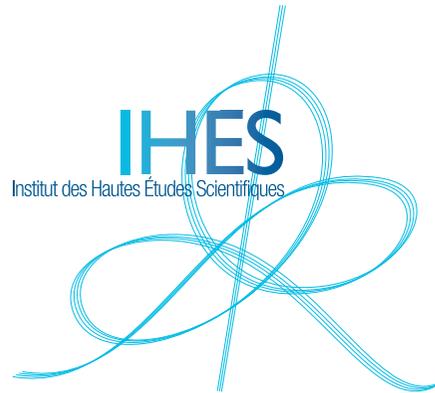


# Analytical Approaches to Coalescing Binary Black Holes

Thibault Damour

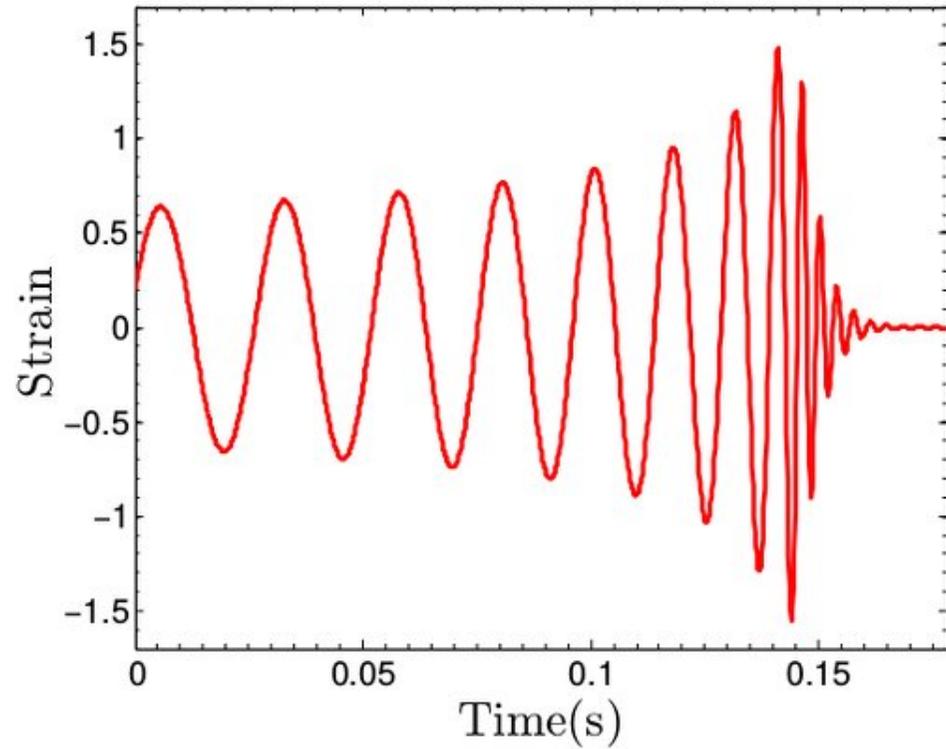
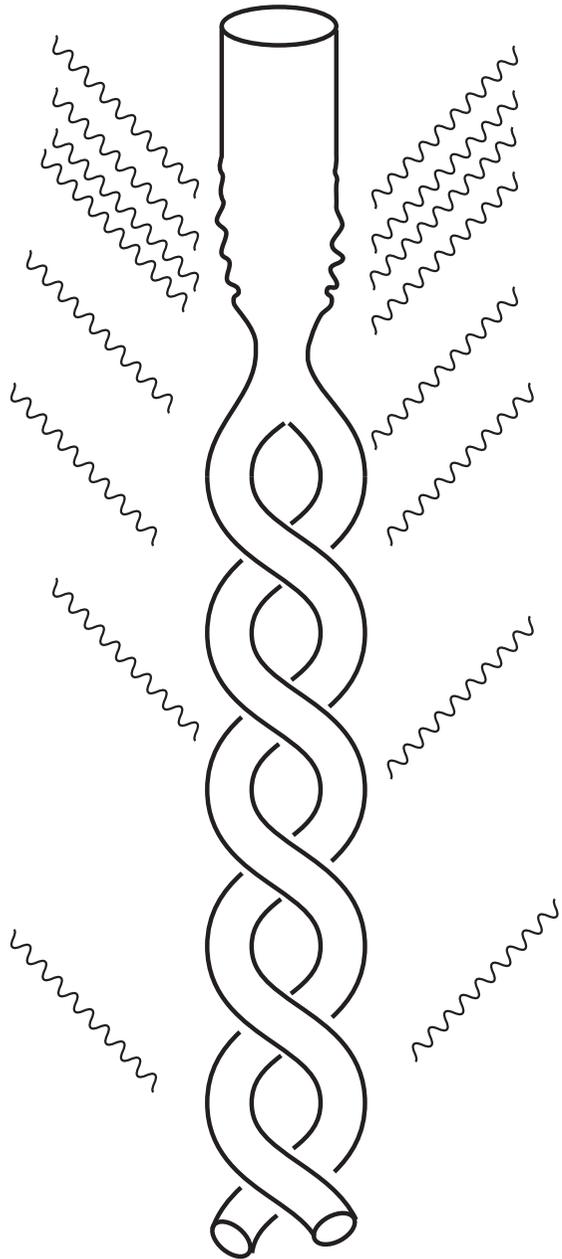
Institut des Hautes Etudes Scientifiques



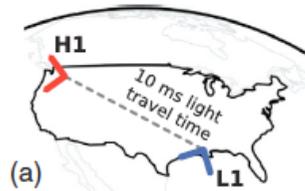
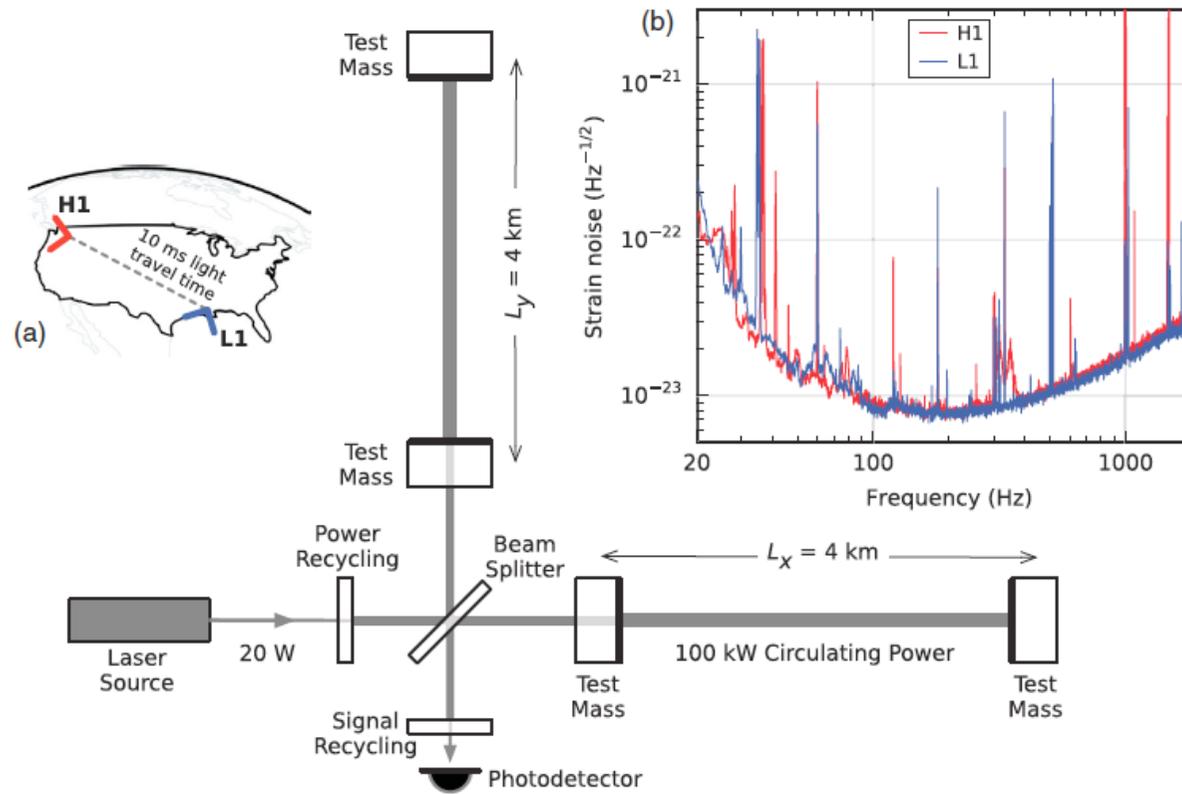
String-Math 2016  
Collège de France, Paris, June 27-July 2, 2016

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$R_{\mu\nu} = 0$$



# LASER INTERFEROMETER GW DETECTORS



LIGO Hanford



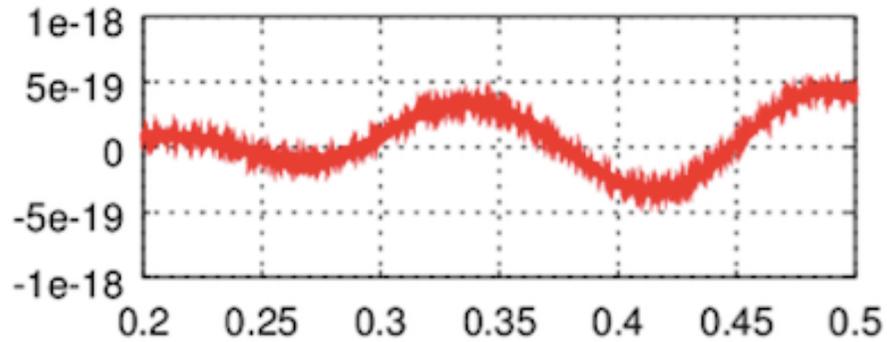
LIGO Livingston



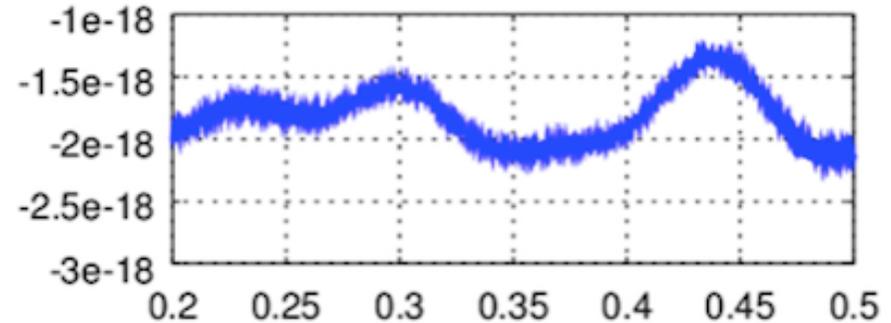
Virgo (IT)

# GW150914: An incredibly small signal lost in the noise

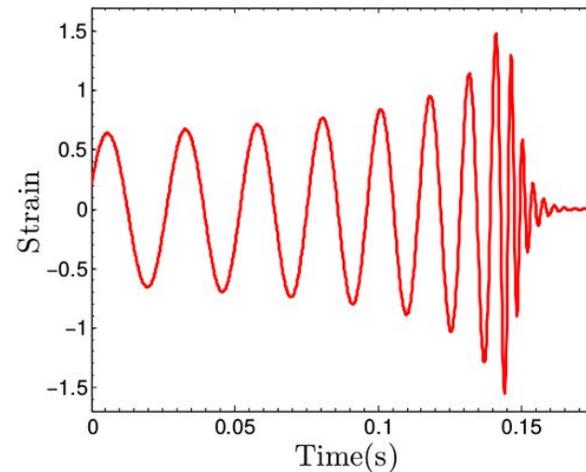
Hanford H1: raw data



Livingston L1: raw data



scale :  $10^{-21}$   
 $500 \times$  smaller



$$\delta L/L = 10^{-21} \rightarrow \delta L \sim 10^{-9} \text{ atom !}$$

# November 25 1915, January 13, 1916, June 1916, January 1918

## Die Feldgleichungen der Gravitation.

VON A. EINSTEIN.

In zwei vor kurzem erschienenen Mitteilungen<sup>1</sup> habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariablen gegenüber kovariant sind.

Der Entwicklungsgang war dabei folgender. Zunächst fand ich Gleichungen, welche die NEWTONSCHE Theorie als Näherung enthalten und beliebigen Substitutionen von der Determinante 1 gegenüber kovariant waren. Hierauf fand ich, daß diese Gleichungen allgemein kovariant entsprechen, falls der Skalar des Energietensors der Materie verschwindet. Das Koordinatensystem war dann nach der einfachen Regel zu spezialisieren, daß  $\sqrt{-g}$  zu 1 gemacht wird, wodurch die Gleichungen der Theorie eine eminente Vereinfachung erfahren. Dabei mußte aber, wie erwähnt, die Hypothese eingeführt werden, daß der Skalar des Energietensors der Materie verschwinde.

Neuerdings finde ich nun, daß man ohne Hypothese über den Energietensor der Materie auskommen kann, wenn man den Energietensor der Materie in etwas anderer Weise in die Feldgleichungen einsetzt, als dies in meinen beiden früheren Mitteilungen geschehen ist. Die Feldgleichungen für das Vakuum, auf welche ich die Erklärung der Perihelbewegung des Merkur gegründet habe, bleiben von dieser Modifikation unberührt. Ich gebe hier nochmals die ganze Betrachtung, damit der Leser nicht genötigt ist, die früheren Mitteilungen unangesehen heranzuziehen.

Aus der bekannten RIEMANNSCHEM Kovariante vierten Ranges leitet man folgende Kovariante zweiten Ranges ab:

$$G_{im} = R_{im} + S_{im} \quad (1)$$

$$R_{im} = -\sum_{\rho} \frac{\partial \{im\}}{\partial x_{\rho}} \left\{ \rho \right\} + \sum_{\rho} \frac{\partial \{i\rho\}}{\partial x_m} \left\{ \rho \right\} \quad (1a)$$

$$S_{im} = \sum_{\rho} \frac{\partial \{i\rho\}}{\partial x_m} - \sum_{\rho} \frac{\partial \{im\}}{\partial x_{\rho}} \left\{ \rho \right\} \quad (1b)$$

<sup>1</sup> Sitzungsber. XLIV, S. 778 und XLVI, S. 799, 1915.

## Über das Gravitationsfeld eines Massenpunktes nach der EINSTEINSCHEM Theorie.

VON K. SCHWARZSCHILD.

(Vorgelegt am 13. Januar 1916 [s. oben S. 42].)

§ 1. Hr. EINSTEIN hat in seiner Arbeit über die Perihelbewegung des Merkur (s. Sitzungsberichte vom 18. November 1915) folgendes Problem gestellt:

Ein Punkt bewege sich gemäß der Forderung

$$\delta \int ds = 0, \quad (1)$$

wobei

$$ds = \sqrt{\sum g_{\alpha\beta} dx_{\alpha} dx_{\beta}}, \quad \alpha, \beta = 1, 2, 3, 4$$

ist,  $g_{\alpha\beta}$  Funktionen der Variablen  $x$  bedeuten und bei der Variation am Anfang und Ende des Integrationswegs die Variablen  $x$  festzuhalten sind. Der Punkt bewege sich also, kurz gesagt, auf einer geodätischen Linie in der durch das Linienelement  $ds$  charakterisierten Mannigfaltigkeit.

Die Ausführung der Variation ergibt die Bewegungsgleichungen des Punktes

$$\frac{d^2 x_{\alpha}}{ds^2} = -\sum_{\beta\gamma} \Gamma_{\alpha\beta\gamma} \frac{dx_{\beta}}{ds} \frac{dx_{\gamma}}{ds}, \quad \alpha, \beta, \gamma = 1, 2, 3, 4 \quad (2)$$

wobei

$$\Gamma_{\alpha\beta\gamma} = -\frac{1}{2} \sum_{\delta} g^{\alpha\delta} \left( \frac{\partial g_{\delta\beta}}{\partial x_{\gamma}} + \frac{\partial g_{\delta\gamma}}{\partial x_{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x_{\delta}} \right) \quad (3)$$

ist und  $g^{\alpha\beta}$  die zu  $g_{\alpha\beta}$  koordinierte und normierte Subdeterminante in der Determinante  $|g_{\alpha\beta}|$  bedeutet.

Dies ist nun nach der EINSTEINSCHEM Theorie dann die Bewegung eines masselosen Punktes in dem Gravitationsfeld einer im Punkt  $x_{\alpha} = x_{\alpha}, x_4 = 0$  befindlichen Masse, wenn die Komponenten des Gravitationsfeldes  $\Gamma$  überall, mit Ausnahme des Punktes  $x_{\alpha} = x_{\alpha}, x_4 = 0$ , den Feldgleichungen

## Über Gravitationswellen.

VON A. EINSTEIN.

(Vorgelegt am 31. Januar 1918 [s. oben S. 79].)

Die wichtige Frage, wie die Ausbreitung der Gravitationsfelder erfolgt, ist schon vor anderthalb Jahren in einer Akademiarbeit von mir behandelt worden<sup>1</sup>. Da aber meine damalige Darstellung des Gegenstandes nicht genügend durchsichtig und außerdem durch einen bedauerlichen Rechenfehler verunstaltet ist, muß ich hier nochmals auf die Angelegenheit zurückkommen.

Wie damals beschränkte ich mich auch hier auf den Fall, daß das betrachtete zeiträumliche Kontinuum sich von einem galileischen nur sehr wenig unterscheidet. Um für alle Indizes

$$g_{\alpha\gamma} = -\delta_{\alpha\gamma} + \gamma_{\alpha\gamma} \quad (1)$$

setzen zu können, wählen wir, wie es in der speziellen Relativitätstheorie üblich ist, die Zeitvariable  $x_4$  rein imaginär, indem wir

$$x_4 = it$$

setzen, wobei  $t$  die »Lichtzeit« bedeutet. In (1) ist  $\delta_{\alpha\alpha} = 1$  bzw.  $\delta_{\alpha\alpha} = 0$ , je nachdem  $\mu = \nu$  oder  $\mu \neq \nu$  ist. Die  $\gamma_{\alpha\beta}$  sind gegen 1 kleine Größen, welche die Abweichung des Kontinuums vom feldfreien darstellen; sie bilden einen Tensor vom zweiten Range gegenüber LORENTZ-Transformationen.

§ 1. Lösung der Näherungsgleichungen des Gravitationsfeldes durch retardierte Potentiale.

Wir gehen aus von den für ein beliebiges Koordinatensystem gültigen<sup>2</sup> Feldgleichungen

$$-\sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left\{ \mu\nu \right\} \left\{ \alpha \right\} + \sum_{\alpha} \frac{\partial}{\partial x_{\nu}} \left\{ \mu\alpha \right\} \left\{ \alpha \right\} + \sum_{\alpha\beta} \left\{ \mu\alpha \right\} \left\{ \nu\beta \right\} \left\{ \alpha \right\} - \sum_{\alpha\beta} \left\{ \mu\nu \right\} \left\{ \alpha\beta \right\} \left\{ \alpha \right\} \quad (2)$$

$$= -\kappa \left( T_{\alpha\nu} - \frac{1}{2} g_{\alpha\nu} T \right).$$

<sup>1</sup> Diese Sitzungsber. 1916, S. 688 ff.

<sup>2</sup> Von der Einführung des »-Billedes« (vgl. diese Sitzungsber. 1917, S. 142) ist dabei Abstand genommen.



