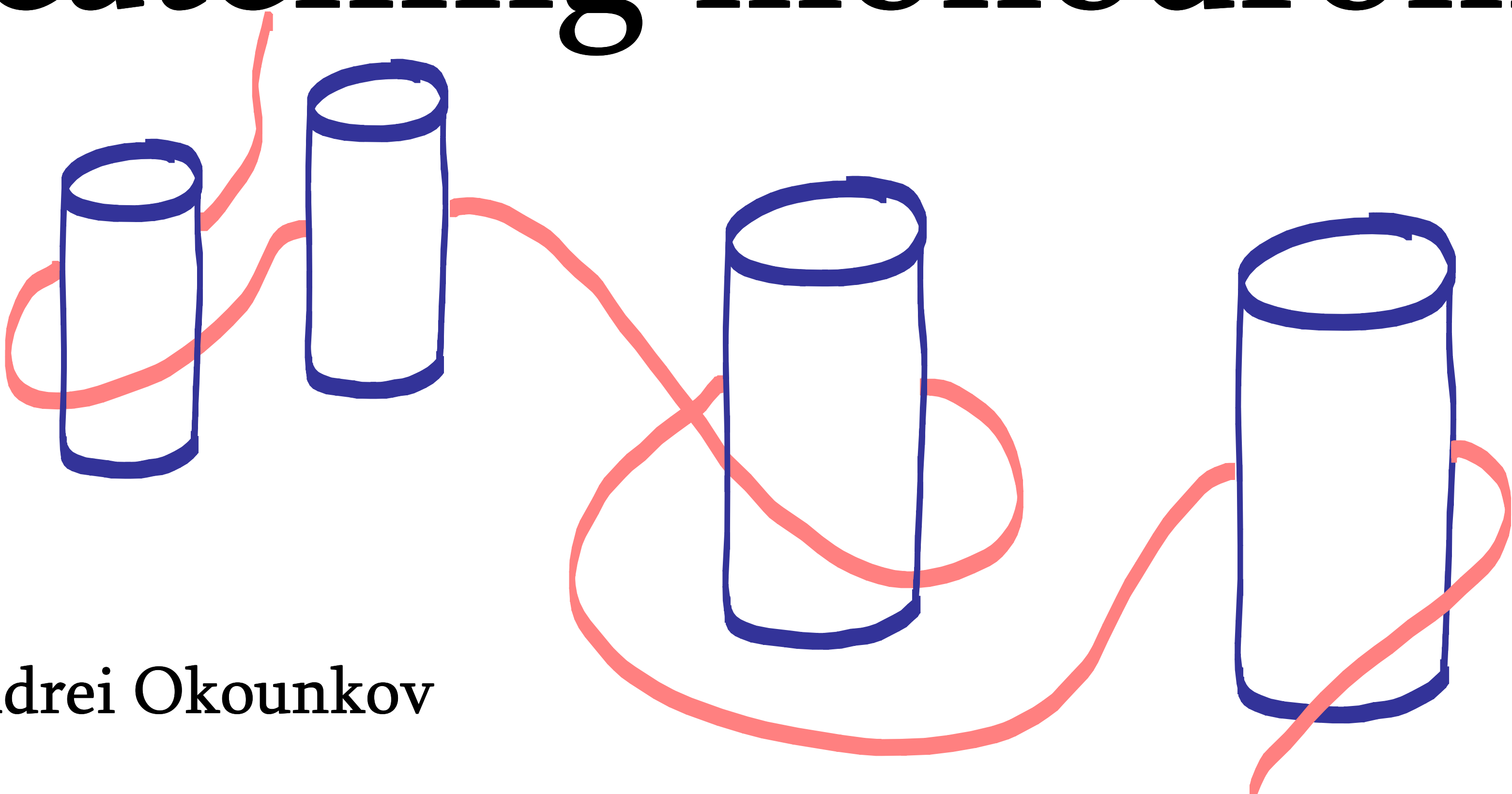
catching monodromy



Andrei Okounkov

Most things in life are not given to us all at once, but rather in installments or successive approximations, for example

$$e = 2.718281828459045\ldots$$

$$= 2 + \frac{7}{10} + \frac{1}{100} + \frac{8}{1000} + \frac{2}{10\,000} + \ldots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

where

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

Even if we want just a number, it is often a value of an important function at a special value of its argument / parameter

$$e = \exp(1), \quad \text{where}$$

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

makes sense for a matrix or a complex number

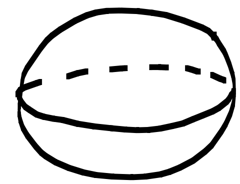
or

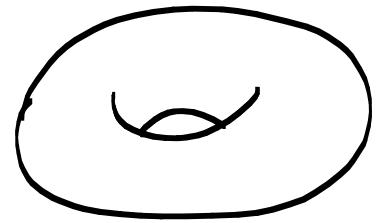
$$\pi = 4 \arctan(1) = 6 \arctan\left(\frac{1}{\sqrt{3}}\right) = \ldots$$

