Umbral symmetry groups and K3 CFTs

Based on work with M. Cheng and R. Volpato

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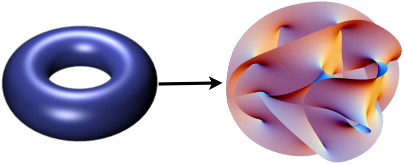
Moonshine is a fascinating connection between 2 mathematical objects:

- Finite groups
- Modular forms,

which has connections to many other areas of mathematics and string theory.
These connections are still in the process of being uncovered and explored.

Today I will focus on connections with sigma models coming from string theory compactified on K3 surfaces.



These probe geometry in interesting way and have interesting connections with lattice theory, number theory, modular forms, and group theory

Elliptic genus

- The elliptic genus of a N=(2,2) 2d CFT is also a useful object in string theory as it is a protected character which counts BPS states

$$\mathbf{EG}(\tau, z) = \mathrm{tr}_{RR}(-1)^{F_L+F_R} y^{J_0} q^{L_0-c/24} \bar{q}^{\overline{L_0}-\bar{c}/24}$$

where F_L, F_R are left- and right-moving Fermion number L_0, J_0 are the zero modes of Virasoro and U(1) R-currents which make up the N=2 superconformal algebra.

- For CFTs with a discrete symmetry g, we will also consider the twining genus:

$$\mathbf{EG}_g(\tau, z) = \mathrm{tr}_{RR}\, g(-1)^{F_L+F_R} y^{J_0} q^{L_0-c/24} \bar{q}^{\overline{L_0}-\bar{c}/24}$$

Mathieu moonshine

Eguchi-Ooguri-Tachikawa, K3 elliptic genus:

$$\mathcal{Z}(\tau,z) = \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)} \left(a\,\underline{\mu(\tau,z)} + q^{-1/8}\big(b + \underline{\sum_{n=1}^{\infty} t_n q^n}\big) \right)$$

24 massless multiplets

Infinite tower of massive multiplets

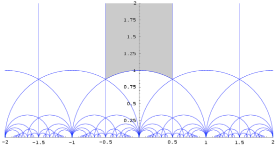
expand this function in N=4 superconformal characters

$$2 \times \underline{45},\ 2 \times \underline{231},\ 2 \times \underline{770},\ 2 \times \underline{2277},\ 2 \times \underline{5796}, \ldots$$

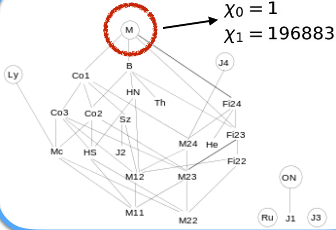
(Sums of) dimensions of irreducible representations of the largest Mathieu group M24!

Compare with...

$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

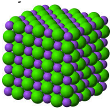


Upper half plane, $\mathbb{H}$.



$\chi_0 = 1$

$\chi_1 = 196883$



$/\mathbb{Z}_2$

Leech

- Mathieu moonshine is a new type of moonshine where the automorphic objects are mock modular forms.
- A holomorphic function $h(\tau)$ on $\mathbb{H}$ is a mock modular form [Ramanujan, Zwegers] of weight k for a discrete group Γ if it has at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$ and if there exists a holomorphic modular form $f(\tau)$ of weight $2-k$ on Γ such that the "completion" of h given by

$$\hat{h}(\tau) = h(\tau) + (4i)^{k-1} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k} \overline{f(-\bar{z})} dz$$

is a (non-holomorphic) modular form of weight k for Γ. The function f is called the shadow of the mock modular form f.

The importance of mock modular forms in physics and mathematics has also begun to be developed. They appear naturally in the context of

- Characters of infinite-dimensional Lie superalgebras [Eguchi-Hikami, Kac-Wakimoto,...]
- Elliptic genera of CFTs with non-compact target spaces [Ashok-Troost, Murthy,...]
- Counting of black hole microstates in string theory [Dabholkar-Murthy-Zagier,...]

Mathieu moonshine is similar to monstrous moonshine, however our understanding is incomplete:

- Representations are governed by a mock modular form
- Supersymmetry is involved in a fundamental way

A few of many reasons why this is interesting:

- K3 surfaces are important in many aspects of string theory, from black hole solutions to AdS/CFT
- Understanding the structure and symmetries of BPS states is of great interest in both physics and mathematics (c.f. many talks at this conference)
- The subject of moonshine unites many areas of mathematics and physics
  - group theory
  - number theory
  - geometry
  - string theory
  - vertex operator algebras/CFTs
  - and more...

Symmetries of K3 CFTs

Moduli space of K3 CFTs has the form

$$\mathcal{M} = O(\Gamma^{4,20})\backslash O(4,20)/O(4)\times O(20)$$

where

- $O(4,20)/O(4)\times O(20)$ corresponds to choice of positive-definite 4-plane $\Pi \subset \mathbb{R}^{4,20}$ specifying Ricci-flat metric and B-field on K3
- $\Gamma^{4,20}$ even unimodular lattice of signature $(4,20)$ corresponding to integral homology of K3; $O(\Gamma^{4,20})$ its group of automorphisms

Symmetries of K3 CFTs

Let G_Π be group of automorphisms of a particular (non-singular) K3 sigma model specified by Π which preserves $\mathcal{N} = (4,4)$ SUSY.