# Basics of Gravitational Waves

In linearized GR (Einstein 1916, 1918):

$$\square h_{\mu\nu} + \partial_\mu H_\nu + \partial_\nu H_\mu = -16\pi G \left( T_{\mu\nu} - \frac{1}{D-2} \eta_{\mu\nu} T \right)$$

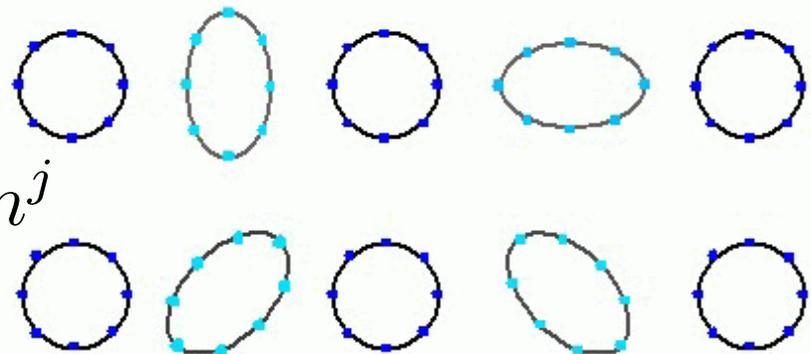
$$H_\mu = \frac{1}{2} \partial_\mu h - \partial^\nu h_{\mu\nu} \quad \partial^\nu T_{\mu\nu} = 0$$

$$ds^2 = -c^2 dt^2 + (\delta_{ij} + h_{ij}) dx^i dx^j$$

$$h_{ij} \simeq \frac{2G}{c^4 r} \ddot{Q}_{ij}^{TT} (t - r/c)$$

Massless, two helicity states  $s=\pm 2$ , i.e. two Transverse-Traceless (TT) tensor polarizations propagating at  $v=c$

$$g_{ij} = \delta_{ij} + h_{ij} \quad h_{ij} = h_+(x_i x_j - y_i y_j) + h_\times (x_i y_j + y_i x_j)$$



$$\frac{\delta L}{L} = \frac{1}{2} h_{ij} n^i n^j$$

Joseph  
Weber  
(1919-  
2000)



# Laser Interferometer GW Detectors

Initial idea: [Weber] Gertsenshtein-Pustovoit 1962

First implementation: Forward ~ 1970

First detailed noise analysis: Rainer Weiss 1972

First search of GW signals: Levine-Hall, Levine-Stebbins 1972

Development of sensitive interferometric detectors ~ 1980's

Garching: Billing H, Maischberger K, Ruediger A, Schilling R, Schnupp L, Winkler, W 1979;  
Shoemaker D et al 1988

Glasgow: Drever, Hough, Pugh, Edelstein, Ward, Ford, Robertson 1979

Caltech: Drever et al 1981

France: Brillet, Man, Vinet 1982

Italy: Brillet- Giazzoto

# BASICS OF BLACK HOLES

1916 Schwarzschild (non rotating) Black Hole (BH)

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

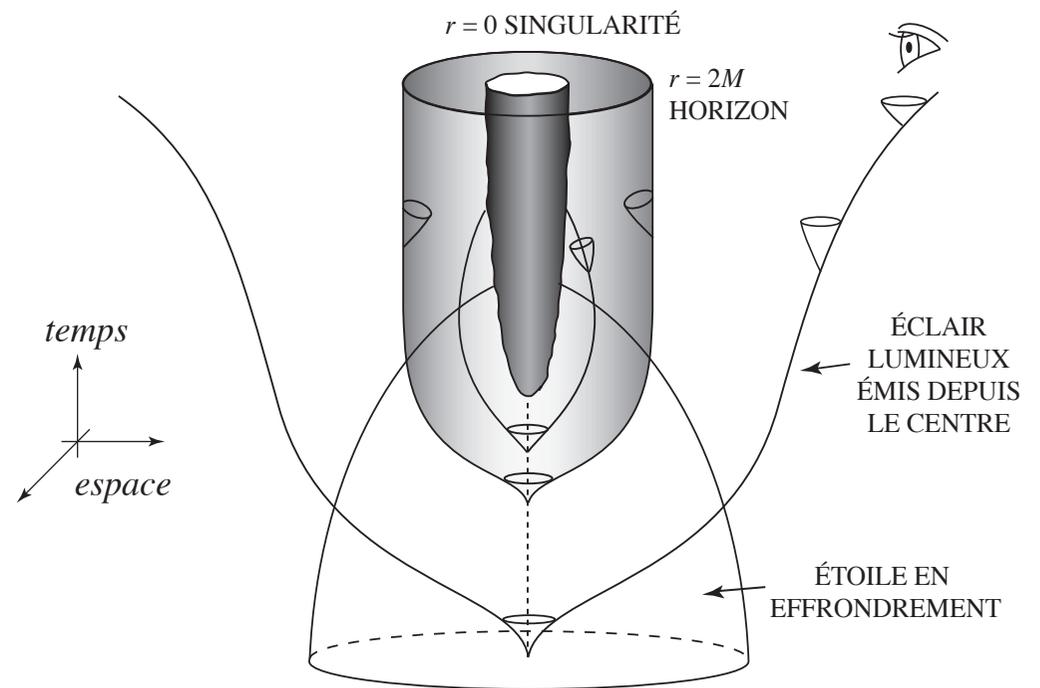
Schwarzschild radius (singularity?):  $r_S = 2GM/c^2$

1939 Oppenheimer-Snyder « continued collapse »

1963 Kerr Rotating BH: M, S

1965 Penrose

Horizon: cylindrical-like regular null hyper-surface whose sectional area is nearly constant, and actually slowly increasing (Christodoulou '70, Christodoulou-Ruffini '71, Hawking '71)

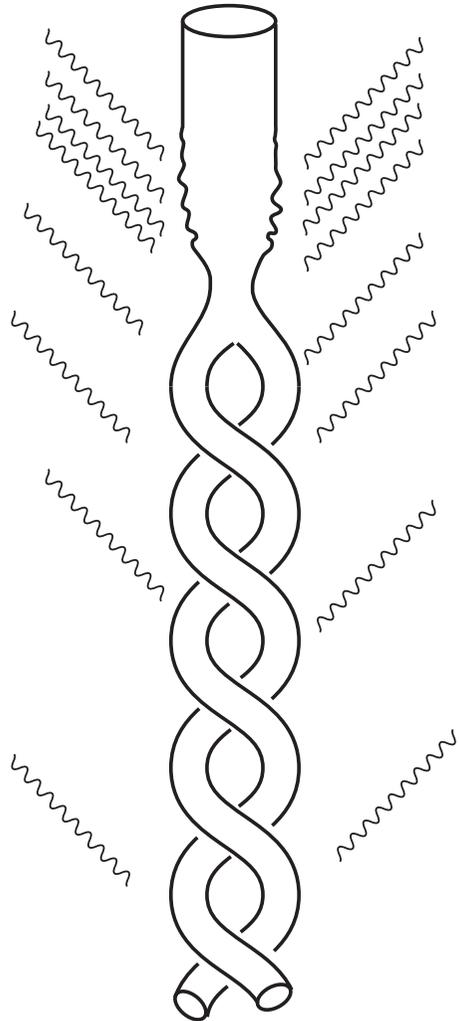


# Einstein Equations and the BBH problem

$$ds^2 = g_{\mu\nu}(x^\lambda) dx^\mu dx^\nu$$

$$R_{\mu\nu} = 0$$

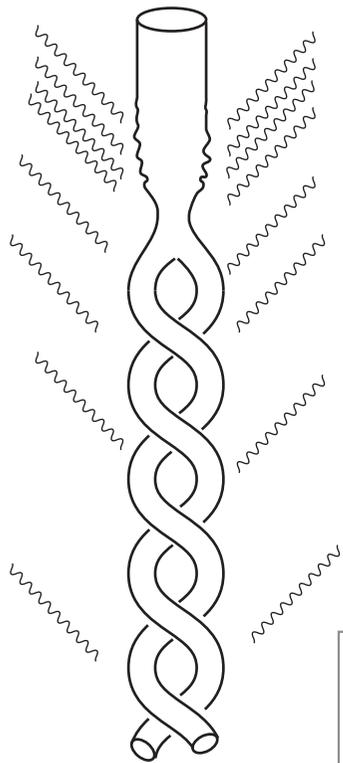
Harmonic coordinates:  $0 = H^\mu := \square_g x^\mu \equiv \frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} g^{\mu\nu})$



$$-g^{\mu\nu} g_{\alpha\beta, \mu\nu} + g^{\mu\nu} g^{\rho\sigma} (g_{\alpha\mu, \rho} g_{\beta\nu, \sigma} - g_{\alpha\mu, \rho} g_{\beta\sigma, \nu} + g_{\alpha\mu, \rho} g_{\nu\sigma, \beta} + g_{\beta\mu, \rho} g_{\nu\sigma, \alpha} - \frac{1}{2} g_{\mu\rho, \alpha} g_{\nu\sigma, \beta}) = 0$$

# (Circular) motion and radiation at lowest order

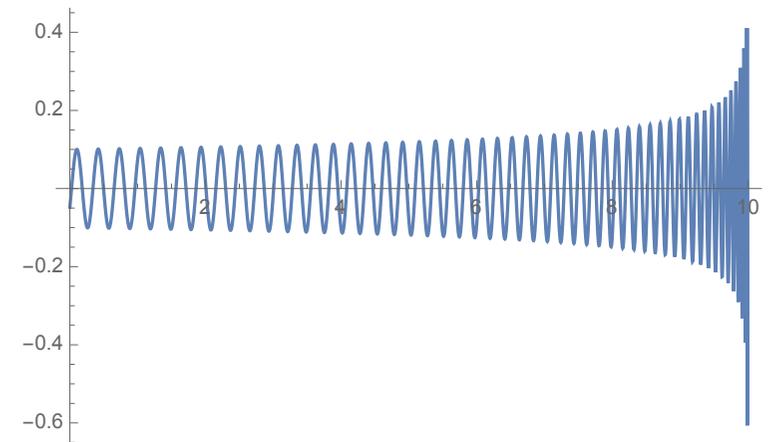
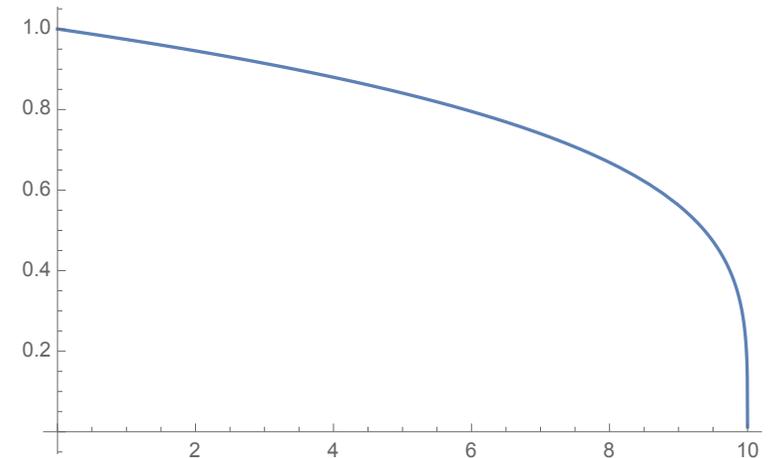
**Dyson 1963**, using Einstein 1918 + Landau-Lifshitz 1951  
first vision of an intense GW flash from coalescing binary NS



$$E = -\frac{G m_1 m_2}{2r}$$

$$\frac{d}{dt} E = -F$$

$$F = \frac{32 G^4}{5 c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{r^5}$$

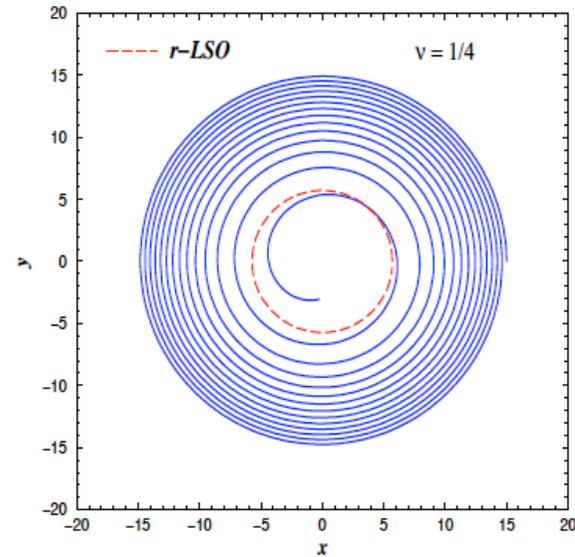
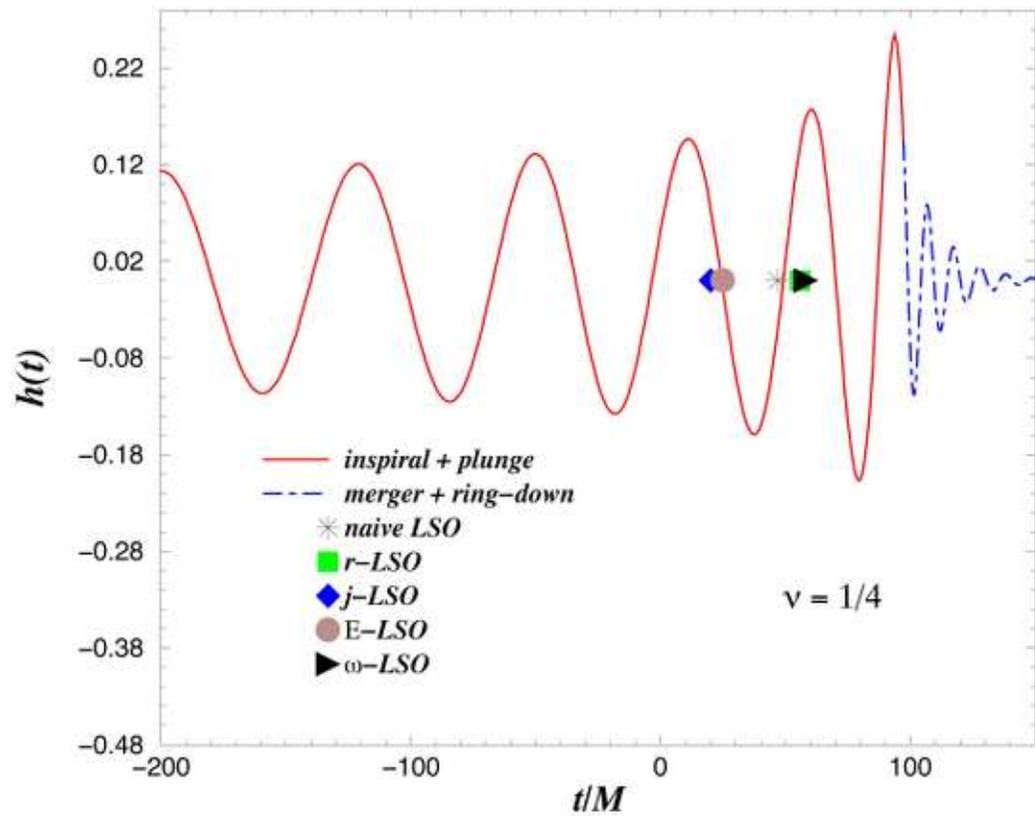


**Challenge:** describe the intense flash of GWs emitted by the last orbits and the merger of a binary BH, when  $v \sim c$  and  $r \sim GM/c^2$

# EFFECTIVE ONE BODY (EOB) FORMALISM

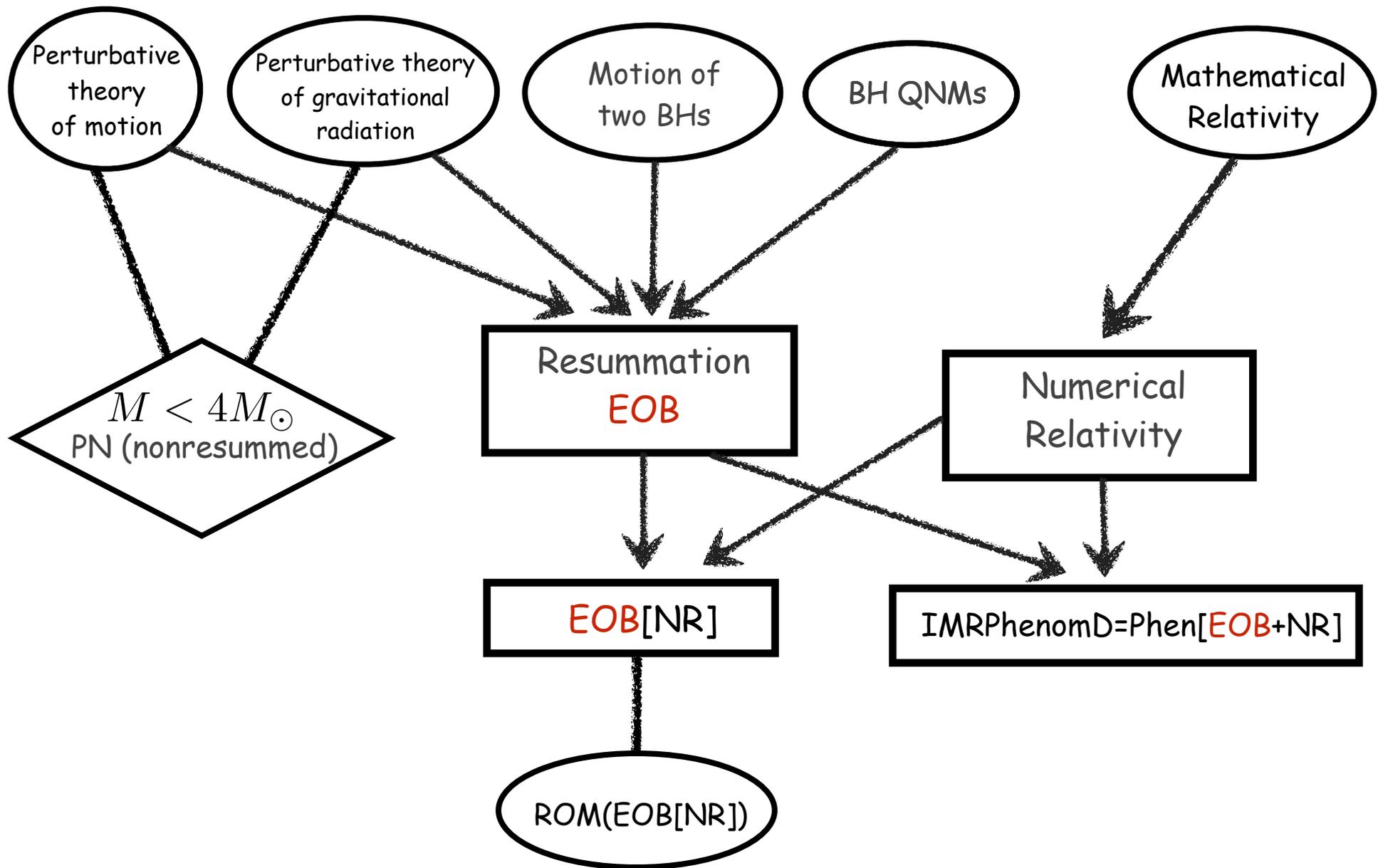
Buonanno-Damour 99,00; Damour-Jaranowski-Schaefer 00; Damour 01