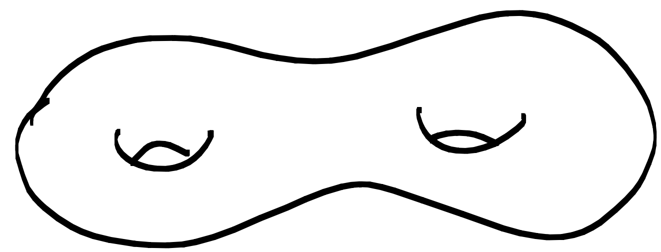
where

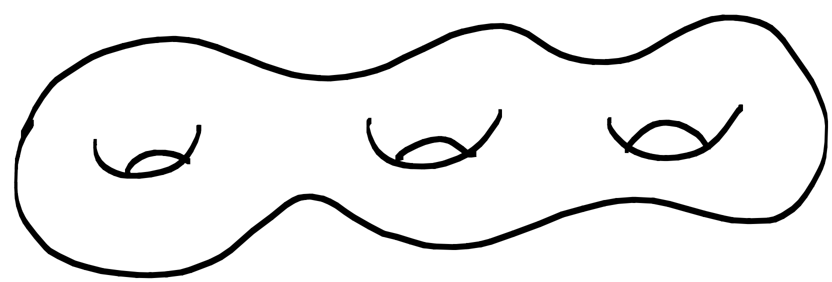
$$\arctan(x) = 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

In particular, in string theory

RESULT = contribution of 

+ g^2 contribution of 

+ g^4 contribution of 

+ g^6 contribution of 

+ ... ,

where g is a parameter (coupling constant)

We are thus facing an ageless problem in mathematics, which is to say something global about a function defined as a power series

