New derived equivalences via a duality of non-abelian gauged linear sigma models

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Introduction

- X variety, $D(X) = D^b(\mathrm{Coh}\, X) =$ derived category of coherent sheaves
- Motivating problem: Construct examples of pairs X, Y with $D(X) \cong D(Y)$ (or with a fully faithful inclusion $D(X) \hookrightarrow D(Y)$).

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Main tool: "LG models and variation of GIT stability"[1]

- Applies to understanding complete intersections in linear GIT quotients, i.e. $X \overset{c.int.}{\subset} \mathbb{C}^n /\!\!/ G$
- Today: One particular case of this.
- Applying LG/VGIT strategy + understanding a proposed duality of GLSM's [Hori] $\rightsquigarrow$ new examples of pairs X, Y with $D(X) \approx D(Y)$

[1] Due to many people, Witten, Herbst–Hori–Page, Herbst–Walcher, Segal, Ballard–Deliu–Favero–Isik–Katzarkov, +++

Pfaffian varieties

- $V \cong \mathbb{C}^n$ vector space, n odd, $k < n$ even
- $\mathbb{P}(\wedge^2 V) = \{(n \times n) \text{ anti-symmetric matrices}\}/\mathbb{C}^*$
- $\mathbb{P}(\wedge^2 V) \supset \mathrm{Pf}(k, n) = \{\text{Rk.} \leq k \text{ matrices}\}/\mathbb{C}^*$
- E.g. $\mathrm{Pf}(2, n) = \mathrm{Gr}(2, n) \overset{\text{Plücker}}{\subset} \mathbb{P}(\wedge^2 V)$

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Pfaffian-Grassmannian equivalence (Borisov-Căldăraru, Kuznetsov 06, LG/VGIT proof by Addington–Donovan–Segal ’14)

- Define 2 smooth Calabi–Yau 3-folds:

$$X \subset \mathrm{Gr}(2,7) \subset \mathbb{P}(\wedge^2\mathbb{C}^7) \text{ \& } Y \subset \mathrm{Pf}(4,7) \subset \mathbb{P}(\wedge^2(\mathbb{C}^7)^\vee)$$

- Choose generic lin. subspace $L \cong \mathbb{C}^{14} \subset \wedge^2\mathbb{C}^7 \rightsquigarrow L^\perp \cong \mathbb{C}^7 \subset \wedge^2(\mathbb{C}^7)^\vee$
- $X := \mathrm{Gr}(2,7) \cap \mathbb{P}(L) \subset \mathbb{P}(\wedge^2\mathbb{C}^7)$
- $Y := \mathrm{Pf}(4,7) \cap \mathbb{P}(L^\perp) \subset \mathbb{P}(\wedge^2(\mathbb{C}^7)^\vee)$
- Theorem: $D(X) \cong D(Y)$

Main result

A generalisation of the Pfaffian–Grassmannian pair

- Choose generic linear subspace $L \subset \wedge^2 V = \wedge^2 \mathbb{C}^n \rightsquigarrow L^\perp \subset \wedge^2 V^\vee$
- Define $X_L = \widetilde{\mathrm{Pf}}(k, n) \cap \mathbb{P}(L) \subset \mathbb{P}(\wedge^2 V)$
- Define $Y_{L^\perp} = \widetilde{\mathrm{Pf}}(n-k-1, n) \cap \mathbb{P}(L^\perp) \subset \mathbb{P}(\wedge^2 V^\vee)$
- $k = 2, n = 7, \dim L^\perp = 7$ recovers Pfaffian–Grassmannian example.
- ($\widetilde{\mathrm{Pf}}$ is some non-commutative resolution of Pf)

Theorem (R.–Segal)

There are fully faithful inclusions/equivalences of derived categories

$$\begin{aligned} \dim L^\perp < kn/2 : &\qquad D(X_L) \hookleftarrow D(Y_{L^\perp}) \\ \dim L^\perp = kn/2 : &\qquad D(X_L) \cong D(Y_{L^\perp}) \\ \dim L^\perp > kn/2 : &\qquad D(X_L) \hookrightarrow D(Y_{L^\perp}) \end{aligned}$$

$X_L = \widetilde{\mathrm{Pf}}(k,n) \cap \mathbb{P}(L)$ & $Y_{L^\perp} = \widetilde{\mathrm{Pf}}(n-k-1,n) \cap \mathbb{P}(L^\perp)$

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Remarks

- Fits into Kuznetsov's "Homological projective duality" – alternative formulation of theorem is "$\widetilde{\mathrm{Pf}}(k,n)$ and $\widetilde{\mathrm{Pf}}(n-k-1,n)$ are homological projectively dual".
- Previously conjectured by Kuznetsov and proved for $k = 2, n \leq 7$.
- In general, X_L and $Y_{L^\perp}$ are not varieties, but instead non-commutative resolutions of singular varieties.