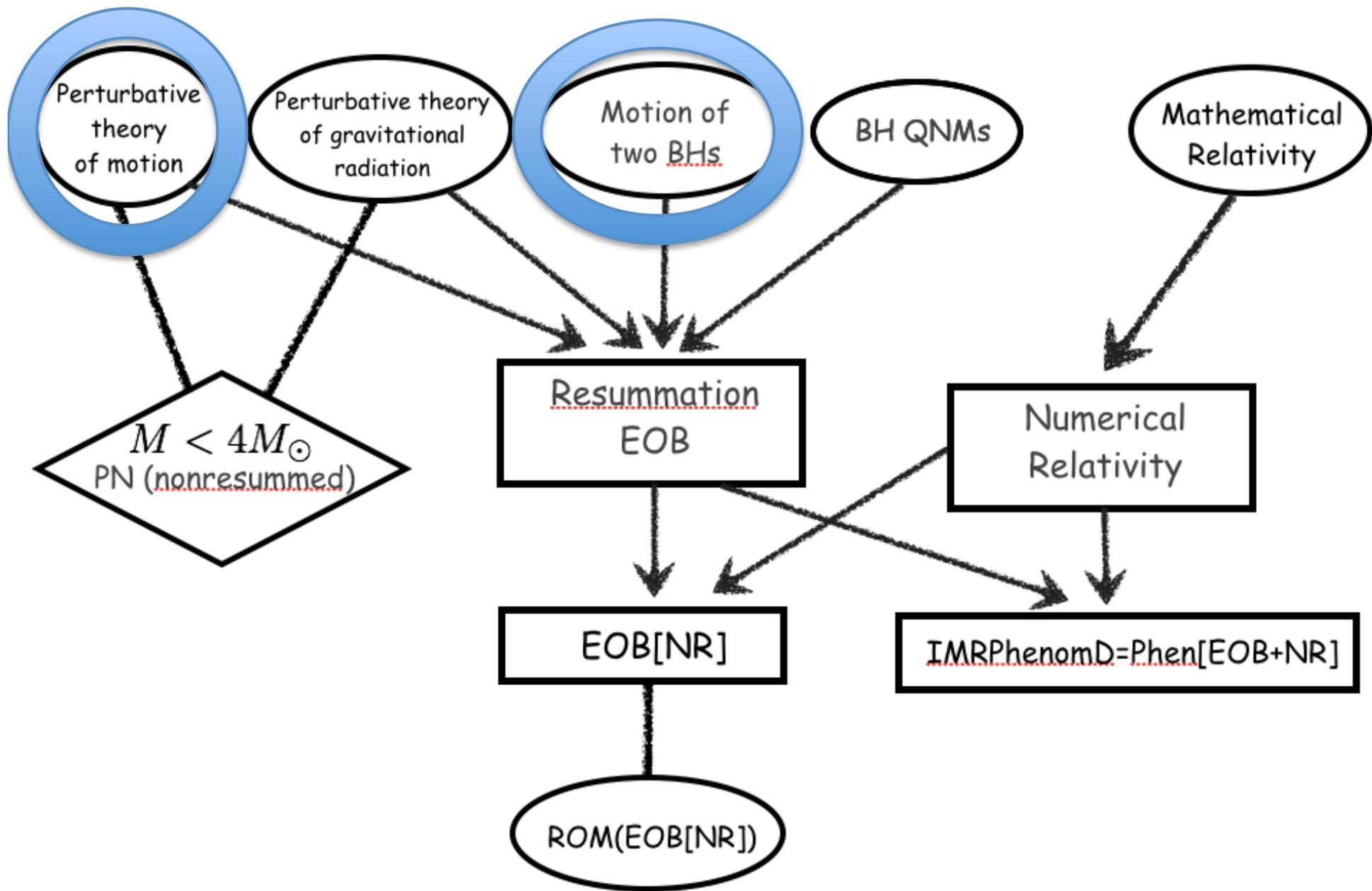
Buonanno-Damour 2000



- Blurred transition from inspiral to plunge
- Final black-hole mass
- Final black hole spin
- Complete waveform

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{\mu}{M}$$





## Long History of the GR Problem of Motion

Einstein 1912 : geodesic principle

$$- \int m \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$$

Einstein 1913-1916 post-Minkowskian

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) , h_{\mu\nu} \ll 1$$

Einstein, Droste : post-Newtonian

$$h_{00} \sim h_{ij} \sim \frac{v^2}{c^2} , h_{0i} \sim \frac{v^3}{c^3} , \partial_0 h \sim \frac{v}{c} \partial_i h$$

Weakly self-gravitating bodies:

$$\nabla_\nu T^{\mu\nu} = 0 \quad ; \quad T^{\mu\nu} = \rho' u^\mu u^\nu + p g^{\mu\nu} \Rightarrow \nabla_u u^\mu = O(\nabla p)$$



Einstein-Grossmann '13,

1916 post-Newtonian: Droste, Lorentz, Einstein (visiting Leiden), De Sitter ;

Lorentz-Droste '17, Chazy '28, Levi-Civita '37 .....

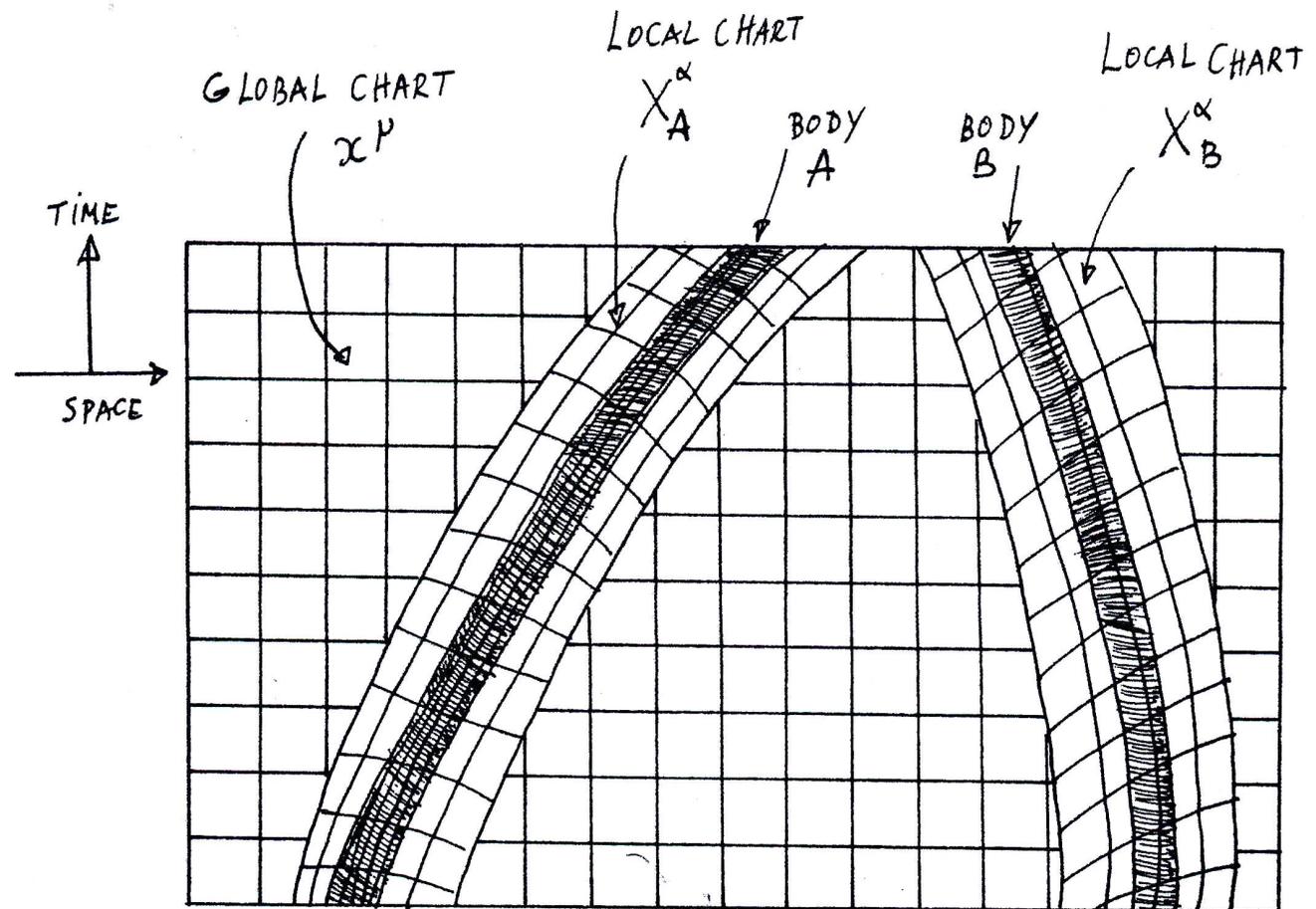
Eddington' 21, ..., Lichnerowicz '39, Fock '39, Papapetrou '51,

... Dixon '64, Bailey-Israel '75, Ehlers-Rudolph '77....

## Strongly Self-gravitating Bodies (NS, BH)

- **Multi-chart** approach and **matched asymptotic expansions**: necessary for strongly self-gravitating bodies (NS, BH)  
Manasse (Wheeler) '63, Demianski-Grishchuk '74, D'Eath '75, Kates '80, Damour '82

Useful even for weakly self-gravitating bodies, i.e. "relativistic celestial mechanics",  
Brumberg-Kopeikin '89,  
Damour-Soffel-Xu '91-94



# Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

**Skeletonization** :  $T_{\mu\nu} \rightarrow$  point-masses

delta-functions in GR : Infeld '54, Infeld-Plebanski '60

justified by Matched Asymptotic Expansions ( « Effacing Principle » Damour '83)

QFT's **analytic** (Riesz '49) or **dimensional regularization** (Bollini-Giambiagi '72, t'Hooft-Veltman '72) imported in GR (Damour '80, Damour-Jaranowski-Schäfer '01, ...)

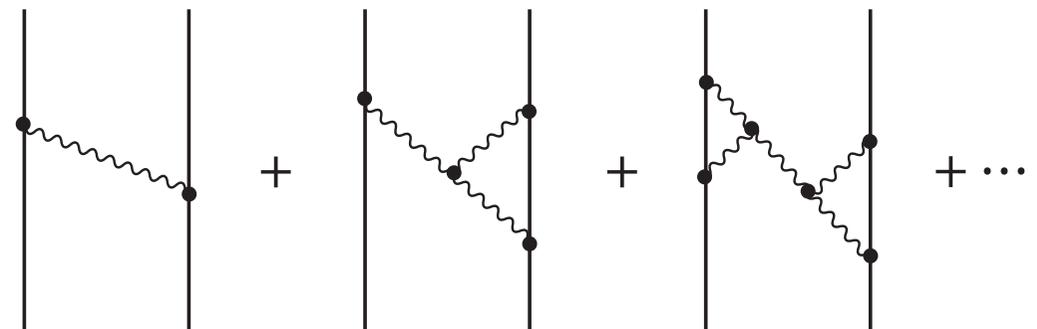
**Feynman-like diagrams** and  
« **Effective Field Theory** » techniques

Bertotti-Plebanski '60,

Damour-Esposito-Farèse '96,

Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,

Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15



## Reduced (Fokker 1929) Action for Conservative Dynamics

Needs gauge-fixed\* action and time-symmetric Green function  $G$ .

\*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.

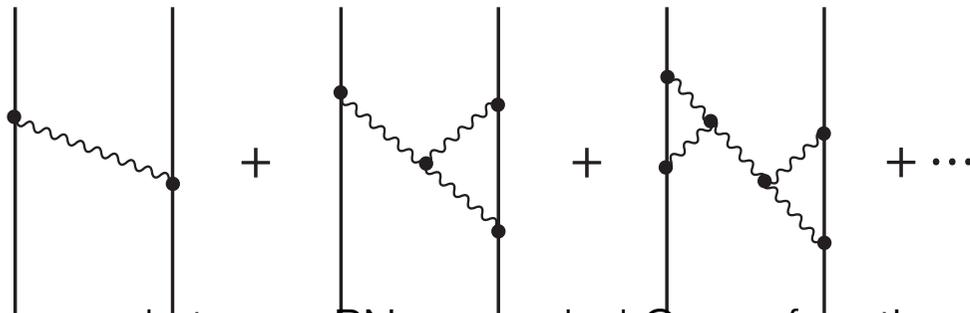
Perturbatively solving (in dimension  $D=4 - \epsilon$ ) Einstein's equations to get the equations of motion and the action for the conservative dynamics

$$g = \eta + h$$

$$S(h, T) = \int \left( \frac{1}{2} h \square h + \partial \partial h h h + \dots + (h + h h + \dots) T \right)$$

$$\square h = -T + \dots \rightarrow h = G T + \dots$$

$$S_{\text{red}}(T) = \frac{1}{2} T G T + V_3(G T, G T, G T) + \dots$$



Beyond 1-loop order needs to use PN-expanded Green function for explicit computations. Introduces **IR** divergences on top of the **UV** divergences linked to the point-particle description. UV is (essentially) finite in dim.reg. and IR is linked to 4PN non-locality (Blanchet-Damour '88).

$$\square^{-1} = \left( \Delta - \frac{1}{c^2} \partial_t^2 \right)^{-1} = \Delta^{-1} + \frac{1}{c^2} \partial_t^2 \Delta^{-2} + \dots$$

# Post-Newtonian Equations of Motion [2-body, wo spins]

---

- 1PN (including  $v^2/c^2$ ) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc.  $v^4/c^4$ ) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81  
Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc.  $v^5/c^5$ ) Damour-Deruelle '81, Damour '82, Schäfer '85,  
Kopeikin '85
- 3 PN (inc.  $v^6/c^6$ ) Jaranowski-Schäfer '98, Blanchet-Faye '00,  
Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,  
Blanchet-Damour-Esposito-Farèse '04, Foffa-Sturani '11
- 3.5 PN (inc.  $v^7/c^7$ ) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,  
Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- 4PN (inc.  $v^8/c^8$ ) Jaranowski-Schäfer '13, Foffa-Sturani '13,  
Bini-Damour '13, Damour-Jaranowski-Schäfer '14

New feature : non-locality in time

## 2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

---

$$H_N(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2} \frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left( -12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \right) \\ + \frac{1}{4} \frac{Gm_1m_2}{r_{12}} \frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2PN}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left( 5 \frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2} \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\ \left. - 6 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right) \\ + \frac{1}{4} \frac{G^2 m_1 m_2}{r_{12}^2} \left( m_2 \left( 10 \frac{\mathbf{p}_1^2}{m_1^2} + 19 \frac{\mathbf{p}_2^2}{m_2^2} \right) - \frac{1}{2} (m_1 + m_2) \frac{27(\mathbf{p}_1 \cdot \mathbf{p}_2) + 6(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) \\ - \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

## 2-body Taylor-expanded 3PN Hamiltonian [JS 98, DJS 01]

$$\begin{aligned}
 c^6 H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{5}{128} \frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32} \frac{Gm_1 m_2}{r_{12}} \left( -14 \frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4 \frac{((\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4\mathbf{p}_1^2 \mathbf{p}_2^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 6 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^2} \right. \\
 & - 10 \frac{(\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 24 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^2} \\
 & + 2 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2 \mathbf{p}_2^2 - 10(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2^3} \\
 & + \frac{(\mathbf{p}_1^2 \mathbf{p}_2^2 - 2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + 15 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \\
 & - 18 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} + 5 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} \left. + \frac{G^2 m_1 m_2}{r_{12}^2} \left( \frac{1}{16} (m_1 - 27m_2) \frac{(\mathbf{p}_1^2)^2}{m_1^4} \right. \right. \\
 & - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48} m_2 \frac{25(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16} \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^3} + \frac{5}{12} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^3} \\
 & - \frac{1}{8} m_1 \frac{(15\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)) (\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \\
 & + \frac{125}{12} m_2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \\
 & - \frac{1}{48} (220m_1 + 193m_2) \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \left. + \frac{G^3 m_1 m_2}{r_{12}^3} \left( -\frac{1}{48} \left( 425m_1^2 + \left( 473 - \frac{3}{4} \pi^2 \right) m_1 m_2 + 150m_2^2 \right) \frac{\mathbf{p}_1^2}{m_1^2} \right. \right. \\
 & + \frac{1}{16} \left( 77(m_1^2 + m_2^2) + \left( 143 - \frac{1}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16} \left( 20m_1^2 - \left( 43 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \\
 & + \frac{1}{16} \left( 21(m_1^2 + m_2^2) + \left( 119 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \\
 & \left. + \frac{1}{8} \frac{G^4 m_1 m_2^3}{r_{12}^4} \left( \left( \frac{227}{3} - \frac{21}{4} \pi^2 \right) m_1 + m_2 \right) + (1 \leftrightarrow 2).
 \end{aligned}$$

# 2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$\begin{aligned}
 c^8 H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256m_1^5} + \frac{Gm_1m_2}{r_{12}} H_{48}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a, \mathbf{p}_a) \\
 &+ \frac{G^3m_1m_2}{r_{12}^3} (m_1^2 H_{441}(\mathbf{x}_a, \mathbf{p}_a) + m_1m_2 H_{442}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^4m_1m_2}{r_{12}^4} (m_1^3 H_{421}(\mathbf{x}_a, \mathbf{p}_a) + m_1^2m_2 H_{422}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^5m_1m_2}{r_{12}^5} H_{40}(\mathbf{x}_a, \mathbf{p}_a) + (1 \leftrightarrow 2), \tag{A3}
 \end{aligned}$$

$$\begin{aligned}
 H_{48}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{16m_1^6m_2^2} \\
 &- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{256m_1^5m_2^2} \\
 &+ \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^2} + \frac{33(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^2} - \frac{85(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2^2} \\
 &- \frac{45(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^5m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^5m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^5m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^5m_2^2} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^5m_2^2} + \frac{55(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^2} \\
 &- \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^2} - \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^2} - \frac{23(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{256m_1^5m_2^2} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^2} \\
 &- \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4m_1^4m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^4m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^2} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^2} \\
 &- \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^2} \\
 &- \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^2} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^2}, \tag{A4a}
 \end{aligned}$$

$$\begin{aligned}
 H_{46}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{369(\mathbf{n}_{12} \cdot \mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{128m_1^5m_2} \\
 &+ \frac{67(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} - \frac{167(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^5m_2} \\
 &+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2} \\
 &- \frac{3571(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^4m_2^2} \\
 &- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} - \frac{13(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{8m_1^3m_2^3} \\
 &+ \frac{191(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3m_2^3} \\
 &+ \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} - \frac{47(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} \\
 &+ \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} \\
 &- \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2m_1^2m_2^4} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4} \\
 &- \frac{133(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} - \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_1^2m_2^4}, \tag{A4b}
 \end{aligned}$$

$$\begin{aligned}
 H_{441}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{5027(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{640m_1^3m_2} \\
 &+ \frac{28561(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{28800m_1^3m_2} \\
 &- \frac{16481(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2400m_1^2m_2^2} \\
 &+ \frac{791(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4}, \tag{A4c}
 \end{aligned}$$

