\vee}$

The Hitchin fibration  
(abelianization)

Ab. variety  
 $\cap$   
 $Jac(S)$

$S = \{\det(\Phi - \lambda \text{Id}) = 0\} \subset \text{Tot } K$   
Spectral curve

$$\begin{array}{ccc} \mathcal{M}_{G_{\mathbb{C}}} & & \mathcal{M}_{^L G_{\mathbb{C}}} \\ h \searrow & & \downarrow \\ \mathcal{A}_{G_{\mathbb{C}}} & \cong & \mathcal{A}_{^L G_{\mathbb{C}}} \end{array}$$

$$\begin{array}{ll} G_{\mathbb{C}} & {}^L G_{\mathbb{C}} \\ GL(n, \mathbb{C}) & GL(n, \mathbb{C}) \\ SL(n, \mathbb{C}) & PGL(n, \mathbb{C}) \\ Sp(2n, \mathbb{C}) & SO(2n+1, \mathbb{C}) \\ SO(2n, \mathbb{C}) & SO(2n, \mathbb{C}) \end{array}$$

E.g.  $G_{\mathbb{C}} = GL(n, \mathbb{C}) \Rightarrow$  generic fibre =  $Jac(S)$

$G_{\mathbb{C}} = SL(n, \mathbb{C}) \Rightarrow$  generic fibre =  $\text{Prym}(S, \Sigma)$

${}^L G_{\mathbb{C}} = PGL(n, \mathbb{C}) \Rightarrow$  generic fibre =  $\text{Prym}(S, \Sigma)^{\vee}$

[ Hausel-Thaddeus, Donagi-Pantev,  
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Kontsevich's  
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$\mathcal{M}_{G_{\mathbb{C}}}$   
Moduli space of  $G_{\mathbb{C}}$ -Higgs bundles

$\mathcal{Z}$   
 $\text{Hom}^{red}(\pi_1(\Sigma), G_{\mathbb{C}})/G_{\mathbb{C}},$   
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[ Corlette, Donaldson, Hitchin, Simpson,  
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$$\mathcal{M}_{G_{\mathbb{C}}}$$

