Poisson-Lie T-duality

Pavol Ševera

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(P/H is the target space of a 2dim σ-model)

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An action of $\mathfrak{g}$ on P extending the $\mathfrak{h}$-action
$+$ vanishing of the 1st Pontryagin class $[\langle F, F\rangle]$ of $P \to P/G$
(multiplicative gerbe over G acts on a gerbe over P)

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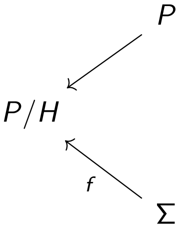

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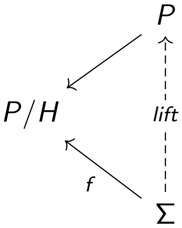

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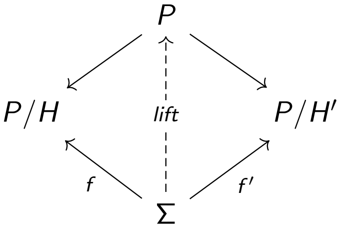

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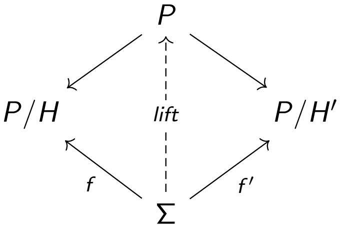

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(Abelian) T-duality: G, H, H' are tori
holonomy constraint = momentum quantization

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$$S(A) = \int_Y \Big(\frac{1}{2} \langle A, dA \rangle + \frac{1}{6} \langle [A, A], A \rangle \Big) \qquad A \in \Omega^1(Y, \mathfrak{g})$$

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needs a pseudo-Riemannian metric on $\Sigma \subset \partial Y$
and a reflection $\mathrm{R} : \mathfrak{g} \to \mathfrak{g}$ with $\mathrm{Tr}\,\mathrm{R} = 0$

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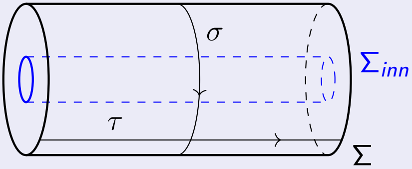
Hollow cylinder: The σ-model with the target G/H



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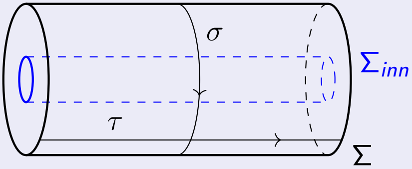

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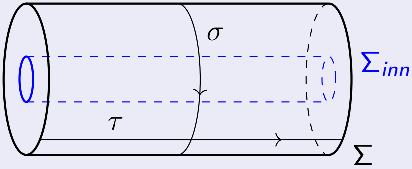

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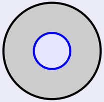
Phase space: moduli space of flat $\mathfrak{g}$-connections on an annulus $\cong T^*(L(G/H))$



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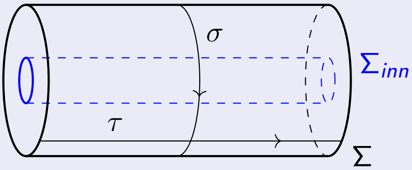

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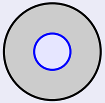

Full cylinder: The duality-invariant part (reduced phase space)

The full story: boundary of the AKSZ model

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AKSZ TFT

symplectic dg manifold $(\mathcal{V}, \omega, Q = \{F, \cdot\}) \rightsquigarrow$ TFT
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Our case: Y a solid cylinder

- $\deg \omega = 2$, Courant algebroid $V \to M$ ($\Gamma(V) = C^\infty(\mathcal{V})^1$)
- Boundary condition: a reflection $\mathrm{R} : V \to V$ (generalized metric) [generally: an exact Lagrangian submanifold of Maps$(T_x[1]\Sigma, \mathcal{V})$]
- Hamiltonian system (phase space = dg maps $T[1]disk \to \mathcal{V}$ mod homotopy rel boundary)

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- a principal dg $\mathbb{R}[2]$-bundle $X \to T[1]M$:
  $Q_X = d + \eta\,\partial_t$, $\eta \in \Omega^3(M)_{\text{closed}}$ (gerbe)
- $\mathcal{V} = T^*[2]X/\!/_1\mathbb{R}[2]$ (exact Courant algebroids)
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summary

(twisted) cotangent bundles = 2-dim σ-models on the boundary; other dg symplectic manifolds appear via symplectic reduction (or Lagrangian relations)
$\rightsquigarrow$ (almost) isomorphisms of Hamiltonian systems

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Hamiltonian systems related by (finite-dimensional) reduction

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AKSZ in $n+1$ dimensions

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boundary condition: exact Lagrangian submanifold
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Non-trivial duality via reduction of dg symplectic manifold (and a good choice of Λ_x's)?

Open ends: quantization

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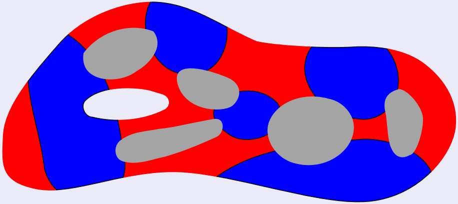

3-dim Y

Σ = gray part of ∂Y

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(Boltzmann weight)

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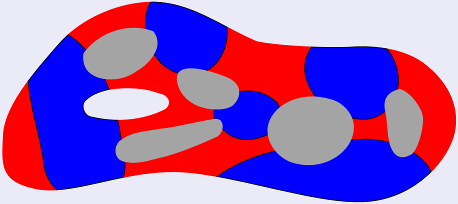

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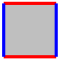
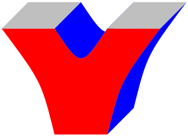
$H = Z($  $)$
Hopf algebra
$\mathfrak{h}, \mathfrak{h}^* \subset \mathfrak{g}$



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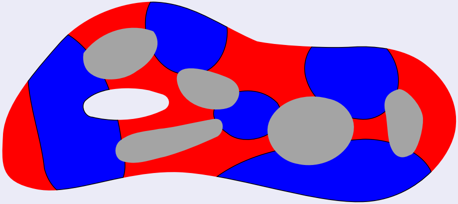

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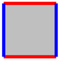
$H = Z($  $)$

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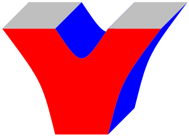

Thanks for your attention!