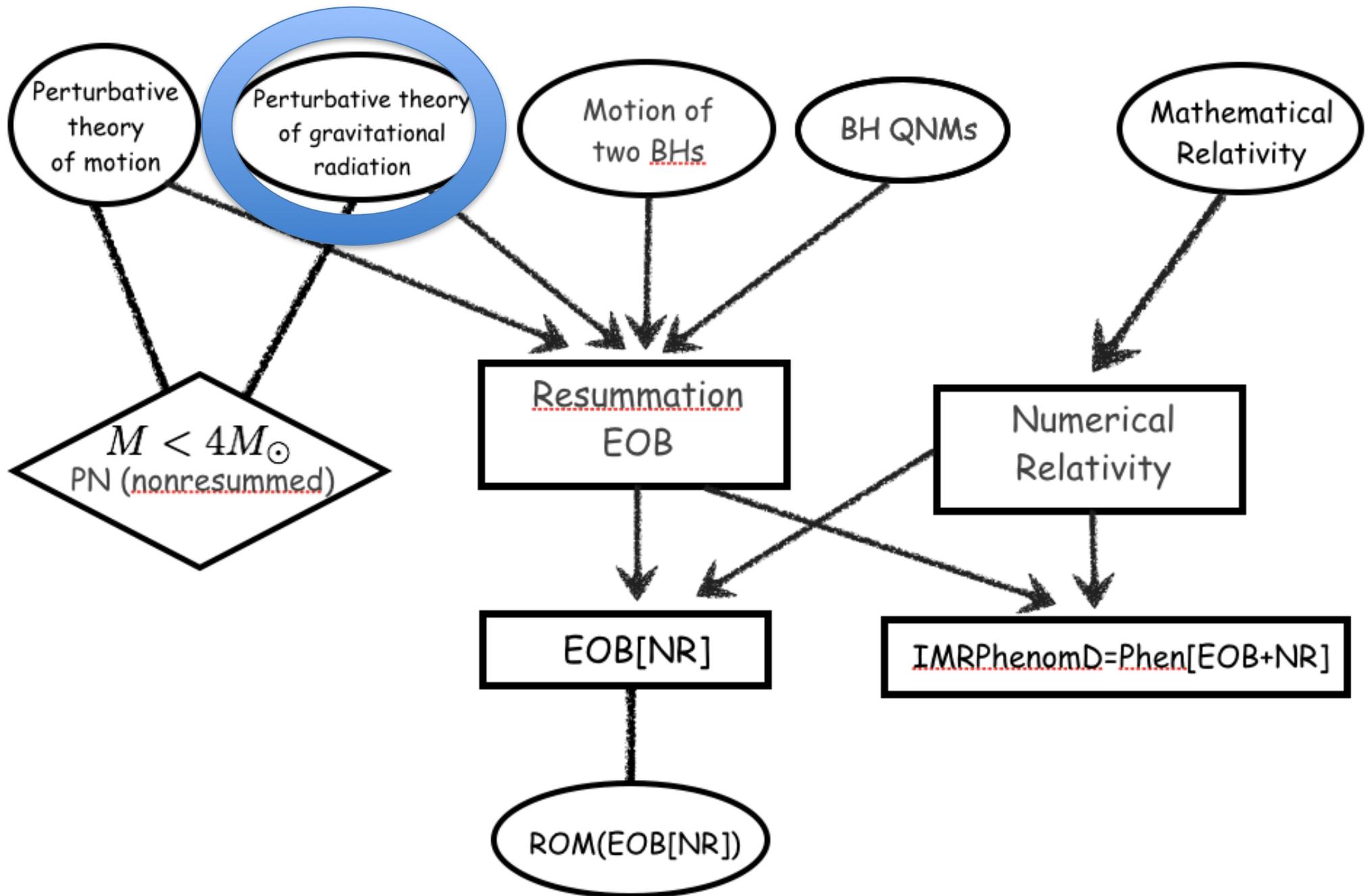
$$\begin{aligned}
 H_{442}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right) \frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^4} \\
 &+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right) \frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} \\
 &+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \left(\frac{2488991}{6400} - \frac{6153\pi^2}{2048}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2^2} \\
 &- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right) \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2m_2} \\
 &+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
 &+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3m_2}, \tag{A4d}
 \end{aligned}$$

$$H_{421}(\mathbf{x}_a, \mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1 \cdot \mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2m_1m_2}, \tag{A4e}$$

$$\begin{aligned}
 H_{422}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right) \frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right) \frac{\mathbf{p}_2^2}{m_2^2} \\
 &+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \\
 &+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}, \tag{A4f}
 \end{aligned}$$

$$H_{40}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right) m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right) m_1^2m_2^2. \tag{A4g}$$

$$\begin{aligned}
 H_{4\text{PN}}^{\text{nonloc}}(t) &= -\frac{1}{5} \frac{G^2M}{c^8} I_{ij}^{(3)}(t) \\
 &\times \text{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v),
 \end{aligned}$$



# Perturbative Theory of Gravitational Radiation

1916, 1918 Einstein :  $h_+$ ,  $h_x$  and **quadrupole formula**

Relativistic, **multipolar extensions** of LO quadrupole radiation :

Mathews '62,

Bonnor-Rotenberg '66,

Campbell-Morgan '71,

Campbell et al '75,

Thorne '80, .., Will et al 00

**MPM Formalism:**

Blanchet-Damour '86,

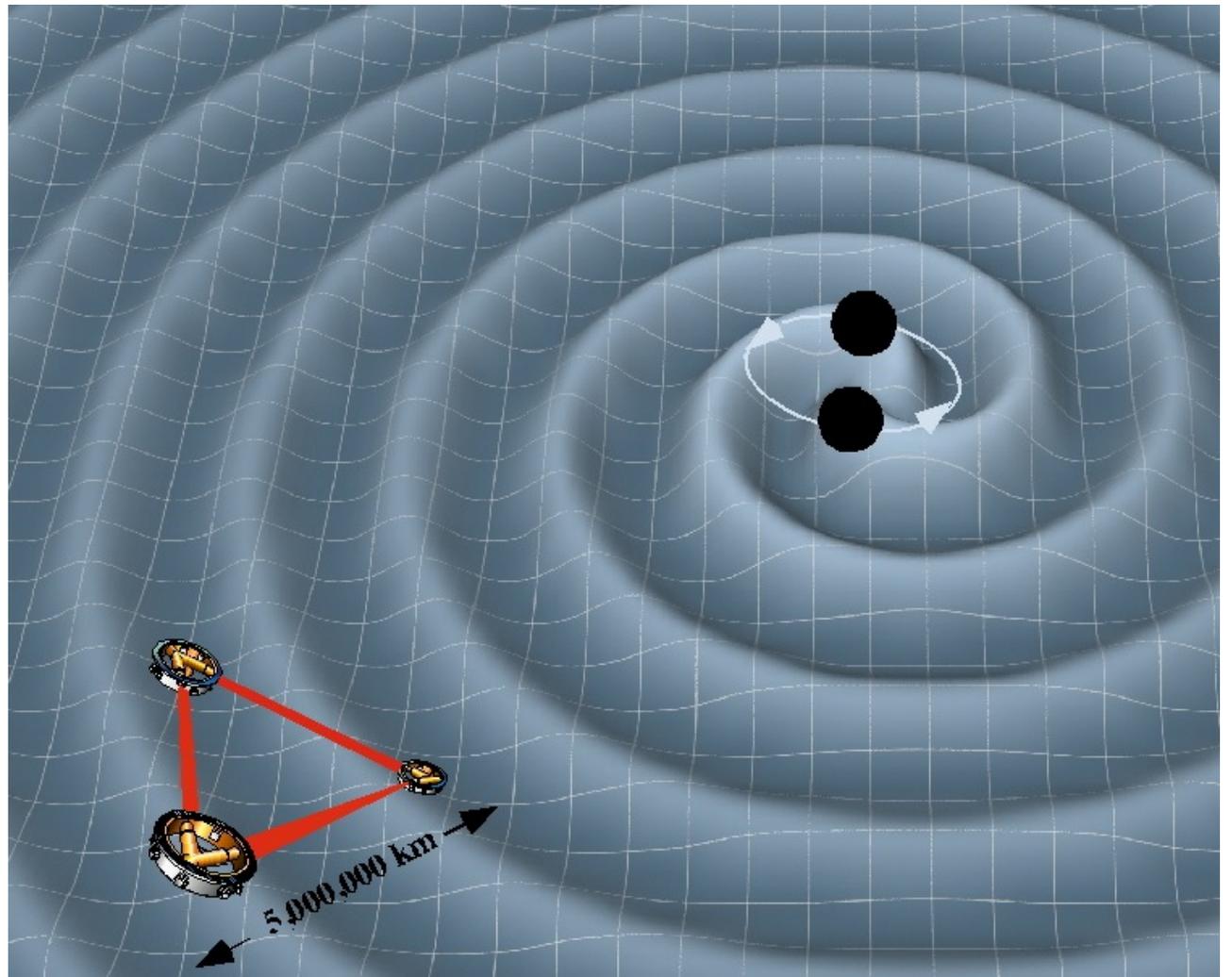
Damour-Iyer '91,

Blanchet '95

Combines multipole exp.

Post Minkowskian exp.

and analytic continuation



## MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER)

Decomposition of space-time in various overlapping regions:

1. near-zone:  $r \ll \lambda$  : PN theory
2. exterior zone:  $r \gg r_{\text{source}}$ : MPM expansion
3. far wave-zone: Bondi-type expansion

followed by **matching** between the zones

in exterior zone, **iterative solution** of Einstein's vacuum field equations by means of a **double expansion** in non-linearity and in multipoles, with crucial use of **analytic continuation** (complex B) for dealing with formal UV divergences at  $r=0$

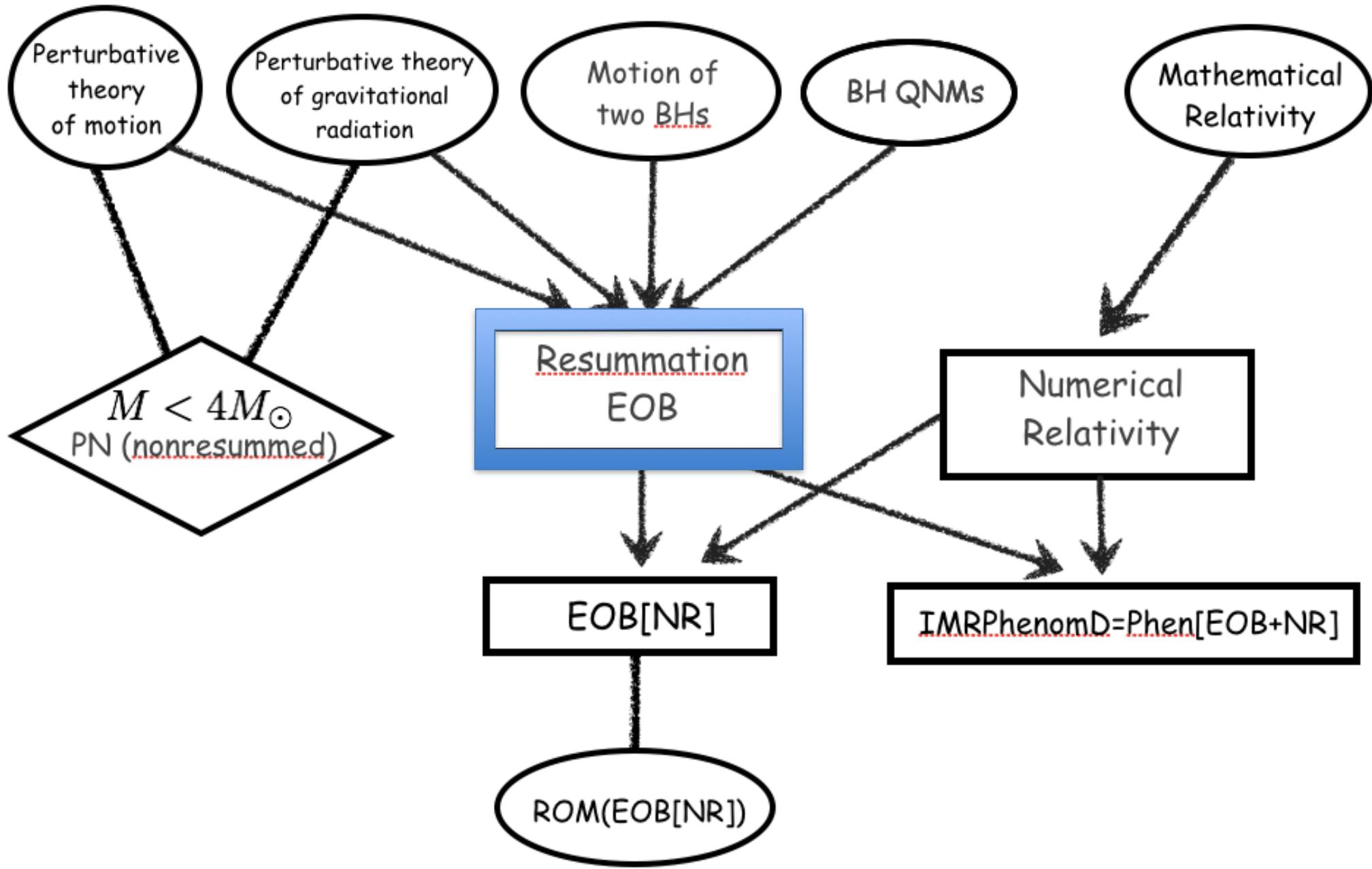
$$\begin{aligned}
 g &= \eta + Gh_1 + G^2h_2 + G^3h_3 + \dots, \\
 \square h_1 &= 0, \\
 \square h_2 &= \partial\partial h_1 h_1, \\
 \square h_3 &= \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2, \\
 h_1 &= \sum_{\ell} \partial_{i_1 i_2 \dots i_{\ell}} \left( \frac{M_{i_1 i_2 \dots i_{\ell}}(t - r/c)}{r} \right) + \partial\partial \dots \partial \left( \frac{\epsilon_{j_1 j_2 k} S_{k j_3 \dots j_{\ell}}(t - r/c)}{r} \right), \\
 h_2 &= FP_B \square_{\text{ret}}^{-1} \left( \left( \frac{r}{r_0} \right)^B \partial\partial h_1 h_1 \right) + \dots, \\
 h_3 &= FP_B \square_{\text{ret}}^{-1} \dots
 \end{aligned}$$

# Perturbative computation of GW flux from binary system

---

- lowest order : Einstein 1918 Peters-Mathews 63
- $1 + (v^2/c^2)$  : Wagoner-Will 76
- ... +  $(v^3/c^3)$  : Blanchet-Damour 92, Wiseman 93
- ... +  $(v^4/c^4)$  : Blanchet-Damour-Iyer Will-Wiseman 95
- ... +  $(v^5/c^5)$  : Blanchet 96
- ... +  $(v^6/c^6)$  : Blanchet-Damour-Esposito-Farèse-Iyer 2004
- ... +  $(v^7/c^7)$  : Blanchet

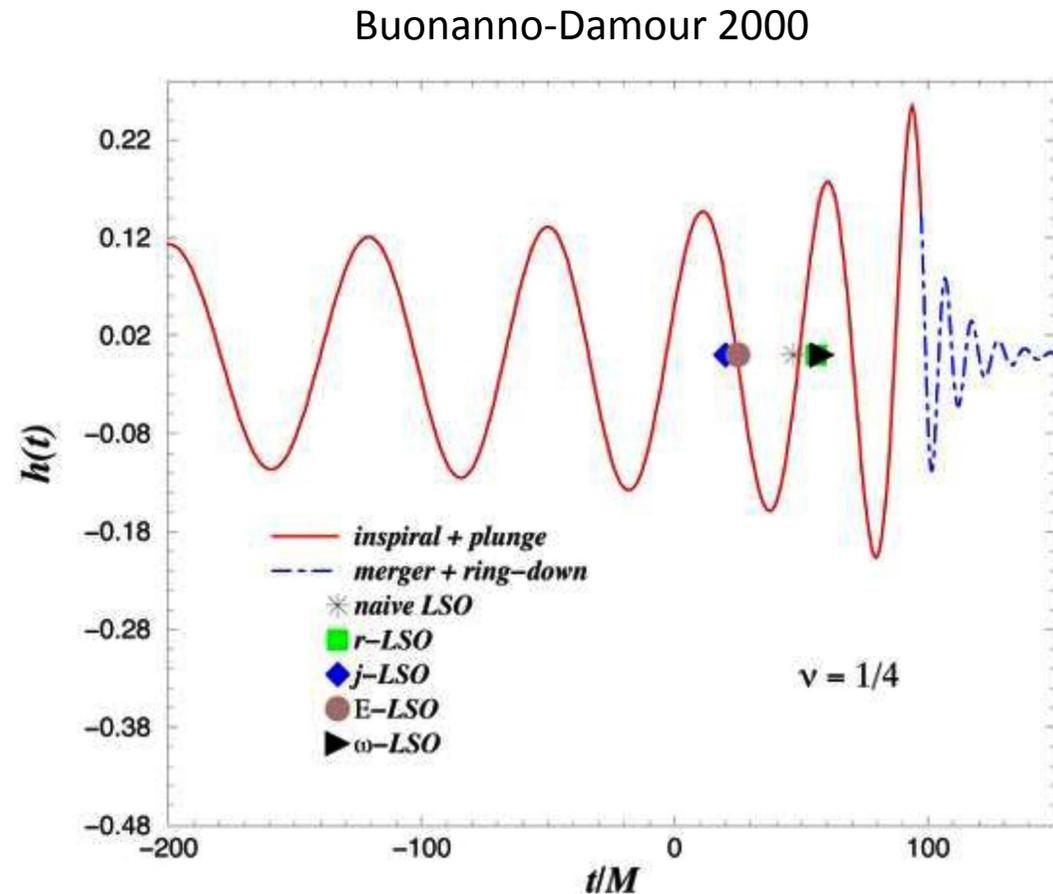
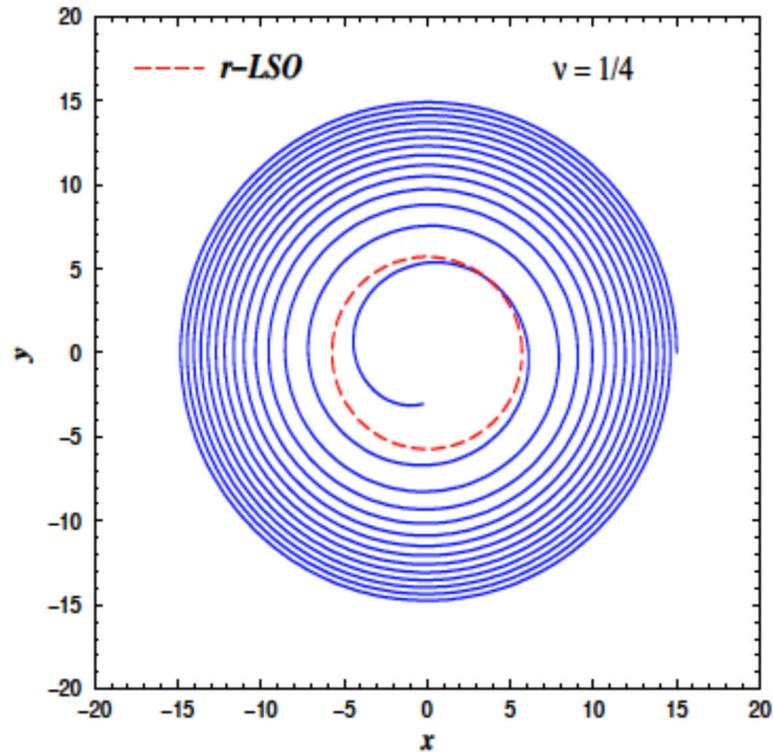
$$\begin{aligned} \mathcal{F} = \frac{32c^5}{5G} \nu^2 x^5 & \left\{ 1 + \left( -\frac{1247}{336} - \frac{35}{12} \nu \right) x + 4\pi x^{3/2} \right. \\ & + \left( -\frac{44711}{9072} + \frac{9271}{504} \nu + \frac{65}{18} \nu^2 \right) x^2 + \left( -\frac{8191}{672} - \frac{583}{24} \nu \right) \pi x^{5/2} \\ & + \left[ \frac{6643739519}{69854400} + \frac{16}{3} \pi^2 - \frac{1712}{105} \gamma_E - \frac{856}{105} \ln(16x) \right. \\ & \quad \left. + \left( -\frac{134543}{7776} + \frac{41}{48} \pi^2 \right) \nu - \frac{94403}{3024} \nu^2 - \frac{775}{324} \nu^3 \right] x^3 \\ & \left. + \left( -\frac{16285}{504} + \frac{214745}{1728} \nu + \frac{193385}{3024} \nu^2 \right) \pi x^{7/2} + \mathcal{O} \left( \frac{1}{c^8} \right) \right\}. \end{aligned}$$



# Effective One Body (EOB) Method

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001

Resummation of perturbative results  $\longrightarrow$  description of the coalescence  
+ addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 1972)



Predictions :

continued transition, non adiabaticity, complete waveform, final spin (OK within 10%)

# EOB THEORY + EOB[NR] + EOB[SF] DEVELOPMENTS

Buonanno,Damour 99

(2 PN Hamiltonian)

Buonanno,Damour 00

(Rad.Reac. full waveform)

Damour, Jaranowski,Schäfer 00

(3 PN Hamiltonian)

Damour 01,  
Buonanno, Chen, Damour 05,  
Damour-Jaranowski,Schäfer 08,  
Barausse, Buonanno, 10,  
Nagar 11,  
Balmelli-Jetzer 12,  
Taracchini et al 12,14,  
Damour,Nagar 14

(spin)

Damour, Nagar 07,  
Damour, Iyer, Nagar 08,  
Pan et al. 11

(factorized waveform)

Damour, Nagar 10  
Bini-Damour-Faye 12

(tidal effects)

Bini, Damour 13, Damour, Jaranowski, Schäfer 15

(4 PN Hamiltonian)