- G_Π is subgroup of $O(\Gamma^{4,20})$ leaving Π fixed pointwise
- Theorem (Gaberdiel-Hohenegger-Volpato): G_Π is subgroup of $Co_0 := Aut(\Lambda_{Leech})$ which fixes pointwise a sublattice of Λ_{Leech} of rank at least 4
- Proof: Orthogonal complement of Π in $\Gamma^{4,20}$ has rank at most 20, and can be embedded in Λ_{Leech} (if no roots)

What does this mean for the EOT observation?

- Symmetries of (non-singular) K3 sigma models have been classified and are those lie within the sporadic group Co_0 and preserve a 4-dimensional subspace in the 24-dimensional representation (GHV)
- In particular, within Co_0 are "extra" symmetries which lie outside of M_{24} at non-generic, isolated points in moduli space (orbifold points)
- ...and within M_{24} are symmetries disallowed by the GHV classification; elements which only preserve a 2-dimensional subspace in the **24**

In the rest of the talk we will consider 2 extensions of the GHV result:

1. Consider singular points in K3 moduli space
2. Take worldsheet parity into account

and the implications for the questions:

- Which groups arise?
- Which twining functions arise?

Why consider singular points?

- Though the CFT is singular and may not be perturbatively well-defined, the full 10d string theory is well-defined. These singular points correspond to 6-dimensional theories with enhanced non-abelian gauge symmetry, coming from massless modes of D-branes on shrinking cycles
- To make connection with a larger structure: Umbral moonshine

Umbral moonshine

Cheng, Duncan, Harvey ('12,'13)

Mathieu moonshine is in fact the first example of a larger phenomenon known as "Umbral moonshine," which relates certain **discrete groups** arising as lattice automorphisms to **mock modular forms**. (Cheng, Duncan, Harvey)

These are specified by the Niemeier lattices:

- Even, unimodular, positive-definite lattices of rank 24
- 24 such lattices, classified by Niemeier: Leech lattice + 23 others which have ADE classification
- Uniquely determined by their root systems, that are all unions of the simply-laced root systems

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4 D_4$	A_6^4	$A_7^2 D_5^2$
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2 D_6$	$A_{11} D_7 E_6$	A_{12}^2	$A_{15} D_9$	$A_{17} E_7$	A_{24}
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10} E_7^2$	D_{12}^2	$D_{16} E_8$	D_{24}
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

For each lattice X, one gets a **group**:

$$G^X = \mathrm{Aut}(L^X)/\mathrm{Weyl}(X),$$

coming from automorphisms of the lattice, and a (unique, vector-valued) **mock modular form**, H_g^X, for each $g \in G^X$, whose modularity properties are specified by the data of the root system and whose coefficients are characters of G^X.

Umbral moonshine and K3 CFTs

Are the other instances of umbral moonshine related to K3 CFT?
Two ways to view decomposition (Cheng-SH)

- Algebraic: The $\mathcal{N}=4$ decomposition of the K3 elliptic genus:

$$\mathbf{EG}(K3) = \frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\Big\{24\,\mu_{2,0}(\tau,z) + \sum_{r\in\mathbb{Z}/4\mathbb{Z}} H_r^{X=A_1^{24}}(\tau)\theta_{2,r}(\tau,z$$

  can be viewed as contributions from BPS and non-BPS $\mathcal{N}=4$ multiplets.

- Geometric:

$$\mathbf{EG}(K3) = 24Z^{A_1,S}(\tau,z) - \frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\sum_{r\in\mathbb{Z}/4\mathbb{Z}} H_r^{A_1^{24}}(\tau)\theta_{2,r}(\tau,z).$$

  Singularity configuration coming from the root system of the Niemeier lattice, and mock modular form contribution encoding characters of the automorphism group

In fact, this type of decomposition holds for all 23 cases of umbral moonshine.

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a}\, \phi^X(\tau, \frac{z + a\tau + b}{m})$$

where

$$\phi^X = \frac{i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)} \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X(\tau)\, \theta_{m,r}(\tau, z)$$

In other words: for the 23 Niemeier lattices L^X we have 23 different ways of separating $\mathbf{EG}(K3)$ into two parts.