is a hyper-Kähler space

I

From the  
Riemann surface  $\Sigma$   
Recover Higgs bundles

J

From the  
complex group  $G_{\mathbb{C}}$   
Recover flat connections

K = I J

$\omega_I$

$\omega_J$

$\omega_K$

# I

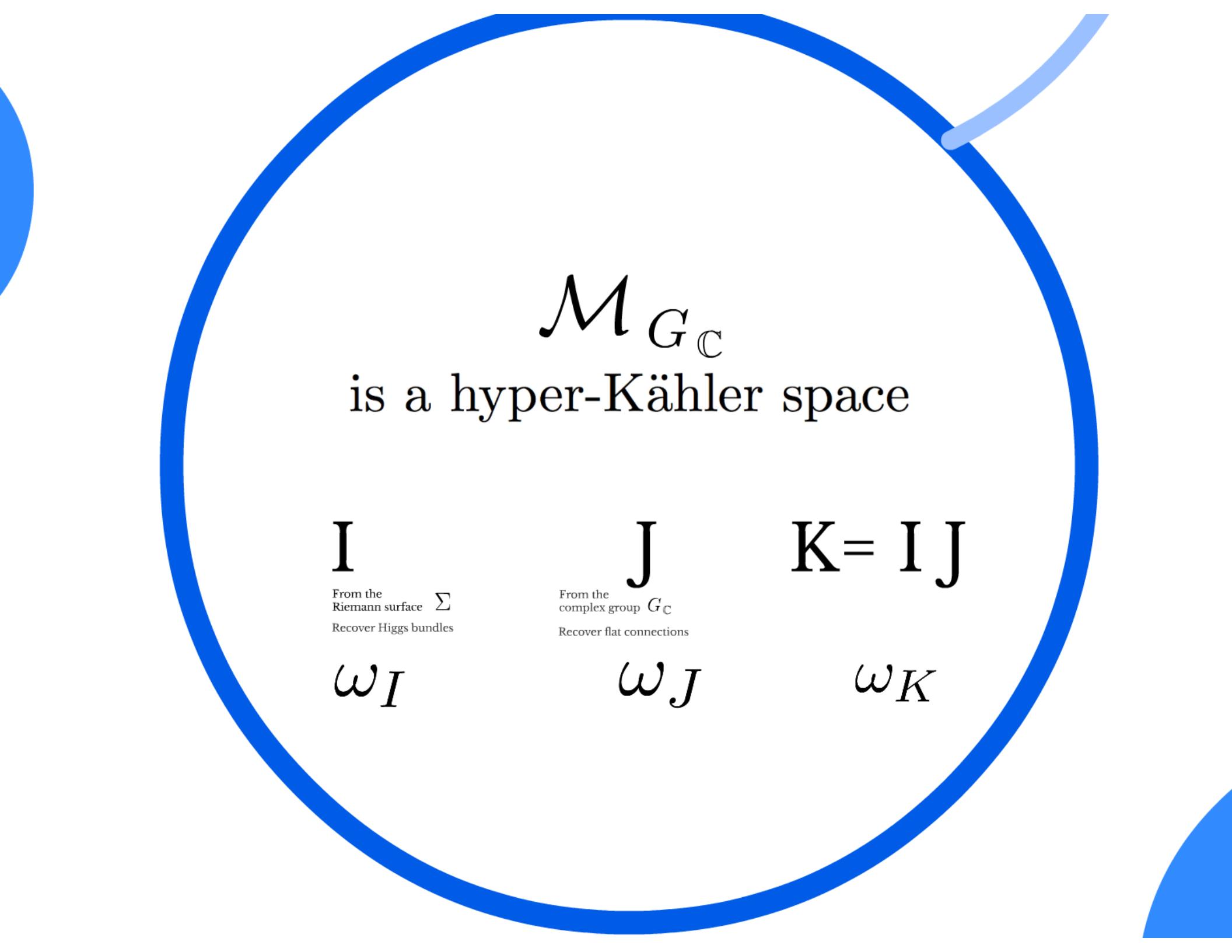
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With respect to each complex /  
symplectic structure...

A holomorphic submanifold  
+ Hol. vector bundle is a  
**B-brane**

A L<sup>A</sup>grangian submanifold  
+ flat connection is an  
**A-brane**

In which ways can one construct families of  
branes of each possible type?

(B,A,A)    (B,B,B)

(A,B,A)    (A,A,B)

See first examples in [Kapustin-Witten, Gukov-Witten]

(B,A,A)

$G$ -Higgs bundles form a Lagrangian fixed by

$$(E, \Phi) \mapsto (\sigma\rho(E), -\sigma\rho(\Phi))$$

- $\sigma \curvearrowright G_{\mathbb{C}}$  anti-holomorphic  
fixing real form  $G$   $\text{SL}(n, \mathbb{R}) = \text{SL}(n, \mathbb{C})^\sigma$   
 $\sigma(X) = \overline{X}$
- $\rho \curvearrowright G_{\mathbb{C}}$  anti-holomorphic  
fixing compact form  $SU(n) = \text{SL}(n, \mathbb{C})^\rho$   
 $\rho(X) = -\overline{X}^t$

[ Hitchin '92 ]

# [Baraglia-S.]

(A,B,A)

is fixed by

$$i_2 : (E, \Phi) \mapsto (f\rho(E), -f\rho(\Phi))$$

- $f \curvearrowright \Sigma$  real structure

Locally  $z \mapsto \bar{z}$

(B,A,A)

$G$ -Higgs bundles form a Lagrangian fixed by

$$i_1 : (E, \Phi) \mapsto (\sigma\rho(E), -\sigma\rho(\Phi))$$

- $\sigma \curvearrowright G_{\mathbb{C}}$  anti-holomorphic  
fixing real form  $G$

$$\begin{aligned} \text{SL}(n, \mathbb{R}) &= \text{SL}(n, \mathbb{C})^\sigma \\ \sigma(X) &= \overline{X} \end{aligned}$$

- $\rho \curvearrowright G_{\mathbb{C}}$  anti-holomorphic  
fixing compact form

$$\begin{aligned} SU(n) &= \text{SL}(n, \mathbb{C})^\rho \\ \rho(X) &= -\overline{X}^t \end{aligned}$$

(A,A,B)

is fixed by

$$i_3 := i_1 \circ i_2$$

Useful for... E.g.