GIT quotient presentation of Pfaffian variety

- Fix $S = \mathbb{C}^k$ & non-degenerate bivector $\omega_S \subset \wedge^2 S$
- $\mathsf{GL}(S) \supset \mathsf{GSp}(S) = \{\sigma \mid \sigma_*(\omega_S) = \lambda\omega_S,\ \lambda \in \mathbb{C}^*\}$, acts on $\mathrm{Hom}(S, V) \cong \mathbb{C}^{kn}$
- $\mathsf{Pf}(k, n) = \mathrm{Hom}(S, V)^{\mathsf{ss}}/\mathsf{GSp}(S) = \mathrm{Hom}(S, V) /\!\!/ \mathsf{GSp}(S)$
- Stack $\mathsf{Pf}^{\mathsf{st}}(k, n) := [\mathrm{Hom}(S, V)^{\mathsf{ss}}/\mathsf{GSp}(S)]$ is a "stacky resolution" of $\mathsf{Pf}(k, n)$

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B-brane category/noncommutative resolution

- Problem: $\mathsf{Pf}^{\mathsf{st}}$ too stacky, has infinite stabiliser groups $\rightsquigarrow D(\mathsf{Pf}^{\mathsf{st}})$ not the right thing to consider (too big)
- Solution: Define a "subcategory of B-branes" $\mathsf{Br}(\mathsf{Pf}^{\mathsf{st}}) \subset D(\mathsf{Pf}^{\mathsf{st}})$ and work with this.
- $\widetilde{\mathsf{Pf}}(k, n) := \mathsf{Br}(\mathsf{Pf}^{\mathsf{st}}(k, n))$ is a non-commutative crepant resolution of $\mathsf{Pf}(k, n)$ (Špenko–Van den Bergh '15)

Proving $D(X_L) \approx D(Y_{L^\perp})$

$$D(X_L) \xrightarrow[\cong]{\text{Knörrer}} \mathrm{Br}(\mathcal{E}, W) \xrightarrow[\approx]{\text{VGIT}} \mathrm{Br}(\mathcal{E}', W) \xrightarrow[\cong]{\text{Hori}^\vee} D(Y_{L^\perp})$$

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Step 1: Variety to LG model, "Knörrer periodicity"

- Idea: Write X_L as a critical locus.
- $X_L \subset \mathrm{Pf}^{\mathrm{st}}(k, n) = [\mathrm{Hom}(S, V)^{\mathrm{ss}}/\mathrm{GSp}(S)]$, cut out by functions f_i
- Define $\mathcal{E} = [\mathrm{Hom}(S, V)^{\mathrm{ss}} \oplus \overset{p_i}{L^\perp} /\mathrm{GSp}(S)]$ w/ potential $W = \sum f_i p_i$
- $\rightsquigarrow X_L = \mathrm{Crit}(W)$ in a nice way (W locally quadratic)
- $\rightsquigarrow$ "Knörrer periodicity" (Isik, Shipman '12, +++):
  $D(X_L) \cong \mathrm{Br}(\mathcal{E}, W) \subseteq D(\mathcal{E}, W)$, category of matrix factorisations

Proving $D(X_L) \approx D(Y_{L^\perp})$

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Step 2: Variation of GIT stability

- $\exists$ 2nd GIT quotient $\mathcal{E}' = [\mathrm{Hom}(S, V) \oplus (L^\perp \setminus 0)/\mathrm{GSp}(S)]$
- Lemma (following Halpern-Leistner, Ballard–Favero–Katzarkov '12): $\mathrm{Br}(\mathcal{E}, W) \approx \mathrm{Br}(\mathcal{E}', W)$