## EOB vs NR and EOB[NR]

Buonanno, Cook, Pretorius 07,  
Buonanno, Pan, Taracchini 08-  
Damour-Nagar 08-

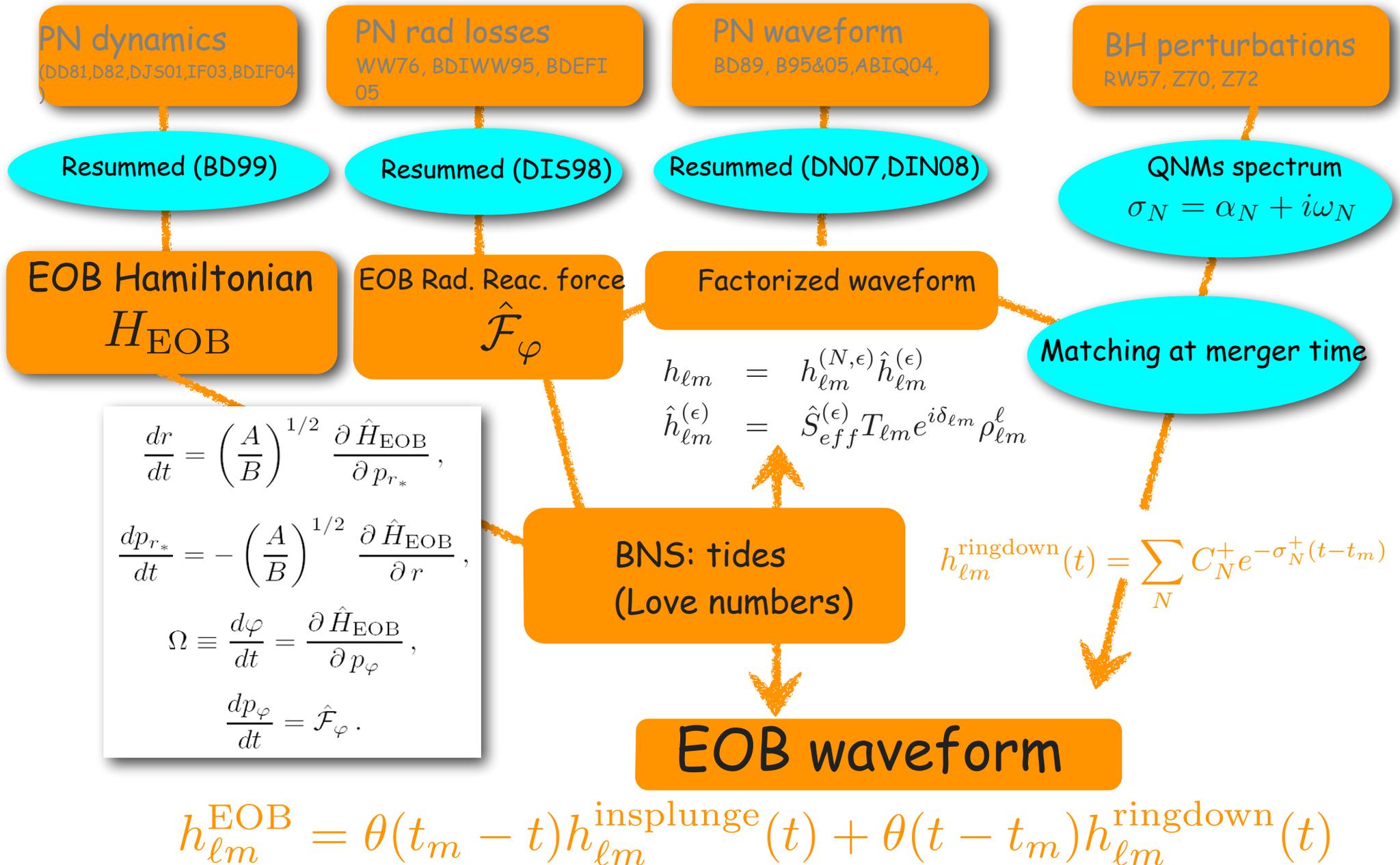
## EOB vs SF and EOB[SF]

Damour 09  
Barack-Sago-Damour 10  
Barausse-Buonanno-LeTiec 12  
Akçay-Barack-Damour-Sago 12  
Bini-Damour 13-16  
LeTiec 15  
Bini-Damour-Geralico 16  
Hopper-Kavanagh-Ottewill 16  
Akçay-vandeMeent 16

Reduced Order Model version (Pürrer 2014, 2016) of  
EOB[NR] (Taracchini et al 2014)

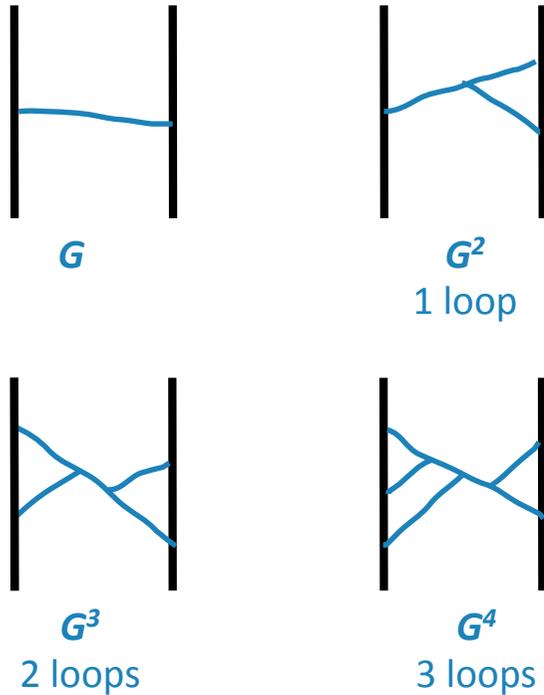
Phenomenological model (Ajith et al 2007, Hannam et  
al 2014, Husa et al 2016, Kahn et al 2016)  
of FFT of hybrids EOB + NR

# STRUCTURE OF THE EOB FORMALISM

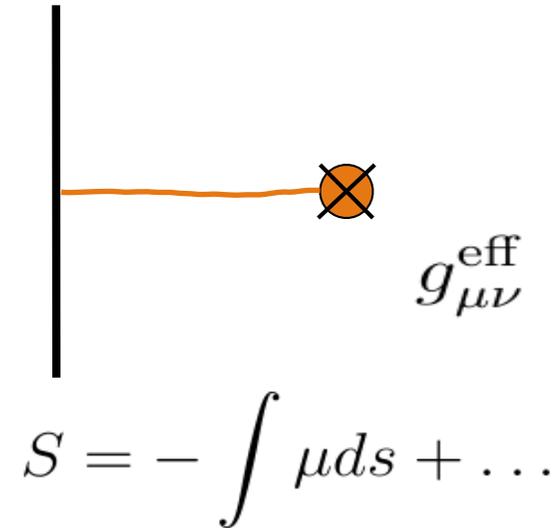


# Real dynamics versus Effective dynamics

## Real dynamics



## Effective dynamics



$$H = H_0 + \left( GH_1 + \frac{G^2}{c^2} H_2 + \frac{G^3}{c^4} H_3 + \frac{G^4}{c^6} H_4 \right) \left( 1 + \frac{1}{c^2} + \dots \right)$$

## Effective metric

$$ds_{\text{eff}}^2 = -A(r)dt^2 + B(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

# TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

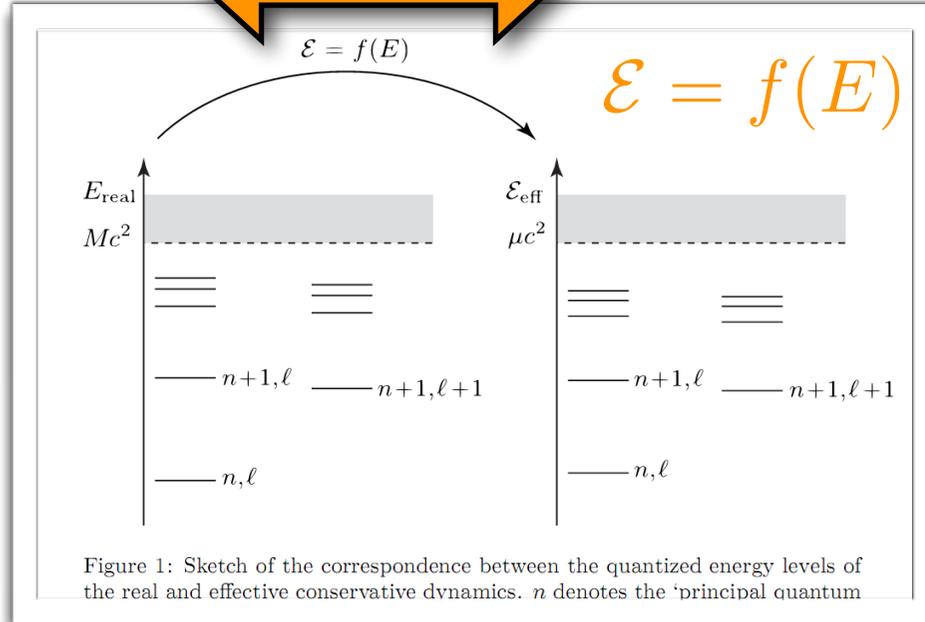
Real 2-body system  
(in the c.o.m. frame)  
 $(m_1, m_2)$



An effective particle  
in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\mu\nu}^{\text{eff}}$$



$$\mu^2 + g_{\text{eff}}^{\mu\nu} \frac{\partial S_{\text{eff}}}{\partial x^\mu} \frac{\partial S_{\text{eff}}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Sommerfeld's  
"Old Quantum Mechanics"  
(action-angle variables &  
Delaunay Hamiltonian)

$$J = \ell \hbar = \frac{1}{2\pi} \oint p_\varphi d\varphi$$

$$N = n \hbar = I_r + J$$

$$I_r = \frac{1}{2\pi} \oint p_r dr$$

$$H^{\text{classical}}(q, p) \longrightarrow H^{\text{classical}}(I_a) \longrightarrow E^{\text{quantum}}(I_a = n_a \hbar) = f^{-1}[\mathcal{E}_{\text{eff}}^{\text{quantum}}(I_a^{\text{eff}} = n_a \hbar)]$$

# Resummed EOB Hamiltonian EOB

$$ds_{\text{eff}}^2 = -A(r)dt^2 + B(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$H_{\text{EOB}} = M \sqrt{1 + 2\nu \left( \frac{1}{\mu} \sqrt{A(r) \left( \mu^2 + \frac{p_r^2}{B(r)} + \frac{p_\phi^2}{r^2} + 2\nu(4 - 3\nu) \left( \frac{GM}{r} \right)^2 \frac{p_r^4}{\mu^2} \right) - 1} \right)}$$

$$A^{3PN}(r; M, \nu) = \text{Pade}_3^1 \left[ 1 - 2 \frac{GM}{c^2 r} + 2\nu \left( \frac{GM}{c^2 r} \right)^2 + \left( \frac{94}{3} - \frac{41}{32} \pi^2 \right) \nu \left( \frac{GM}{c^2 r} \right)^3 \right]$$

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{\mu}{M}$$

# 2-body CoM Newton + 1PN + 2PN + 3PN Hamiltonian

---

$$\begin{aligned}\widehat{H}_N(\mathbf{r}, \mathbf{p}) &= \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \\ \widehat{H}_{1\text{PN}}(\mathbf{r}, \mathbf{p}) &= \frac{1}{8}(3\nu - 1)(\mathbf{p}^2)^2 - \frac{1}{2}\left\{(3 + \nu)\mathbf{p}^2 + \nu(\mathbf{n} \cdot \mathbf{p})^2\right\}\frac{1}{r} + \frac{1}{2r^2}, \\ \widehat{H}_{2\text{PN}}(\mathbf{r}, \mathbf{p}) &= \frac{1}{16}(1 - 5\nu + 5\nu^2)(\mathbf{p}^2)^3 + \frac{1}{8}\left\{(5 - 20\nu - 3\nu^2)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n} \cdot \mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n} \cdot \mathbf{p})^4\right\}\frac{1}{r} \\ &\quad + \frac{1}{2}\left\{(5 + 8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n} \cdot \mathbf{p})^2\right\}\frac{1}{r^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{r^3}, \\ \widehat{H}_{3\text{PN}}(\mathbf{r}, \mathbf{p}) &= \frac{1}{128}(-5 + 35\nu - 70\nu^2 + 35\nu^3)(\mathbf{p}^2)^4 \\ &\quad + \frac{1}{16}\left\{(-7 + 42\nu - 53\nu^2 - 5\nu^3)(\mathbf{p}^2)^3 + (2 - 3\nu)\nu^2(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 + 3(1 - \nu)\nu^2(\mathbf{n} \cdot \mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n} \cdot \mathbf{p})^6\right\}\frac{1}{r} \\ &\quad + \left\{\frac{1}{16}(-27 + 136\nu + 109\nu^2)(\mathbf{p}^2)^2 + \frac{1}{16}(17 + 30\nu)\nu(\mathbf{n} \cdot \mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5 + 43\nu)\nu(\mathbf{n} \cdot \mathbf{p})^4\right\}\frac{1}{r^2} \\ &\quad + \left\{\left(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48}\right)\nu - \frac{23\nu^2}{8}\right)\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\right)\nu(\mathbf{n} \cdot \mathbf{p})^2\right\}\frac{1}{r^3} \\ &\quad + \left\{\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\nu\right\}\frac{1}{r^4}.\end{aligned}$$

# Resummed EOB waveform EOB

(Damour-Iyer-Sathyaprakash 1998) Damour-Nagar 2007, Damour-Iyer -Nagar 2008

$$h_{\ell m} \equiv h_{\ell m}^{(N, \epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\text{NQC}}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\text{eff}}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{k})}{\Gamma(\ell + 1)} e^{\pi\hat{k}} e^{2i\hat{k} \ln(2kr_0)}$$

$$\begin{aligned} \rho_{22}(x; \nu) = & 1 + \left( \frac{55\nu}{84} - \frac{43}{42} \right) x + \left( \frac{19583\nu^2}{42336} - \frac{33025\nu}{21168} - \frac{20555}{10584} \right) x^2 \\ & + \left( \frac{10620745\nu^3}{39118464} - \frac{6292061\nu^2}{3259872} + \frac{41\pi^2\nu}{192} - \frac{48993925\nu}{9779616} - \frac{428}{105} \text{eulerlog}_2(x) + \frac{1556919113}{122245200} \right) x^3 \\ & + \left( \frac{9202}{2205} \text{eulerlog}_2(x) - \frac{387216563023}{160190110080} \right) x^4 + \left( \frac{439877}{55566} \text{eulerlog}_2(x) - \frac{16094530514677}{533967033600} \right) x^5 + \mathcal{O}(x^6), \end{aligned}$$

$$\mathcal{F}_{\varphi} \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=1}^{\ell} (m\Omega)^2 |Rh_{\ell m}^{(\epsilon)}|^2$$

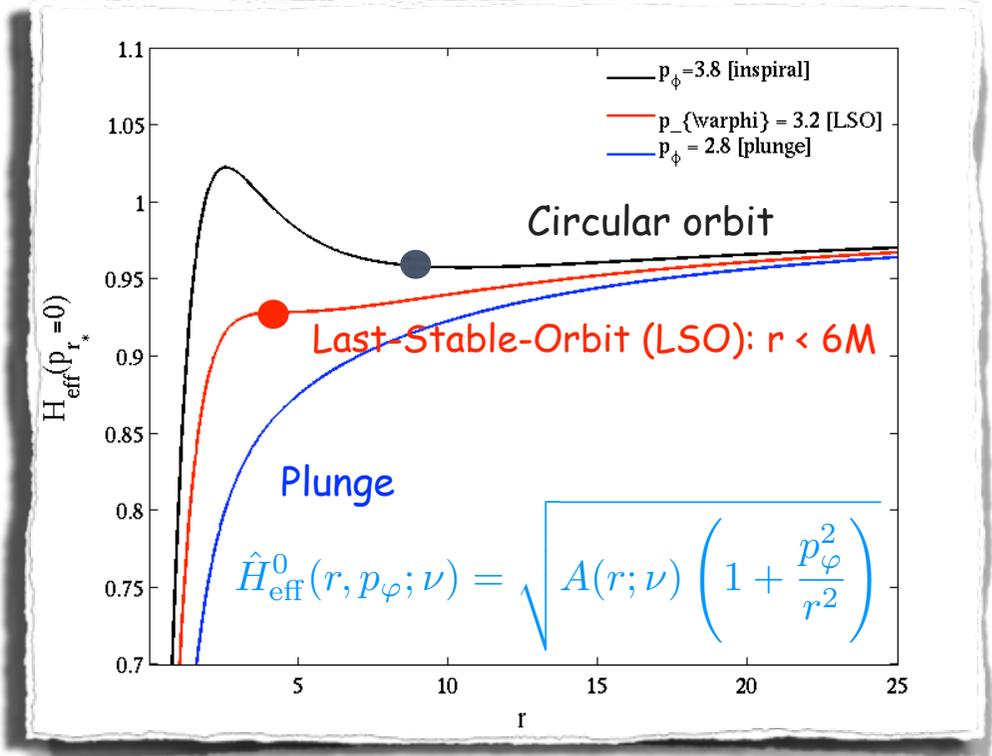
# HAMILTON'S EQUATIONS & RADIATION REACTION

$$\dot{r} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_{r_*}}$$

$$\dot{\varphi} = \frac{\partial \hat{H}_{\text{EOB}}}{\partial p_{\varphi}} \equiv \Omega$$

$$\dot{p}_{r_*} = - \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\text{EOB}}}{\partial r} + \hat{\mathcal{F}}_{r_*}$$