... [Franco-Jardim-Marchesi]  
for framed instantons

... [Schaffhauser-Hoskins]  
for quiver varieties

## What should the dual branes be?

[Kapustin-Witten] give the dual type

[Gukov] gives first examples of duality from Physics

$$\mathcal{M}_{G_{\mathbb{C}}}$$

$$\mathcal{M}_{L G_{\mathbb{C}}}$$

$$(B, A, A) \sim (B, B, B)$$

$G$ -Higgs bundles

## [ Baraglia - S. '13] Conjecture.

The support of the dual (B,B,B) brane to the (B,A,A) brane of G-Higgs bundles is

$$\mathcal{M}_{\check{H}} \subset \mathcal{M}_{^L G_{\mathbb{C}}}$$

For  $\check{H}$  Nadler's group associated to G.

E.g.  $G = \text{split form of } G_{\mathbb{C}} \Rightarrow \check{H} = {}^L G_{\mathbb{C}}$

$G = Sp(2m, \mathbb{R}) \Rightarrow \check{H} = SO(2m + 1, \mathbb{C})$

More Maths support N. Hitchin '13:  $G = U(n, n)$   
for non-compact... L. Branco '16:  $G = \text{SL}(n, \mathbb{H}), \text{SO}(2n, \mathbb{H}), \text{Sp}(2n, 2n)$

How do the (B,A,A)-branes  
lie inside the Hitchin  
fibration?

# (B,A,A) - branes of G-Higgs bundles

## Abelian

[ S. ]  $G$  split real form of  $G_{\mathbb{C}}$

E.g.  $G_{\mathbb{C}} = SL(n, \mathbb{C})$  and  $G = SL(n, \mathbb{R})$

**Theorem.** The (B,A,A)-brane intersects the smooth fibres in torsion 2-points.

E.g.  $G = SL(n, \mathbb{C}) \rightarrow$  generic fibre = Prym( $S, \Sigma$ )  
 $G = SL(n, \mathbb{R}) \rightarrow$  generic intersection =  $\{L \in \text{Prym}(S, \Sigma) : L^2 \cong \mathcal{O}\}$

Useful for...

... monodromy of the Gauss Manin connection.[S.],[Baraglia - S.]  
... understanding other relations between branes.[Brandlow - S.]  
... understanding characteristic classes.[Hitchin], [S.]

## Nonabelian

[ Hitchin - S. ]  $G = SL(n, \mathbb{H}), SO(2n, \mathbb{H}), Sp(2n, 2n)$

**Theorem.** The (B,A,A)-brane intersects the fibres generically as certain subspaces of semi-stable rank-2 bundles with fixed determinant.

Useful for...

... understanding the dual (B,B,B)-branes.[Lucas Branco]