Proving $D(X_L) \approx D(Y_{L^\perp})$

$$D(X_L) \xrightarrow[\cong]{\text{Knörrer}} \mathrm{Br}(\mathcal{E}, W) \xrightarrow[\approx]{\text{VGIT}} \mathrm{Br}(\mathcal{E}', W) \xrightarrow[\cong]{\text{Hori}^\vee} D(Y_{L^\perp})$$

Step 3: Analysing $\mathrm{Br}(\mathcal{E}', W)$ (or failing at that)

- $$\begin{array}{ccc} \widetilde{\mathrm{Pf}}(n-k-1, n) \cap \mathbb{P}(L^\perp) = Y_{L^\perp} & & (\mathcal{E}', W) \\ & \searrow \quad \swarrow & \\ & \mathbb{P}(L^\perp) & \end{array}$$
- Could hope to prove $D(Y_{L^\perp}) \cong \mathrm{Br}(\mathcal{E}', W)$ fibre-wise over $\mathbb{P}(L^\perp)$
- Problem: Hard to analyse fibres, i.e. for $x \in \mathbb{P}(L^\perp)$

$$\mathrm{Br}(\mathcal{E}'|_x, W|_x) = \mathrm{Br}(\mathrm{Hom}(S, V)/\mathrm{Sp}(S), W_x)$$

where W_x is some quadratic potential.

Hori's duality

- Recall $S = \mathbb{C}^k$, and let $Q = \mathbb{C}^{n-k-1}$
- Consider gauged linear sigma models

$$GLSM_{S,L^\perp} : \quad (\mathrm{Hom}(S,V) \oplus L^\perp / \mathsf{GSp}(S), W_S)$$

$$GLSM_{Q,L} : \quad (L \oplus \mathrm{Hom}(V,Q) / \mathsf{GSp}(Q), W_Q)$$

- Claim: $GLSM_{S,L^\perp} \cong GLSM_{Q,L}$.

Theorem (R.–Segal)

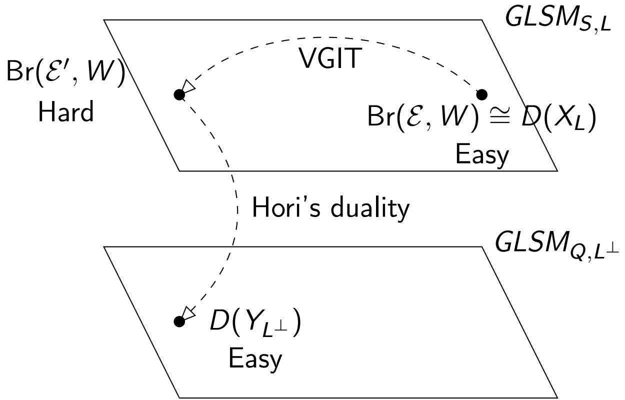
This holds at the level of categories:

$$Br(\mathrm{Hom}(S,V) \oplus L^\perp / GSp(S), W_S)$$
$$\cong Br(L \oplus \mathrm{Hom}(V,Q) / GSp(Q), W_Q)$$

Completing the proof

$$
\begin{aligned}
D(X_L) &\overset{\text{Knörrer}}{\cong} \mathrm{Br}(\mathcal{E}, W_S) \overset{\text{VGIT}}{\sim} \mathrm{Br}(\mathcal{E}', W_S) \\
&= \mathrm{Br}(\mathrm{Hom}(S, V) \oplus (L^{\perp} \setminus 0)/\mathrm{GSp}(S), W_S) \\
&\overset{\text{Hori's duality}}{\cong} \mathrm{Br}(L \oplus \mathrm{Hom}(V, Q)^{\mathrm{ss}}/\mathrm{GSp}(Q), W_Q) \\
&\overset{\text{Knörrer}}{\cong} D(Y_{L^{\perp}})
\end{aligned}
$$

A picture of GIT stability parameter spaces



Extensions

- This duality exchanges $\mathrm{Sp}(k) \leftrightarrow \mathrm{Sp}(n-k-1)$.
- Hori also gives analogous duality $\mathrm{SO}(k) \leftrightarrow \mathrm{O}(n-k+1)$
- $\rightsquigarrow$ categorical duality for loci of symmetric matrices of given rank in $\mathbb{P}(\mathrm{Sym}^2 V)$, instead of antisymmetric ones in $\mathbb{P}(\wedge^2 V)$.
- Combining Sp/SO-dualities, adding on $\mathbb{C}^*$-factors $\rightsquigarrow$ more examples.