$$\dot{p}_{\varphi} = \hat{\mathcal{F}}_{\varphi}$$

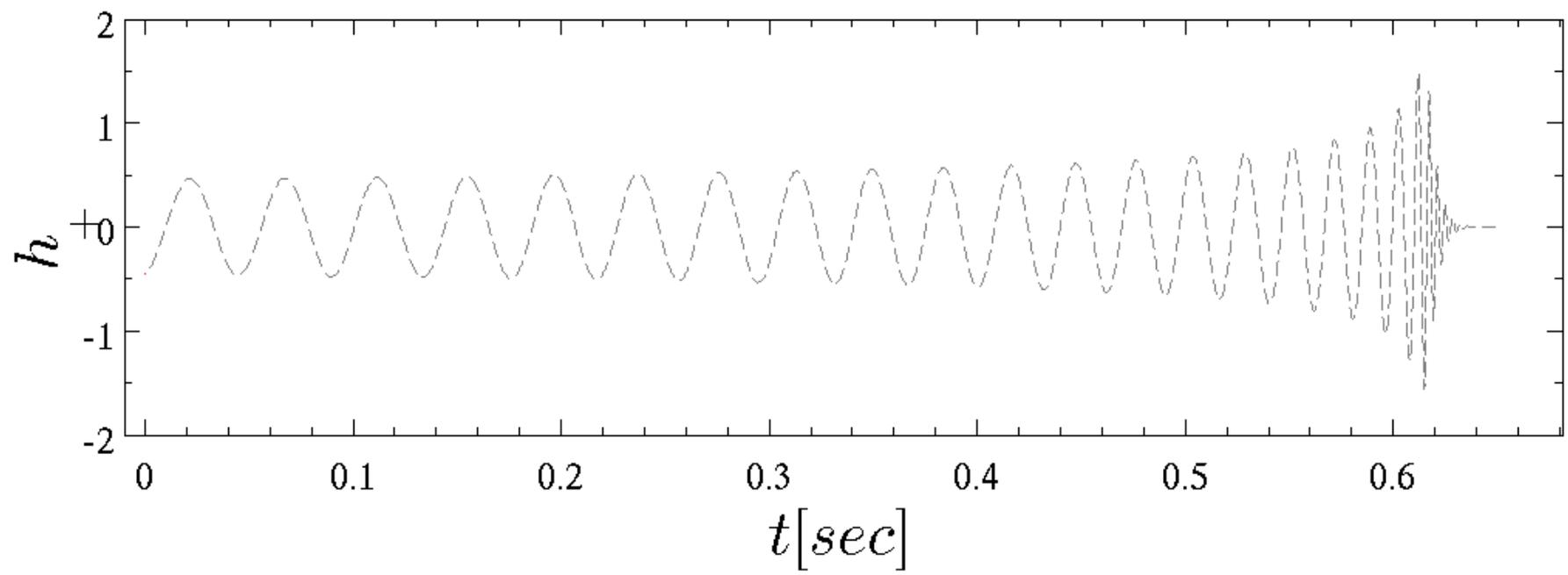
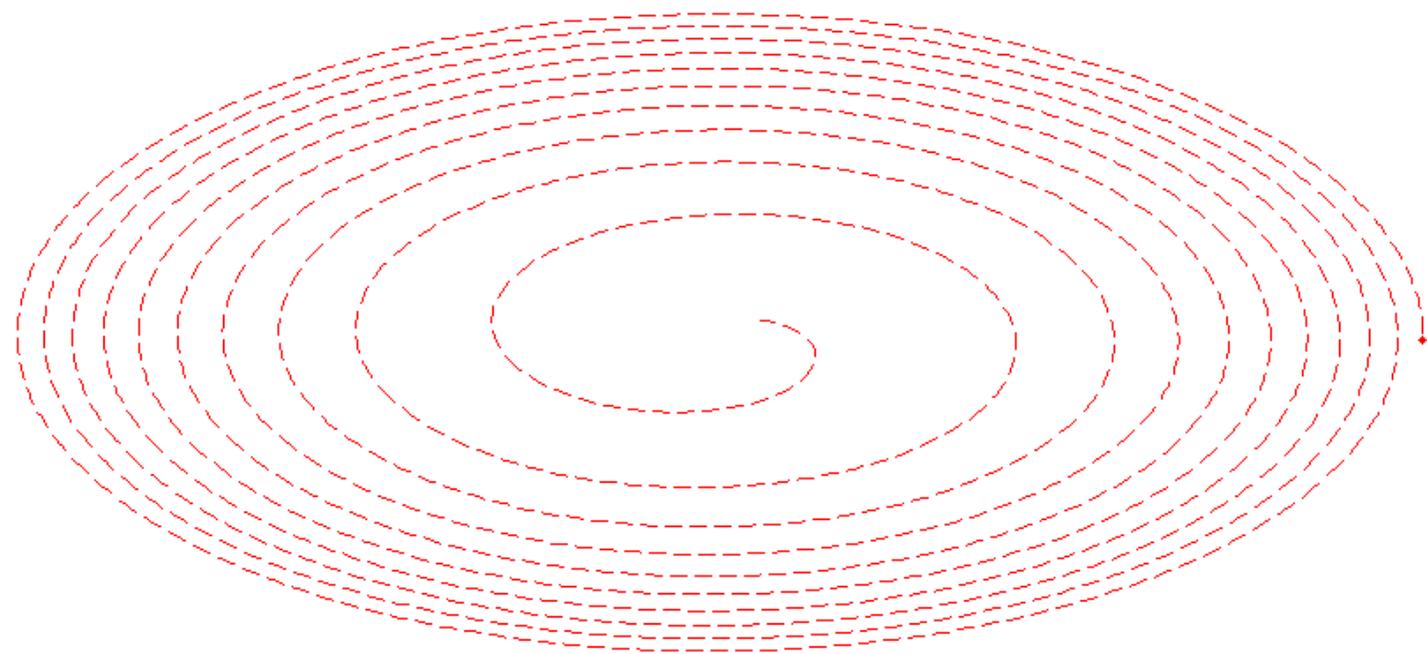


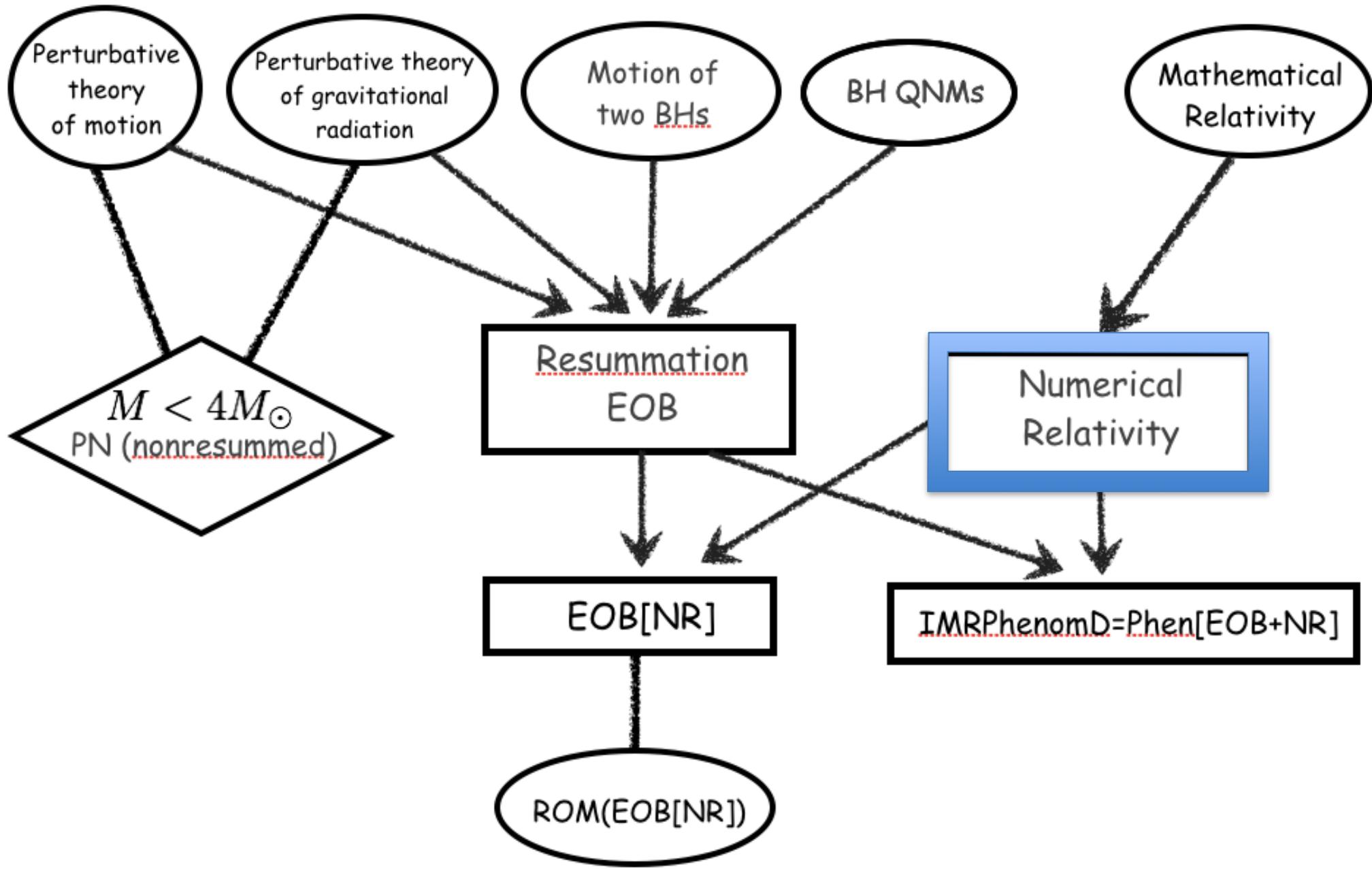
- ▶ The system must radiate angular momentum
- ▶ How? Use PN-based (Taylor-expanded) radiation reaction force (ang-mom flux)
- ▶ Need flux resummation

$$\hat{\mathcal{F}}_{\varphi}^{\text{Taylor}} = -\frac{32}{5} \nu \Omega^5 r_{\Omega}^4 \hat{F}^{\text{Taylor}}(\nu_{\varphi}) \rightarrow$$

Resummation multipole by multipole  
(Damour&Nagar 2007,  
Damour, Iyer & Nagar 2008,  
Damour & Nagar, 2009)

Plus horizon contribution [Nagar&Akcaay2012]





# Numerical Relativity (NR)

Mathematical foundations : Darmois 27, Lichnerowicz 43, Choquet-Bruhat 52-

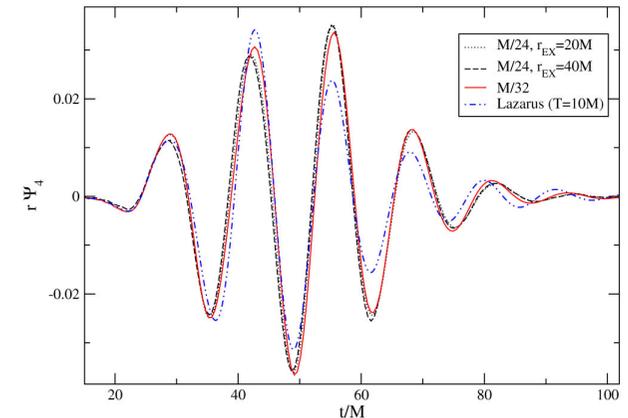
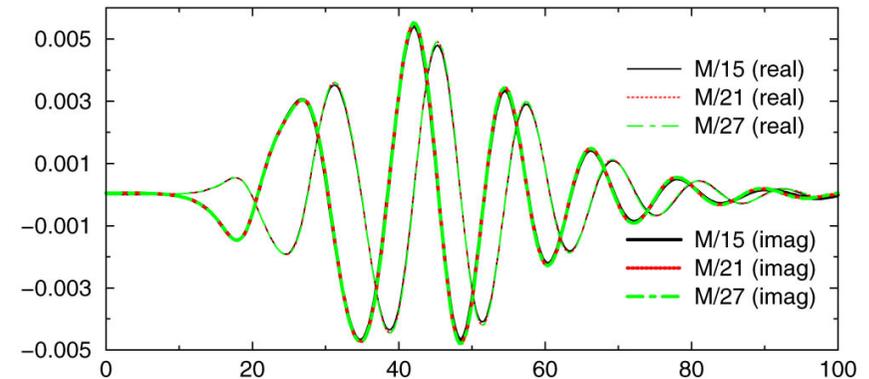
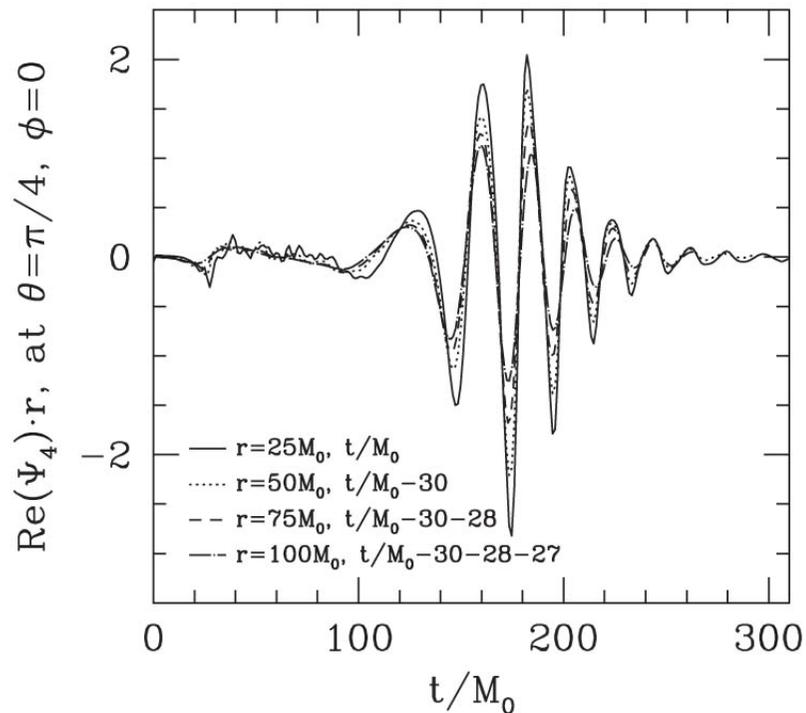
Breakthrough:

**Pretorius 2005** generalized harmonic coordinates, constraint damping, excision

Campanelli-Lousto-Maronetti-Zlochover 2006

Baker-Centrella-Choi-Koppitz-van Meter 2006

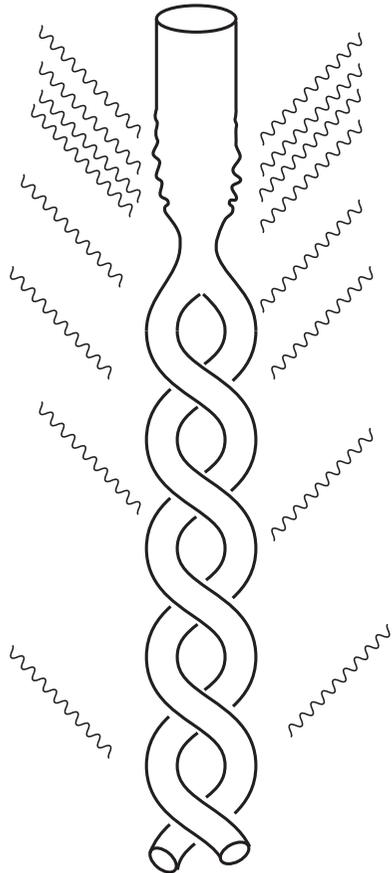
## Moving punctures



Excision + generalized harmonic coordinates (Friedrich, Garfinkle)

$$C_a \equiv g_{ab} (H^a - \square x^a) = 0.$$

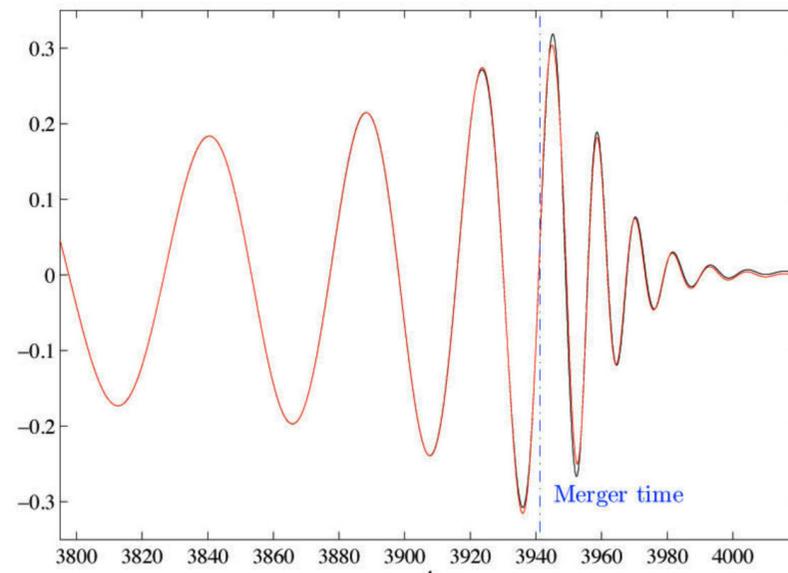
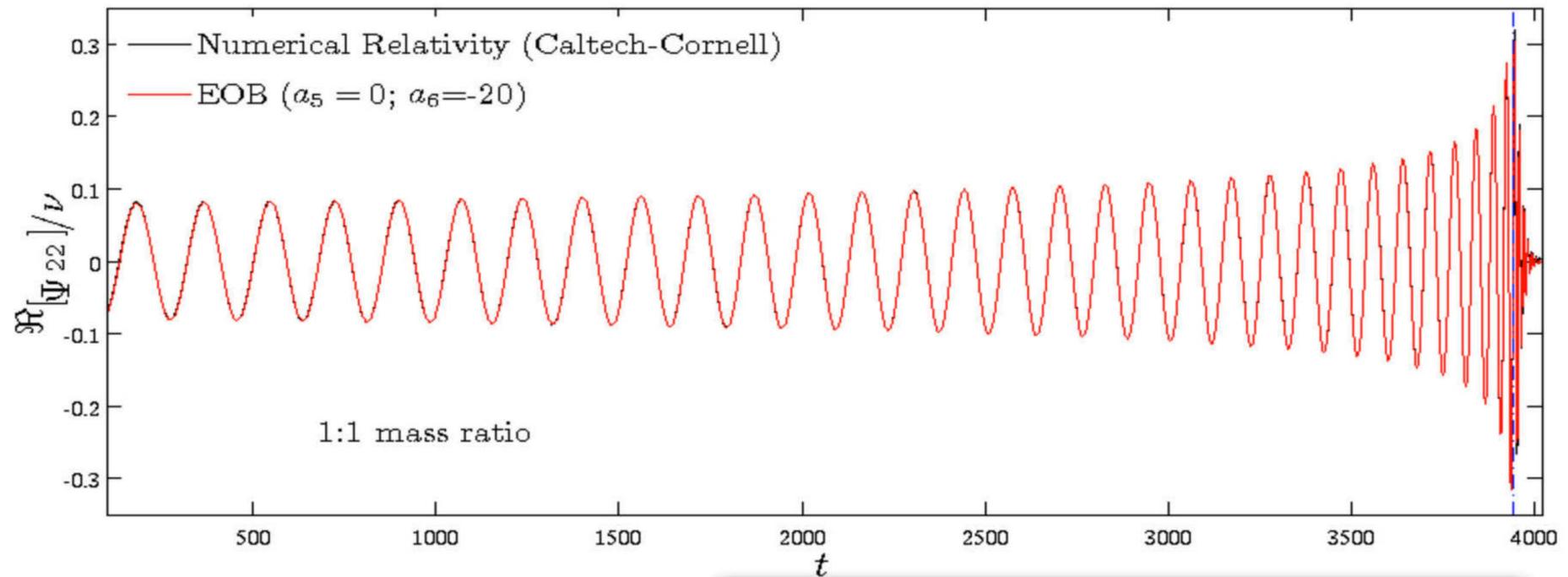
+ Constraint damping (Brodbeck et al., Gundlach et al., Pretorius, Lindblom et al.)