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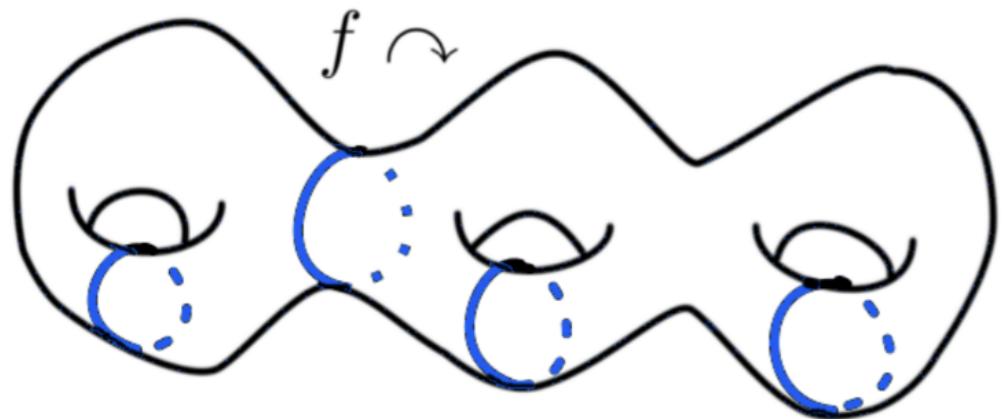
... understanding the dual (B,B,B)-branes. [Lucas Branco]

How do the (A,B,A)-branes lie in the  
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# (A,B,A)-branes

fixed by  $i_2 : (E, \Phi) \mapsto (f\rho(E), -f\rho(\Phi))$

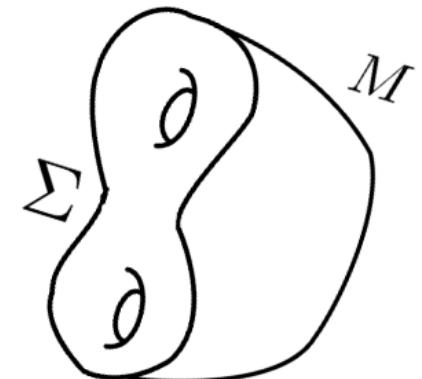
[ Baraglia - S. ]



**Theorem.** The (A,B,A) brane is a Lagrangian fibration giving a real intergrable system inside the complex Hitchin fibration, whose fibres are finite collections of tori.

What is the relation between the  
(A,B,A) brane and  $\text{Hom}^{\text{red}}(\pi_1(\Sigma), G_{\mathbb{C}})/G_{\mathbb{C}}$ ?

$$M = \frac{\Sigma \times [-1, 1]}{(x, t) \mapsto (f(x), -t)} \quad \partial M = \Sigma$$



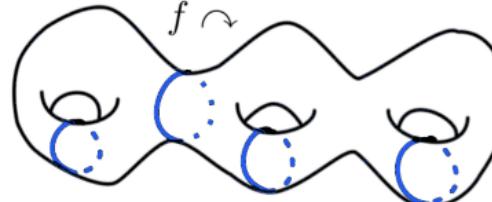
**Theorem.** Let  $(E, \Phi)$  be a fixed simple point in the (A,B,A) brane. Then the associated connection extends over  $M$  as a flat connection if and only if the class

$$[E] \in \tilde{K}_{\mathbb{Z}_2}^0(\Sigma)$$

in reduced equivariant K-theory is trivial.

## (A,B,A)-branes

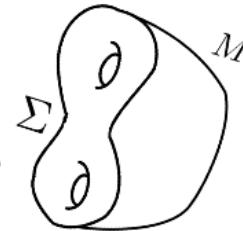
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[Kapustin-Witten]

What should the d  
branes be?

$\mathcal{M}_{G_{\mathbb{C}}}$   $\mathcal{M}$   
 $(A.B.A) \curvearrowright (A$

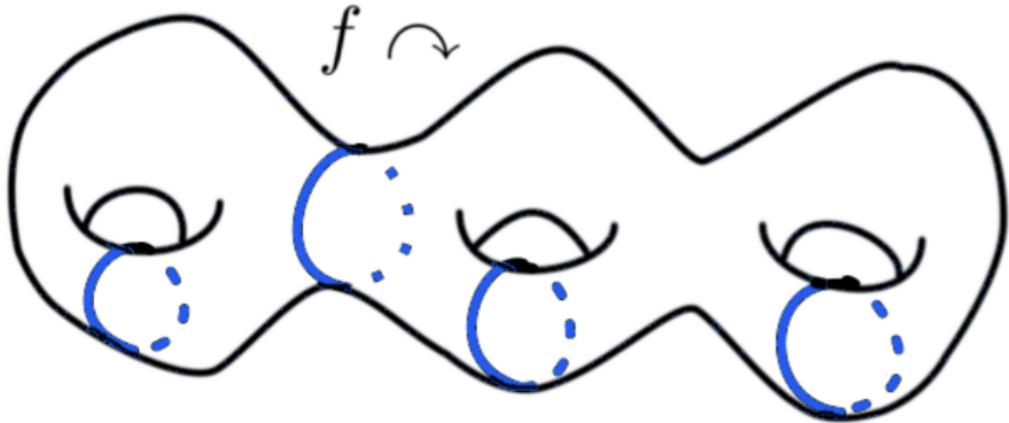
[Kapustin-Witten]