1. Replace the Niemeier root system X with the corresponding configuration of singularities to obtain a contribution to the $K3$ elliptic genus by the singularities.
2. Use the umbral moonshine construction for the mock modular form associated to each L^X to get the rest of $\mathbf{EG}(K3)$ after a summation procedure

Example:

$$\mathbf{EG}(\tau, z; K3) = 12Z^{A_2,S}(\tau, z) + \frac{1}{6} \sum_{\substack{a,b\in\mathbb{Z}/3\mathbb{Z} \\ z+a\tau+b}} q^{a^2} y^{2a} \frac{i\theta_1(\tau, 3z)\theta_1(\tau, 2z)}{\eta^3(\tau)\theta_1(\tau, z)} \sum_{r\in\mathbb{Z}/6\mathbb{Z}} H_r^{A_2^{12}}(\tau)\,\theta_{3,r}(\tau, z)$$

where

$$H^{A_2^{12}}(\tau) = q^{-1/12}(-2 + 32q + 110q^2 + 288q^3 + \dots) + q^{-1/3}(20q + 88q^2 + 220q^3 + \dots)$$

encoding irreps of $2.M_{12}$

- We can also define a set of twining genera for each $g \in G^X$ for all Niemeier lattices X:

$$Z_g^X(\tau, z) = Z_g^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \, \phi_g^X\left(\tau, \frac{z + a\tau + b}{m}\right).$$

- Let the Frame shape π_g encode the eigenvalues of g in its 24-dimensional representation
- For g a symplectic automorphism with Frame shape π_g, one is guaranteed to get the same Z_g^X no matter which umbral group g comes from.
- For g a (non-geometric) sigma-model symmetry, Z_g^X can differ depending on which umbral group g is in, even if it has the same π_g.
- Do these twining genera as derived from Umbral moonshine have any relation to symmetries of K3 sigma models?

Symmetries of K3 CFTs, revisited

Inclusion of singular points in moduli space. If G_Π is orthogonal to a root, then replace Leech with corresponding Niemeier, and Co_0 with corresponding umbral group.

- Theorem (Cheng-SH-Volpato): Γ_{G_Π} can be embedded in at least one Niemeier N^X and G_Π is subgroup of $O(N^X) := Aut(N^X)$ which fixes pointwise a sublattice of N^X of rank at least 4
- Conjecture (all Niemeier are important): For all N^X, there exists a $G_\Pi \subset O(\Gamma^{4,20})$ fixing pointwise a sublattice Γ^{G_Π} of rank $4 + d$, $d \geq 0$ s.t. Γ_{G_Π} can only be embedded in N^X

Worldsheet parity

We briefly comment on symmetries g which yield twining genera with "multiplier." Consider full twining partition function

$$\psi_g^{\mathcal{C}}(\tau, z, \bar{u}) = \mathrm{tr}_{RR}\left(g q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} e^{2\pi i z J_0} e^{-2\pi i \bar{u} \bar{J}_0} (-1)^{F+\bar{F}}\right) ,$$

which reduces to the twined elliptic genus for $u = 0$:

$$\phi_g^{\mathcal{C}}(\tau, z) = \psi_g^{\mathcal{C}}(\tau, z, 0). \tag{0.1}$$

This should transform as

$$\begin{aligned} \psi_g^{\mathcal{C}}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \frac{\bar{u}}{c\bar{\tau} + d}\right) &= C\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\ldots)\psi_g^{\mathcal{C}}(\tau, z, u) , \\ & \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N) . \end{aligned}$$

Note that $C\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$ ("trivial multiplier") unless twined Witten index $\psi_g^{\mathcal{C}}(\tau, 0, 0) = 0$.

Worldsheet parity

Suppose $\mathcal{C}$ and $\mathcal{C}'$ are two K3 sigma models related by some duality $h \in O(\Gamma^{4,20})$ that exchanges the left- and the right-moving sector. This means that h maps the fields of $\mathcal{C}$ to the fields of $\mathcal{C}'$ and maps the $\mathcal{N} = (4,4)$ algebras to one each other as

$$hL_nh^{-1} = \bar{L}'_n \qquad hJ_nh^{-1} = \bar{J}'_n\ , \tag{0.2}$$

For g a symmetry of $\mathcal{C}$ and $g' = hgh^{-1}$ a symmetry of $\mathcal{C}'$, after a bit of algebra, one can show that if the multiplier of $\phi_g^{\mathcal{C}}$ is C, then the multiplier of $\phi_{g'}^{\mathcal{C}'}$ is $\overline{C}$, the complex conjugate.

Worldsheet parity

Implications:

- If a given function Z_g^N has multiplier system $C : \Gamma_g \to \mathbb{C}^*$ with image not lying in $\mathbb{R}$, then Z_g^N can only arise from sigma model elliptic genus twined by symmetries acting differently on the left- and the right-moving Hilbert space.
- If a theory $\mathcal{C}$ leads to the twining function ϕ_g with a complex multiplier system, $\exists$ a theory $\mathcal{C}'$ with a twining function $\phi_{g'}$ with the complex conjugate multiplier system

Summary

Classification of symmetries of (singular) K3 CFTs, revisited:

- When taking into account singular points in K3 moduli space, umbral groups are relevant for classification of symmetries
- Certain non-geometric symmetries with complex multiplier must arise from actions which are not invariant under worldsheet parity symmetry

Many interesting questions remain; here are two:

- What implications do our better understanding of sporadic group symmetries in K3 sigma models have for other string theory compactifications involving K3?
- What about umbral symmetries which do not preserve a 4-plane?–necessary condition for resolving mystery of Mathieu/umbral moonshine