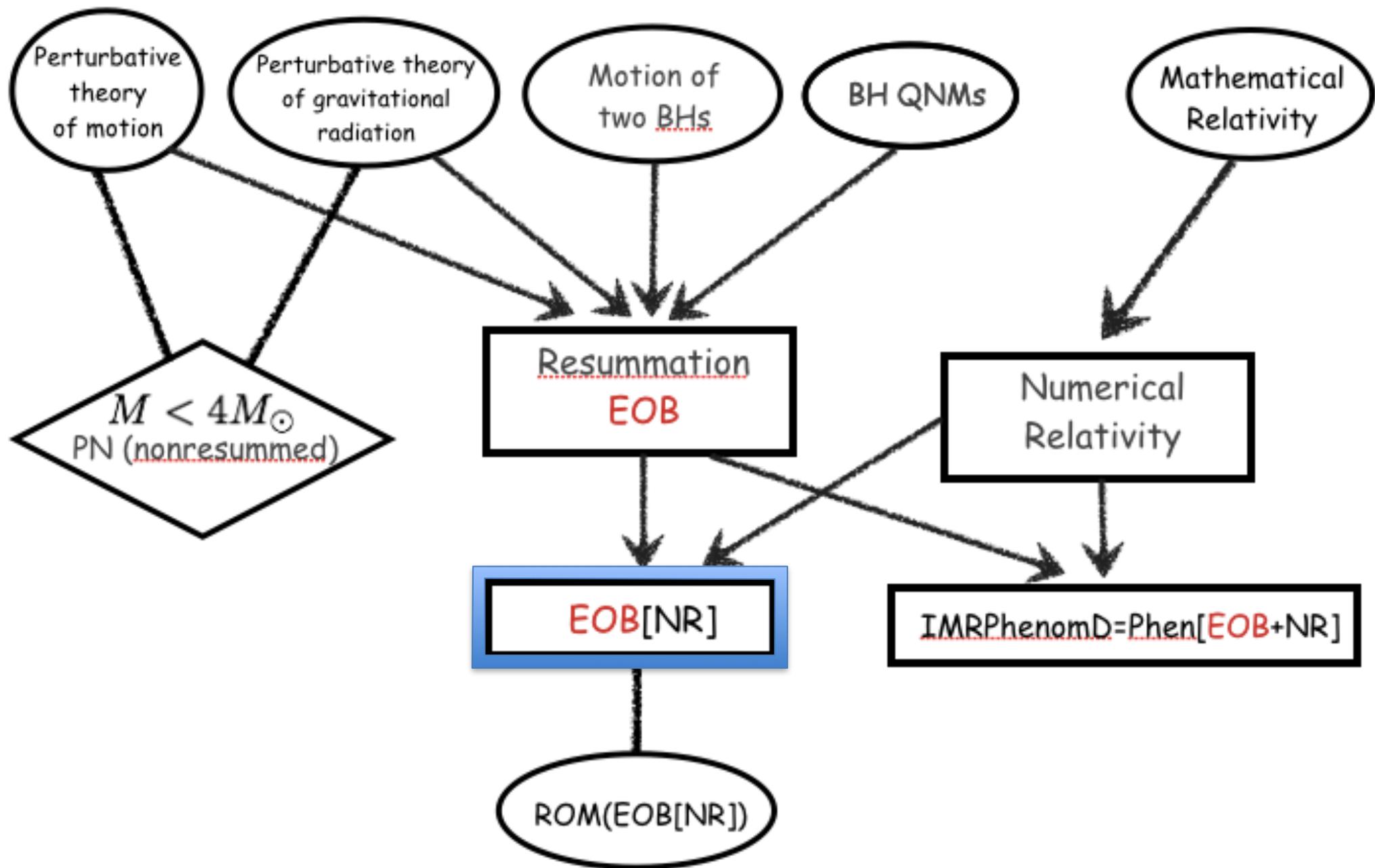


$$\begin{aligned} & \frac{1}{2} g^{cd} g_{ab,cd} + \\ & g^{cd} ({}_{,a} g_{b)d,c} + H_{(a,b)} - H_d \Gamma_{ab}^d + \Gamma_{bd}^c \Gamma_{ac}^d \\ & + \kappa [n_{(a} C_{b)} - \frac{1}{2} g_{ab} n^d C_d] \\ & = -8\pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right). \end{aligned}$$

$$\square C^a = -R^a_b C^b + 2\kappa \nabla_b [n^{(b} C^{a)}],$$

# Numerical Relativity Waveform (Caltech-Cornell, SXS)





# NR-completed resummed 5PN EOB radial A potential

4PN analytically complete + 5 PN logarithmic term in the  $A(u, \nu)$  function,

With  $u = GM/R$  and  $\nu = m_1 m_2 / (m_1 + m_2)^2$

[Damour 09, Blanchet et al 10, Barack-Damour-Sago 10, Le Tiec et al 11,

Barausse et al 11, Akcay et al 12, Bini-Damour 13,

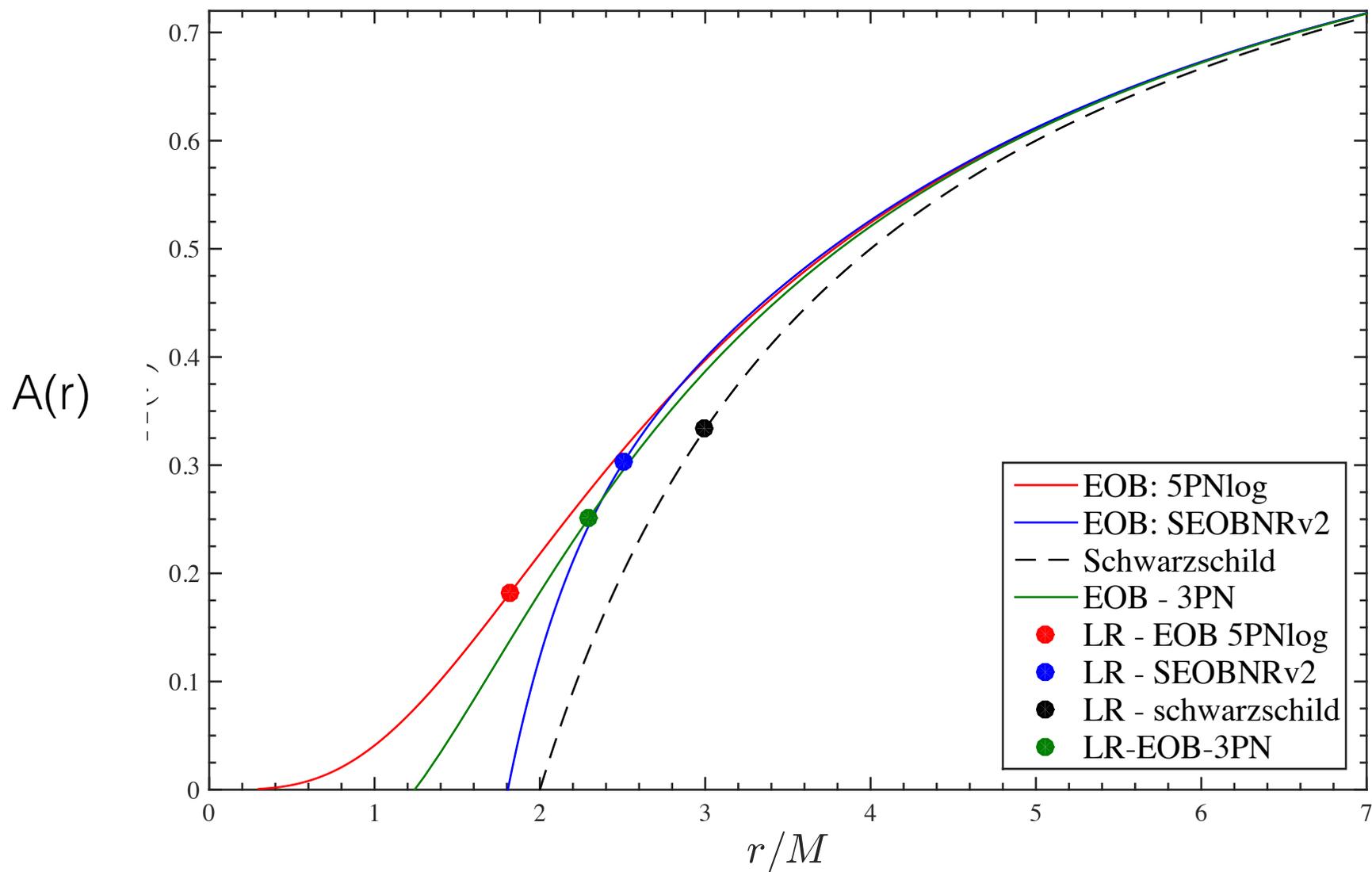
Damour-Jaranowski-Schäfer 14, Nagar-Damour-Reisswig-Pollney 15]

$$\begin{aligned} A(u; \nu, a_6^c) &= P_5^1 \left[ 1 - 2u + 2\nu u^3 + \nu \left( \frac{94}{3} - \frac{41}{32} \pi^2 \right) u^4 \right. \\ &+ \nu \left[ -\frac{4237}{60} + \frac{2275}{512} \pi^2 + \left( -\frac{221}{6} + \frac{41}{32} \pi^2 \right) \nu + \frac{64}{5} \ln(16e^{2\gamma} u) \right] u^5 \\ &+ \left. \nu \left[ a_6^c(\nu) - \left( \frac{7004}{105} + \frac{144}{5} \nu \right) \ln u \right] u^6 \right] \end{aligned}$$

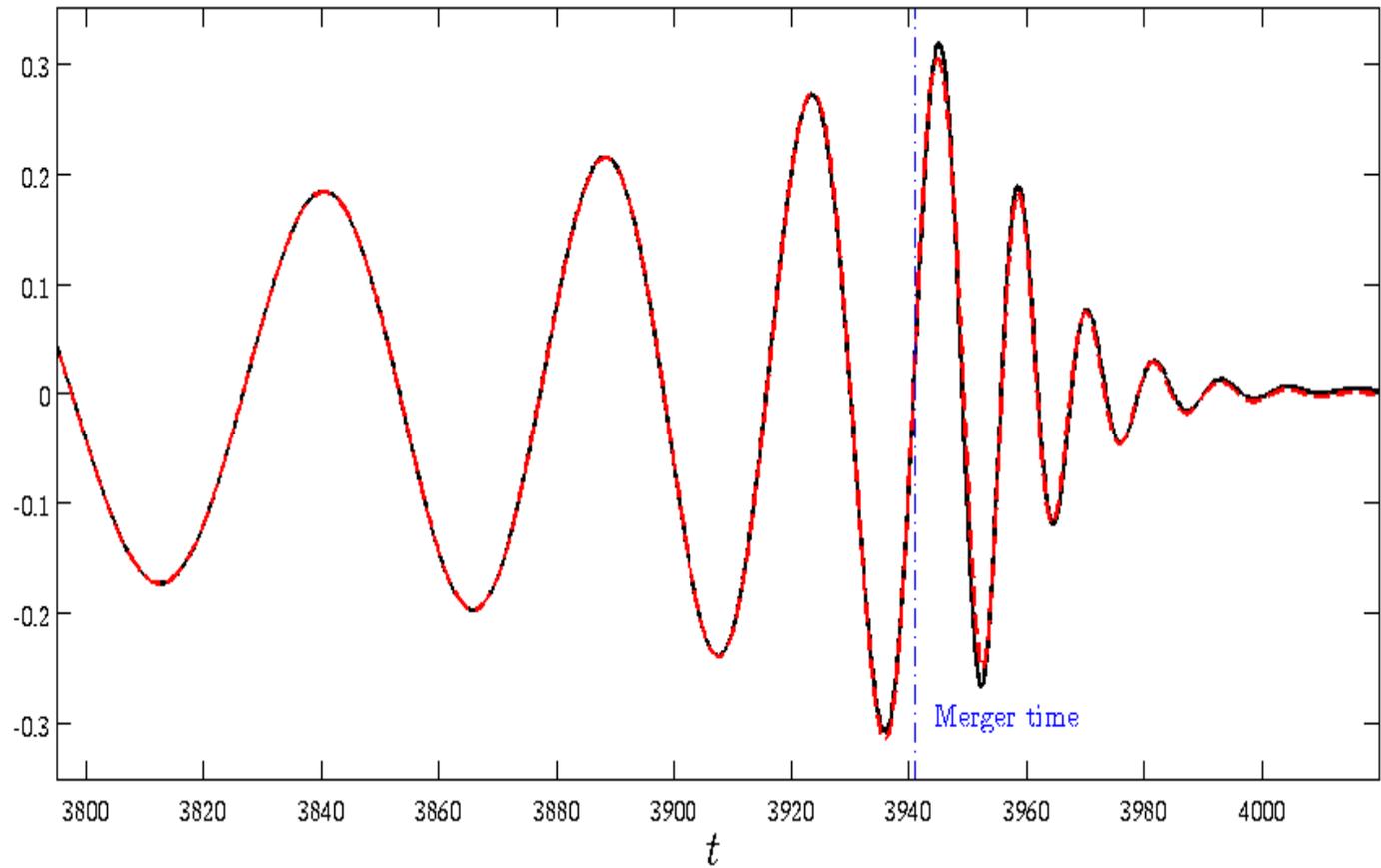
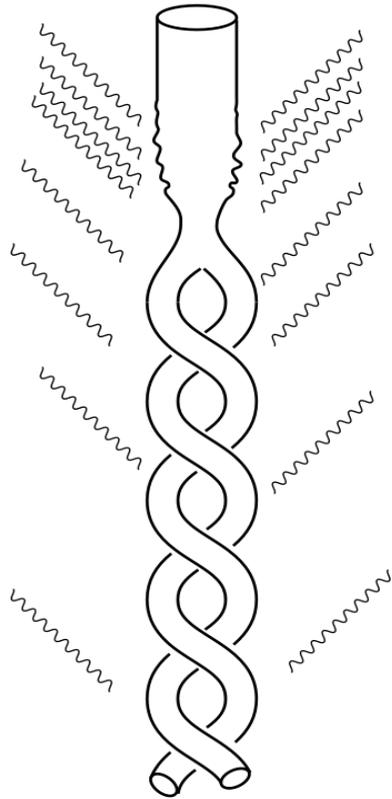
$$a_6^{c \text{ NR-tuned}}(\nu) = 81.38 - 1330.6 \nu + 3097.3 \nu^2$$

# MAIN RADIAL RADIAL EOB POTENTIAL A(R)

m1=m2 case



# EOB / NR Comparison

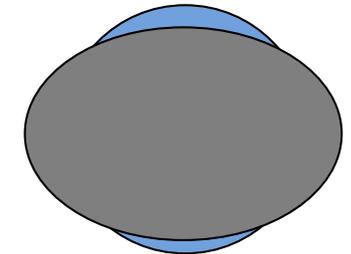


Inspiral + « plunge »



Two orbiting point-masses:  
Resummed dynamics

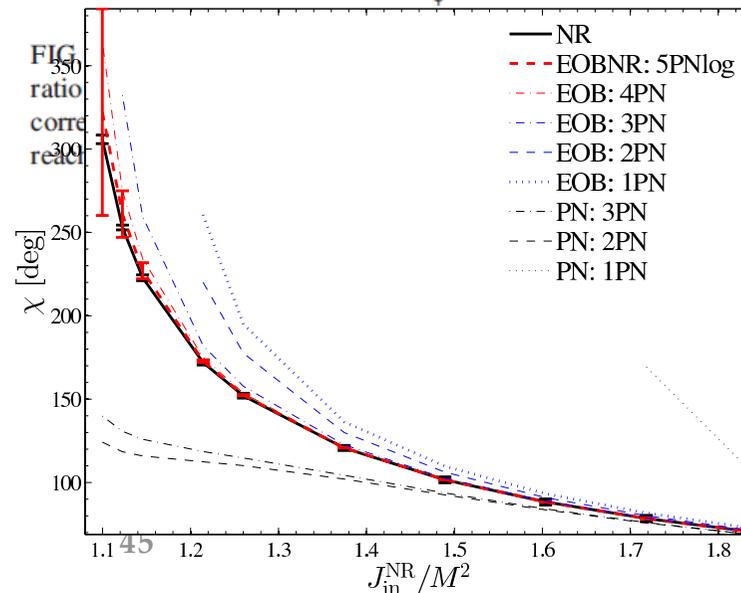
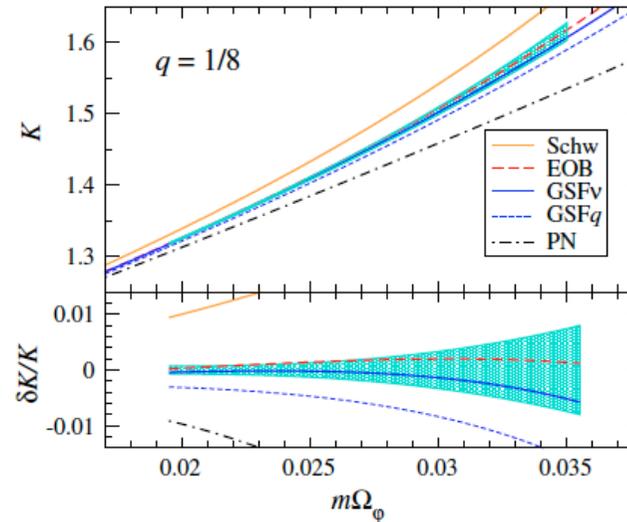
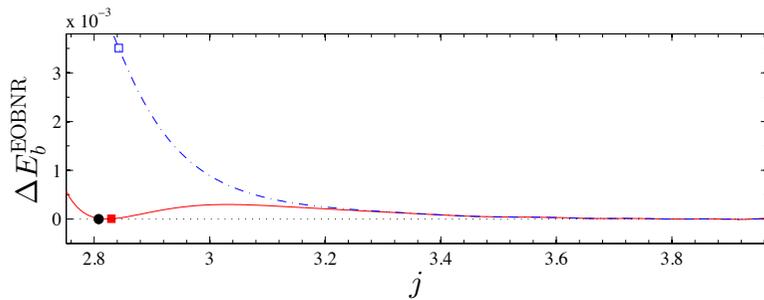
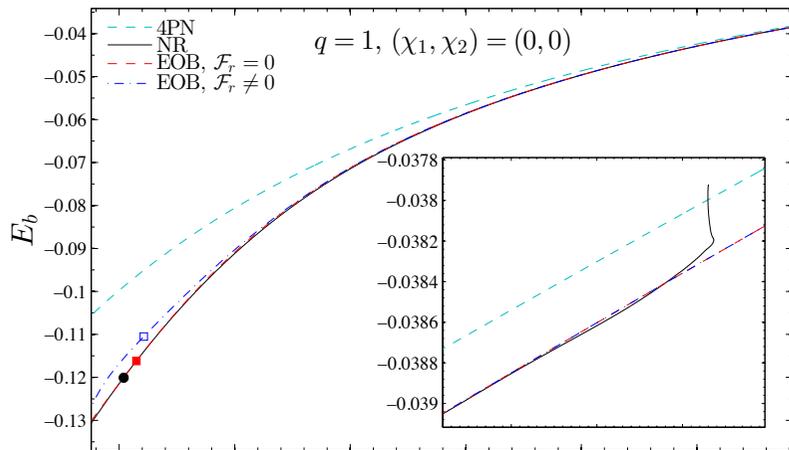
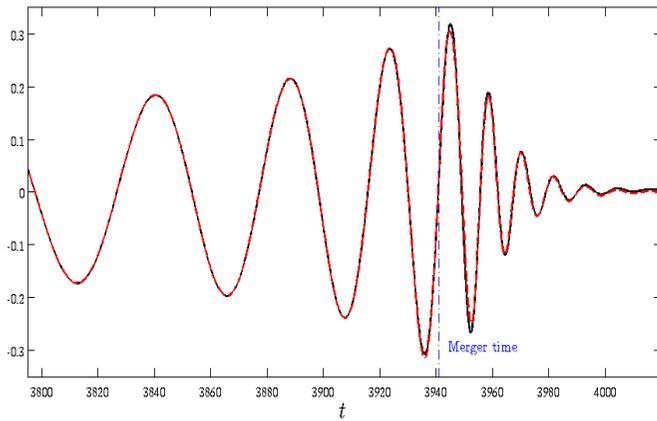
Ringing BH



Peak emitted power  $\sim 3 \times 10^{56}$  erg/s  $\sim 0.001 c^5/G$

# EOB VS NR

waveform (Damour-Nagar 09),  
energetics (Nagar-Damour-Resswig-Pollney 16),  
periastron precession (LeTiec-Mroue-Barack-  
Buonanno-Pfeiffer-Sago-Tarachini 11, and  
scattering angle (Damour-Guercilena-Hinder-Hopper-  
Nagar-Rezzolla 14)