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$$\mathcal{M}_{^L G_{\mathbb{C}}}$$

$$(A, B, A) \sim (A, B, A)$$

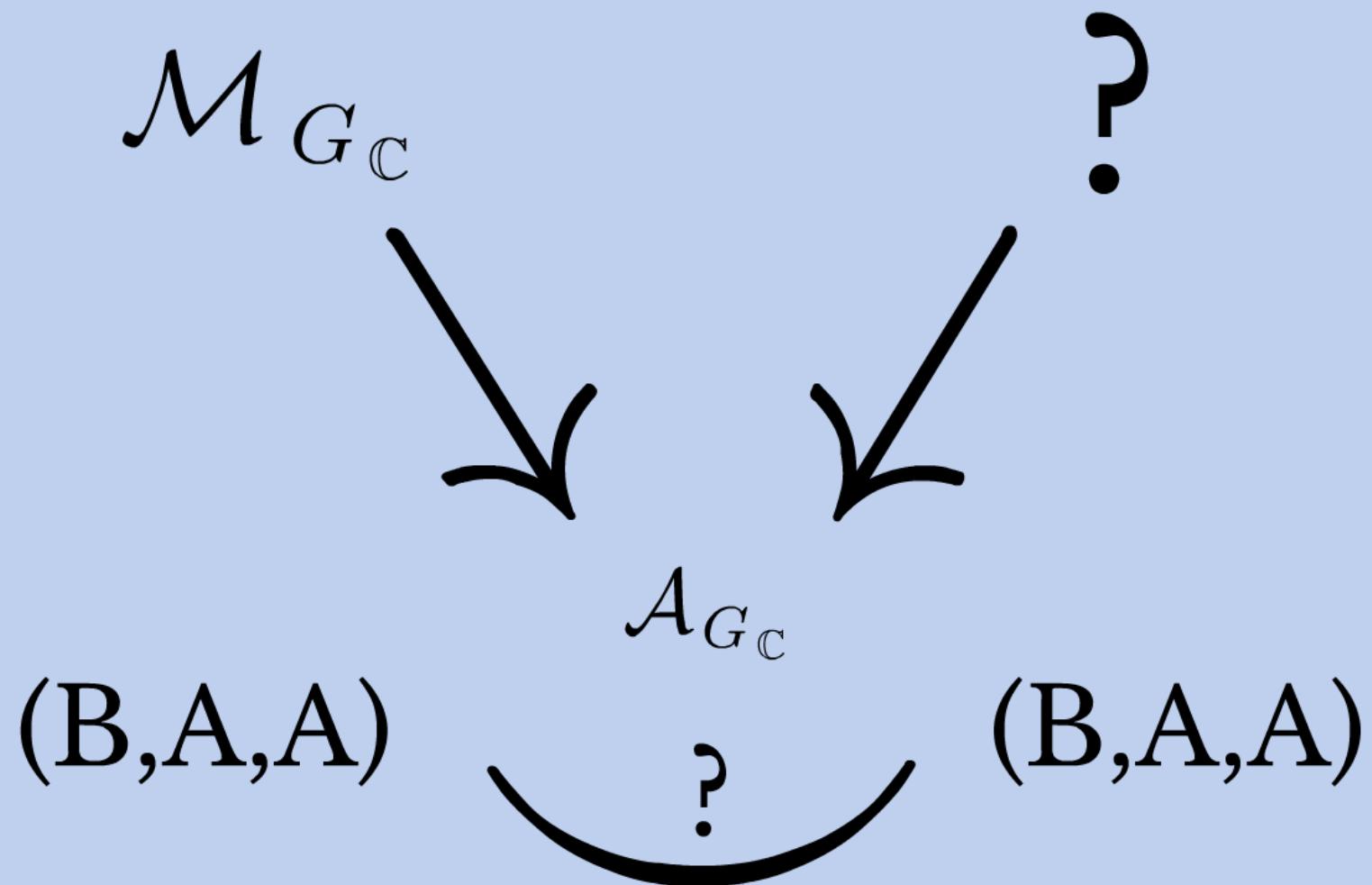


## [ Baraglia - S. '13] Conjecture.

For  ${}^L\rho \curvearrowright {}^LG_c$  compact anti-holomorphic involution,  
the support of the dual  $(A, B, A)$  brane is  
the fixed point set in  $\mathcal{M}_{LG_c}$  of

$$(E, \Phi) \mapsto (f \cdot {}^L\rho(E), -f \cdot {}^L\rho(\Phi))$$

Which other correspondances between  
fibrations and branes are there?

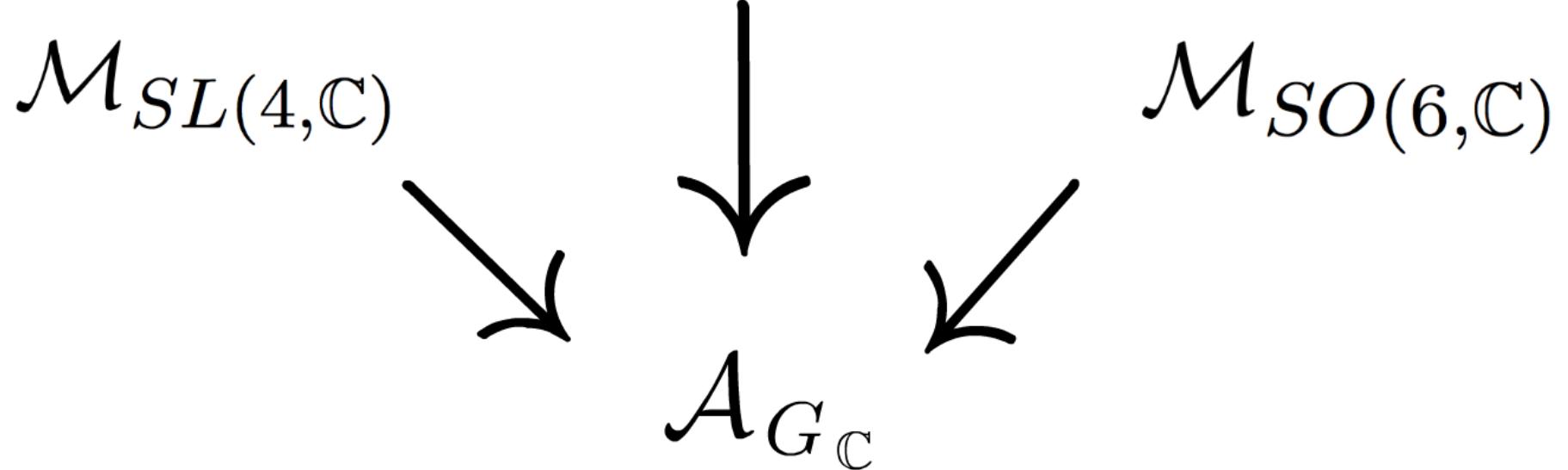


# [ Bradlow-S. ]

Symmetric and Anti-symmetric Higgs bundles

Structure groups  $G_{\mathbb{C}} \times G_{\mathbb{C}}$ ,  $G_{\mathbb{C}} \times G_{\mathbb{C}} \times G_{\mathbb{C}}$ , ...