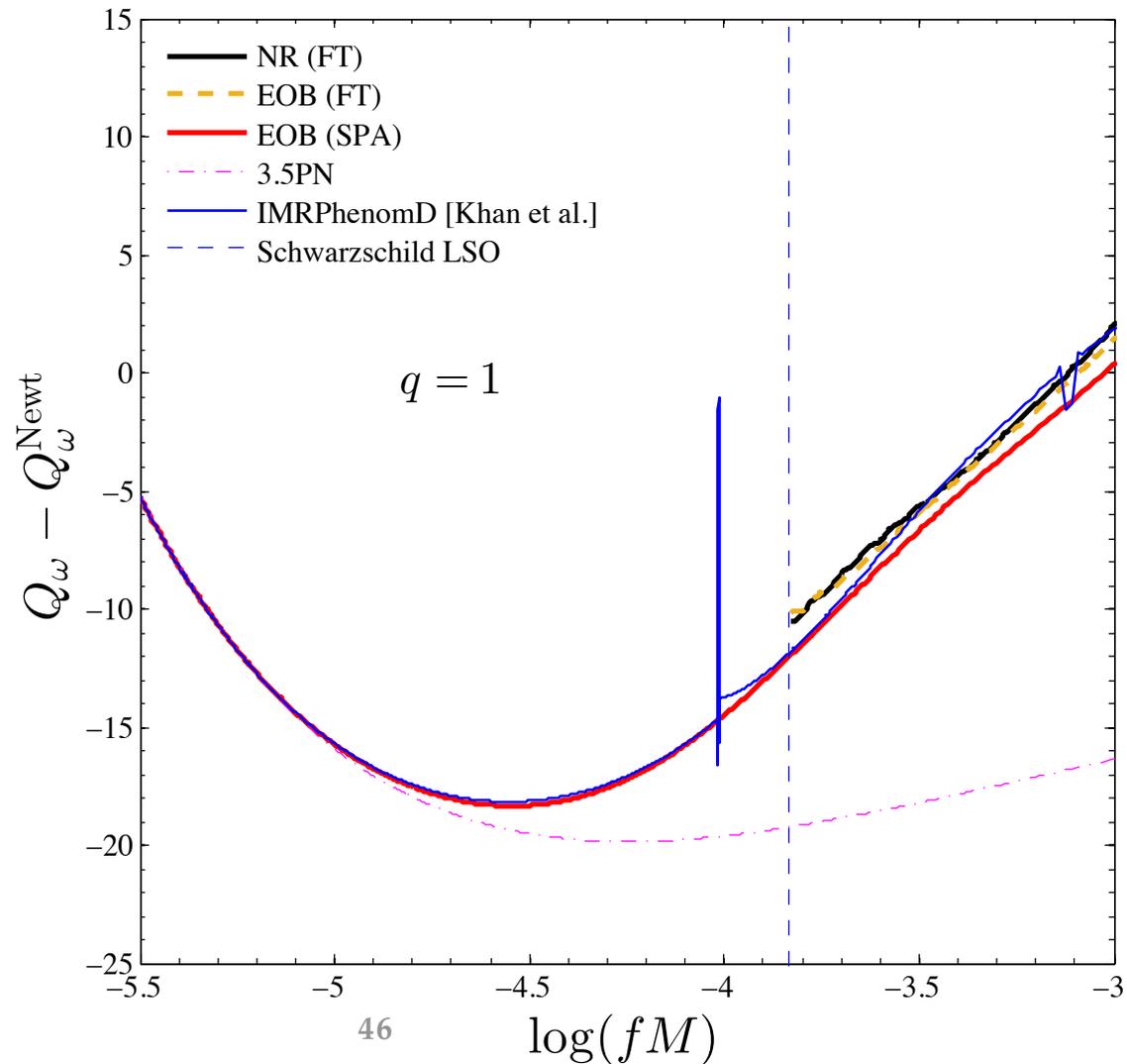


# PN, EOB, NR, PHENOMD

PN accuracy loss during inspiral

Dimensionless « quality factor » of GW phase  $Q_\omega = f^2 \frac{d^2 \psi(f)}{df^2} \approx \frac{\omega^2}{\dot{\omega}}$

$$Q_\omega - Q_\omega^N$$



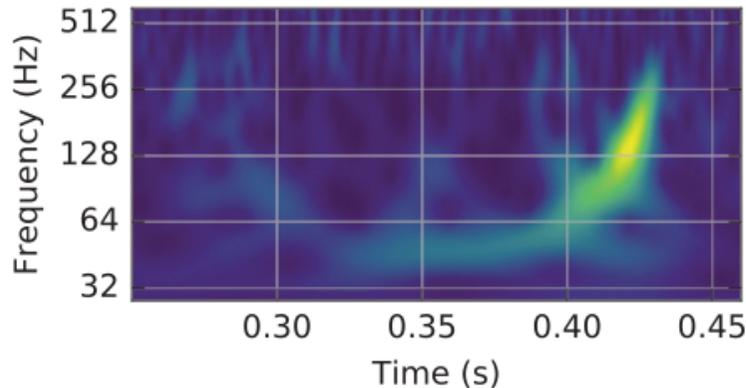
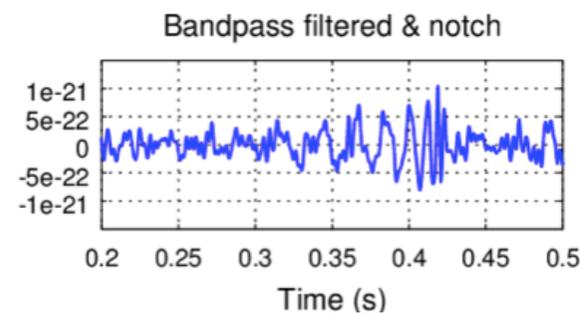
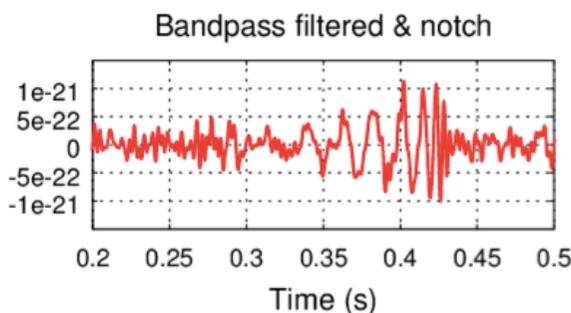
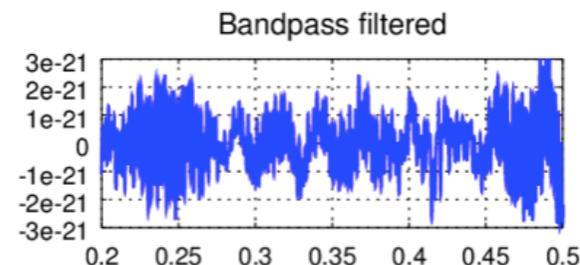
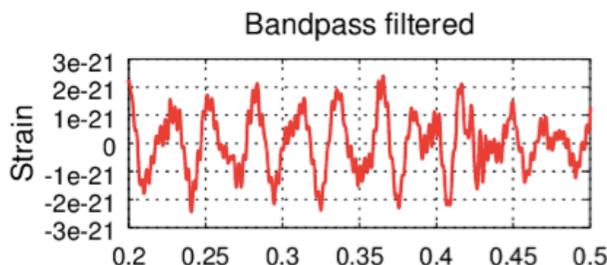
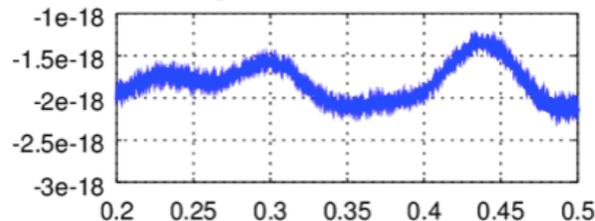
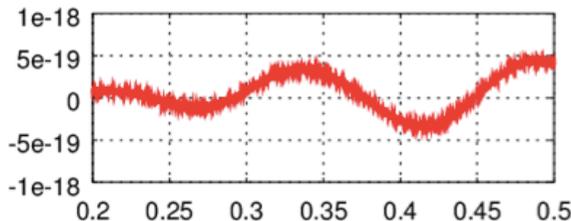
# GW150914: An incredibly small signal lost in the noise

$h(t)$



Chassande-Mottin,  
Acad Sciences,  
5 April 2016

÷ 500

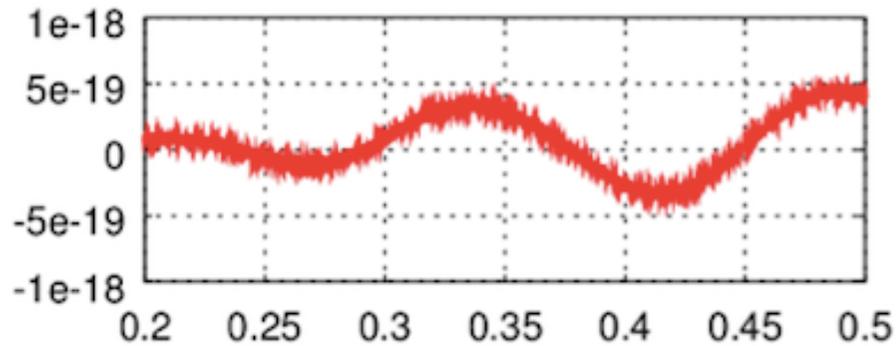


Two levels of signal search:

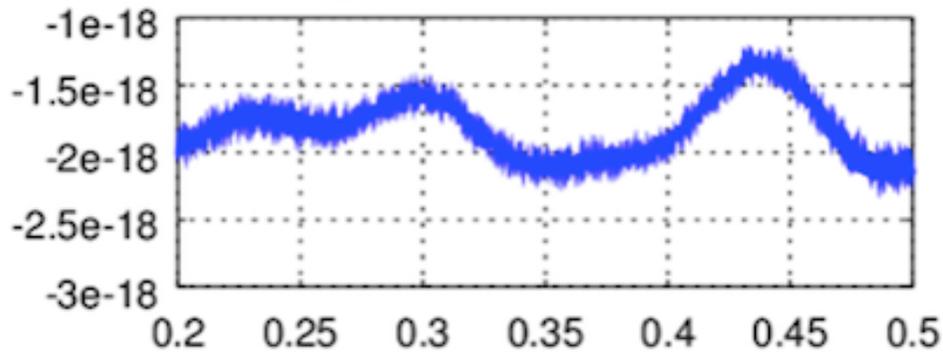
1. **time-frequency** analysis (Wilson, Meyer, Daubechies-Jaffard-Journe, Klimenko et al.)
2. Wiener's **matched filter** analysis (EOB[NR] and Phenom[EOB+NR])

# GW150914 vs EOB[NR]

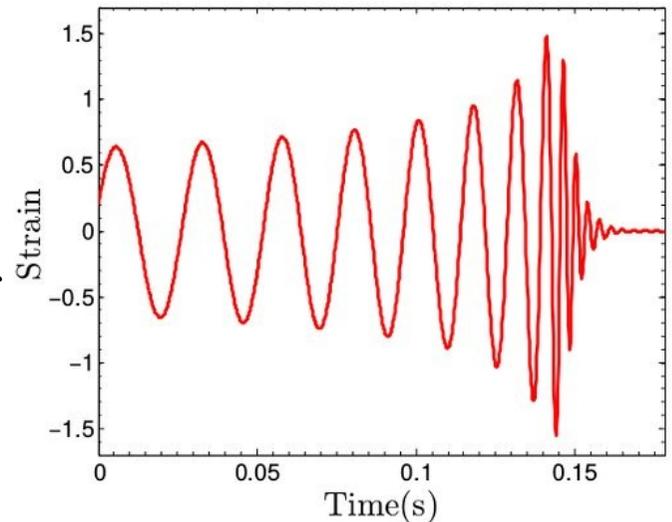
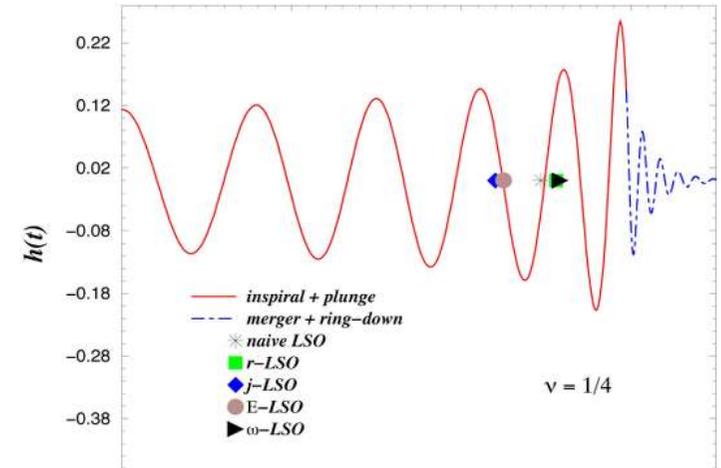
Hanford H1: raw data



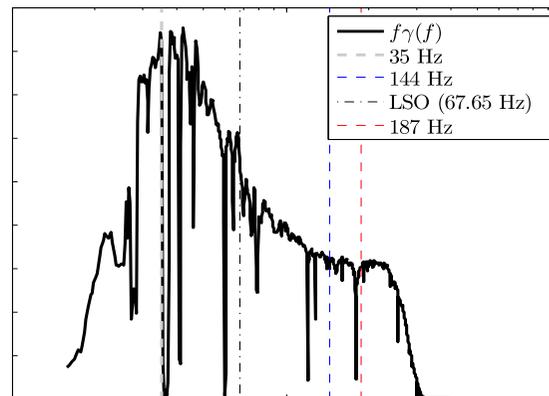
Livingston L1: raw data



scale :  $10^{-21}$   
500 × smaller



$$\frac{d\rho^2}{d \ln f} = \frac{f |\tilde{h}(f)|^2}{S_n(f)}$$



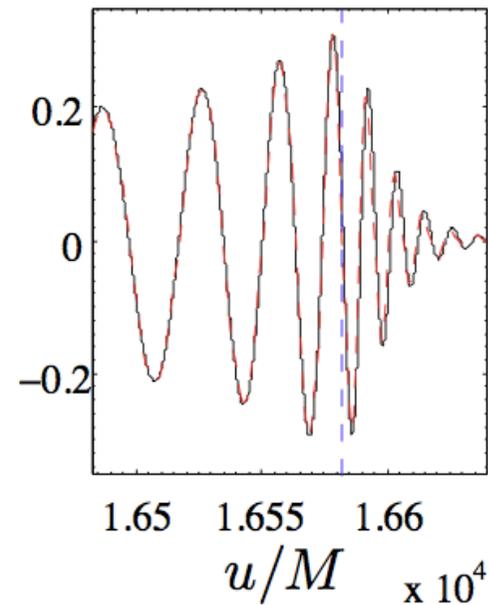
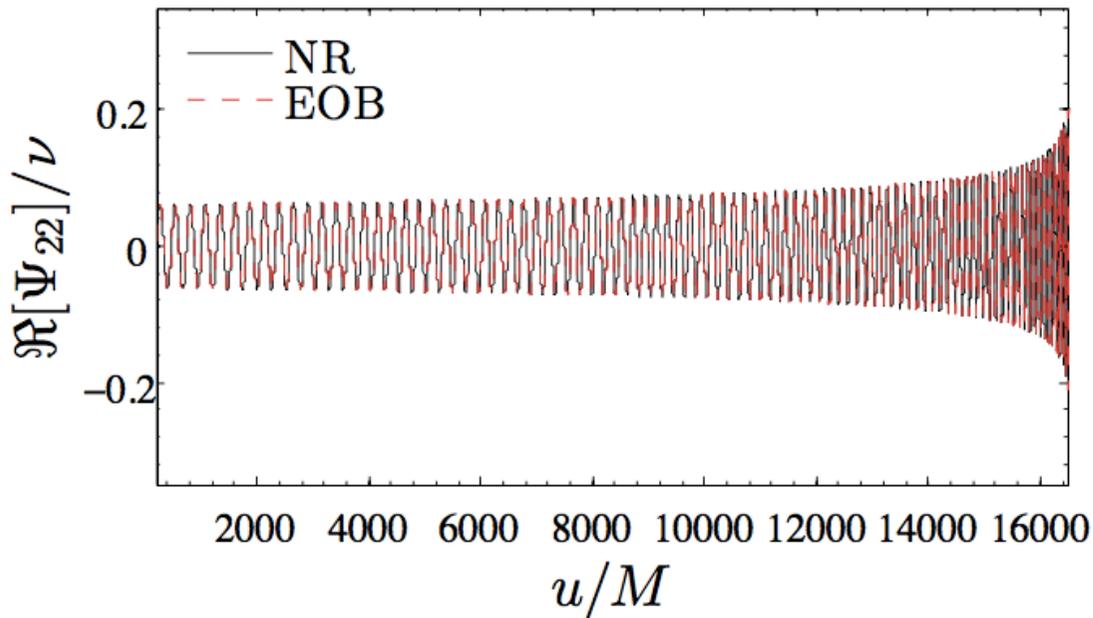
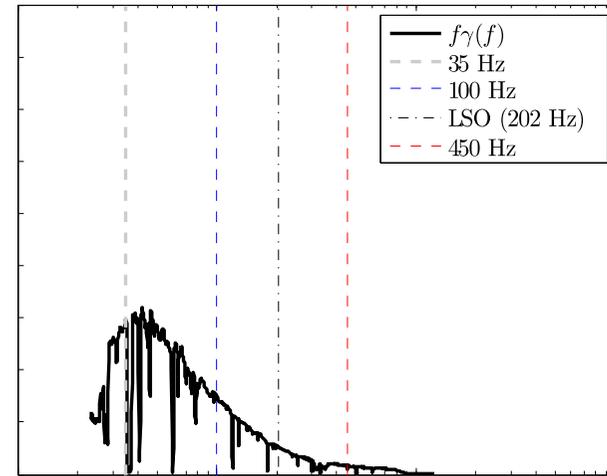
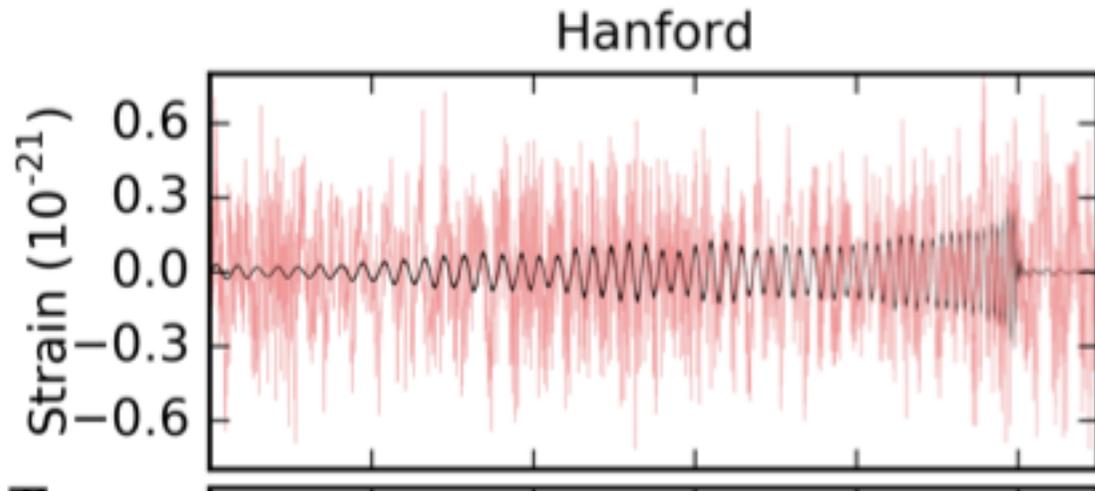
$$m_1 = 36_{-4}^{+5} M_{\odot}$$

$$m_2 = 29_{-4}^{+4} M_{\odot}$$

$$\chi_{\text{eff}} = -0.06_{-0.18}^{+0.17}$$

$$D_L = 410_{-180}^{+160} \text{Mpc}$$

# GW151226: only detected via accurate matched filters



$$m_1 = 14.2^{+8.3}_{-3.7} M_{\odot}$$

$$m_2 = 7.5^{+2.3}_{-2.3} M_{\odot}$$

$$\chi_{\text{eff}} = +0.21^{+0.20}_{-0.10}$$

$$D_L = 440^{+180}_{-190} \text{Mpc}$$

# Conclusions

---

- One century after the work of Einstein (and Schwarzschild), thanks to the LIGO observations (+ the LIGO-Virgo analysis):
  1. We have seen vibrations of the space geometry pass by the Earth, and
  2. We have got the first (almost) direct evidence for the existence of BHs.
- Theory had predicted (analytically  $\geq 2000$  and numerically  $\geq 2005$ ) the GW signal emitted by the coalescence of two black holes. This theoretical prediction  $[f(m_1, m_2, S_1, S_2)]$  is crucial for extracting the signal by matched filtering, and for measuring the source parameters. Several aspects of the analytical work have played a key role: perturbative theory of motion, perturbative theory of GW generation, EOB formalism. The union between Analytical Relativity and Numerical Relativity (particularly EOB+NR) has been crucial.
- The detailed study of BBH coalescence signals opens a new window for probing relativistic gravity:  $v/c = 1/2$ ;  $GM/(c^2 r) = 1/2$ . We hope to have soon indirect confirmations of the GR-predicted physical properties of BH (QNMs ...), and of the (TT) structure of GWs.
- Opening of a new window on the universe: GW astronomy: might be dominated by BBH (Belczynski et al 2010); waiting for BNS + EM signal (GRB ?), and for LIGO/Virgo/Kagra/Indigo network. The detailed study of coalescences involving NS will open a window on the EOS of nuclear matter (tidal polarizability).