$$\mathcal{M}_{SL(4,\mathbb{C})} \times \mathcal{M}_{SL(4,\mathbb{C})}$$

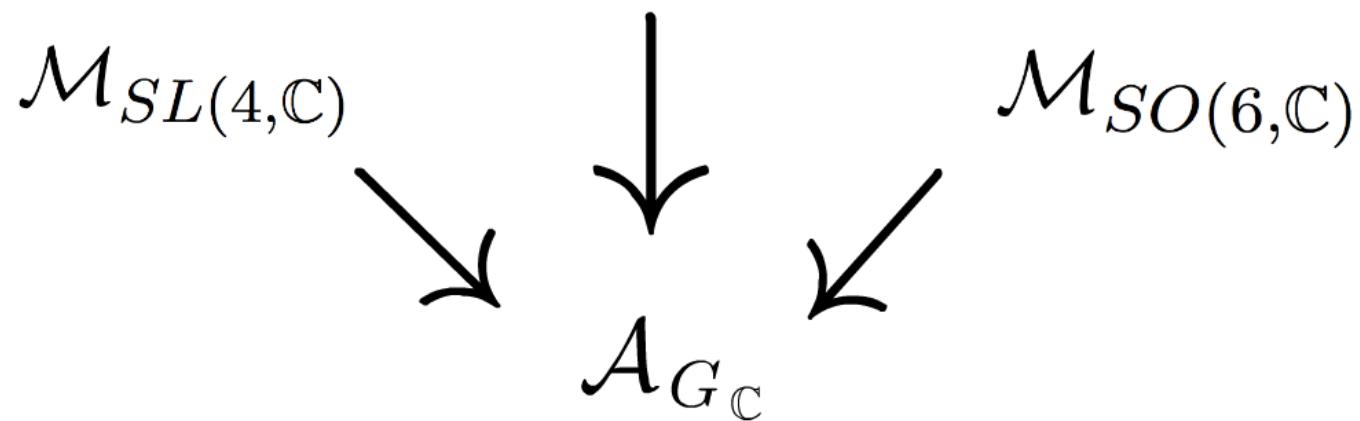


## [ Bradlow-S. ]

Symmetric and Anti-symmetric Higgs bundles

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$$S \mapsto (S \times_{\Sigma} S)_0 \underset{\tau}{\curvearrowright} (S \times_{\Sigma} S)_0 / \tau \mapsto Sym((S \times_{\Sigma} S)_0 / \tau)$$

## [ Bradlow-S. ]

Symmetric and Anti-symmetric Higgs bundles  
 Structure groups  $G_{\mathbb{C}} \times G_{\mathbb{C}}$ ,  $G_{\mathbb{C}} \times G_{\mathbb{C}} \times G_{\mathbb{C}}$ , ...

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{SL}(4,\mathbb{C})} \times \mathcal{M}_{\mathrm{SL}(4,\mathbb{C})} & & \\ \downarrow & & \downarrow \\ \mathcal{M}_{SL(4,\mathbb{C})} & & \mathcal{M}_{SO(6,\mathbb{C})} \\ \searrow & & \swarrow \\ & \mathcal{A}_{G_{\mathbb{C}}} & \end{array}$$

$$S \mapsto \underbrace{(S \times_{\Sigma} S)_0}_{\tau} \mapsto (S \times_{\Sigma} S)_0 / \tau \mapsto Sym((S \times_{\Sigma} S)_0 / \tau)$$

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{SL}(4,\mathbb{R})} & & \mathcal{M}_{SO_0(3,3)} \\ (B,A,A) & \xrightarrow{2^{2g}} & (B,A,A) \end{array}$$

**Useful for...** E.g.  $SU(2) \times SL(2, \mathbb{R}) \subset SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$   
 ... minimal surfaces & Higgs bundles  
 [Bradlow-Loftin-MacIntosh-S.]

For v.b on elliptic fibrations  
 see [Guerra '07]

**Gracias!**

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