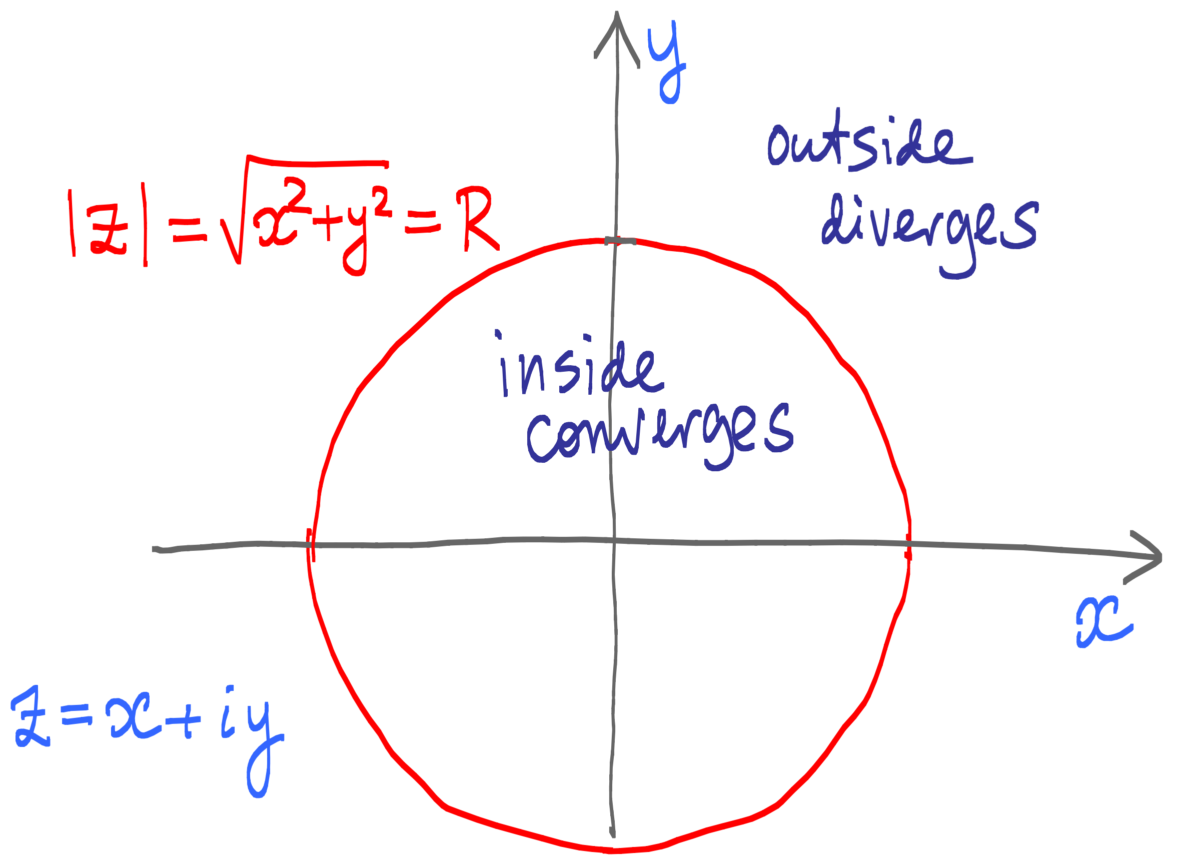
$$f(z) = \sum_{n \geqslant 0} c_n z^n$$

may be complex $z = x + iy$

$i^2 = -1$

even if one plans to plug in only real values of the argument, it is still very useful to know what the function does for complex z

A power series converges for z inside its radius of convergence R, which may be $R=0$ or $R=\infty$



The radius R reflects the exponential growth rate of the coefficients in $f(z)=\sum_n c_n z^n$

Basically

$$|c_n| \sim \left(\frac{1}{R}\right)^n,$$

think of

$$\frac{1}{1-z/R} = 1 + \frac{z}{R} + \frac{z^2}{R^2} + \frac{z^3}{R^3} + \ldots\ldots$$

For example

$$\exp(z) = \sum_{n \geqslant 0} \frac{z^n}{n!}$$

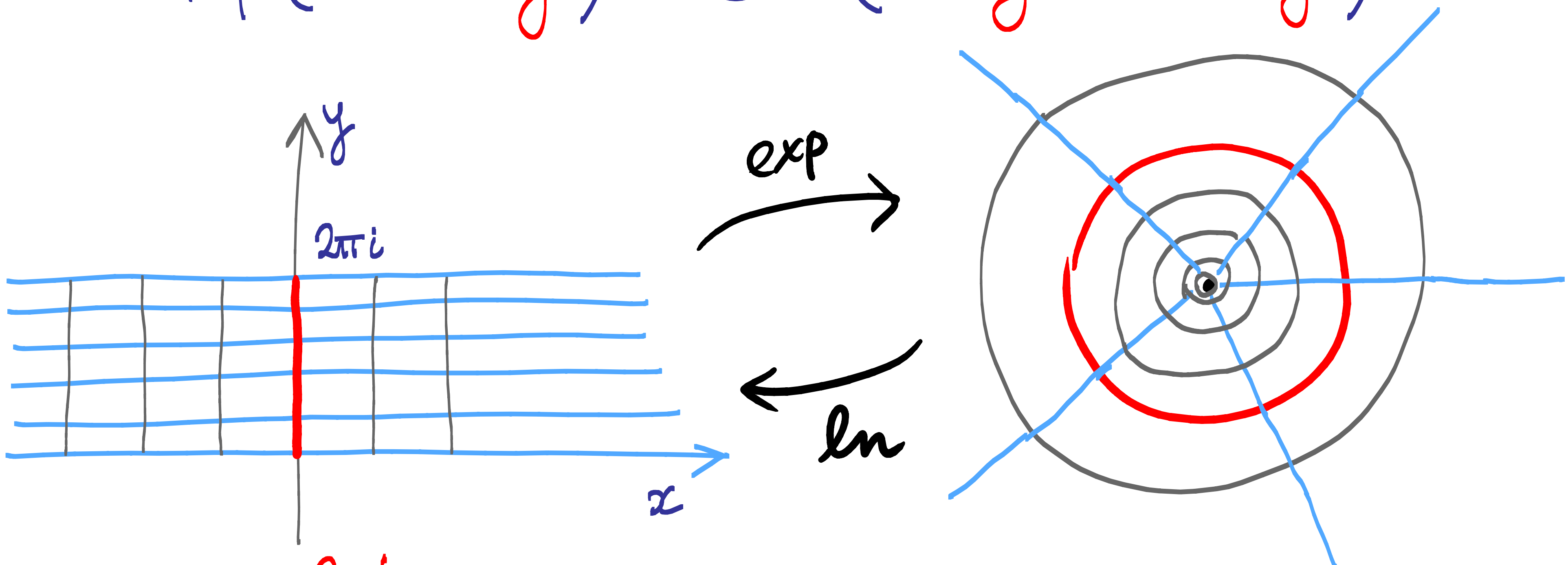
has an infinite radius of convergence because

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

grows faster than any exponential. This won't happen in string theory....

Incidentally, comparing series we conclude

$$\exp(x + iy) = e^{x}(\cos y + i \sin y)$$



And so $e^{2\pi i} = 1$ and, more generally, the logarithm of any complex number $z \neq 0$ is defined up to addition of $2\pi i$

Our next example is

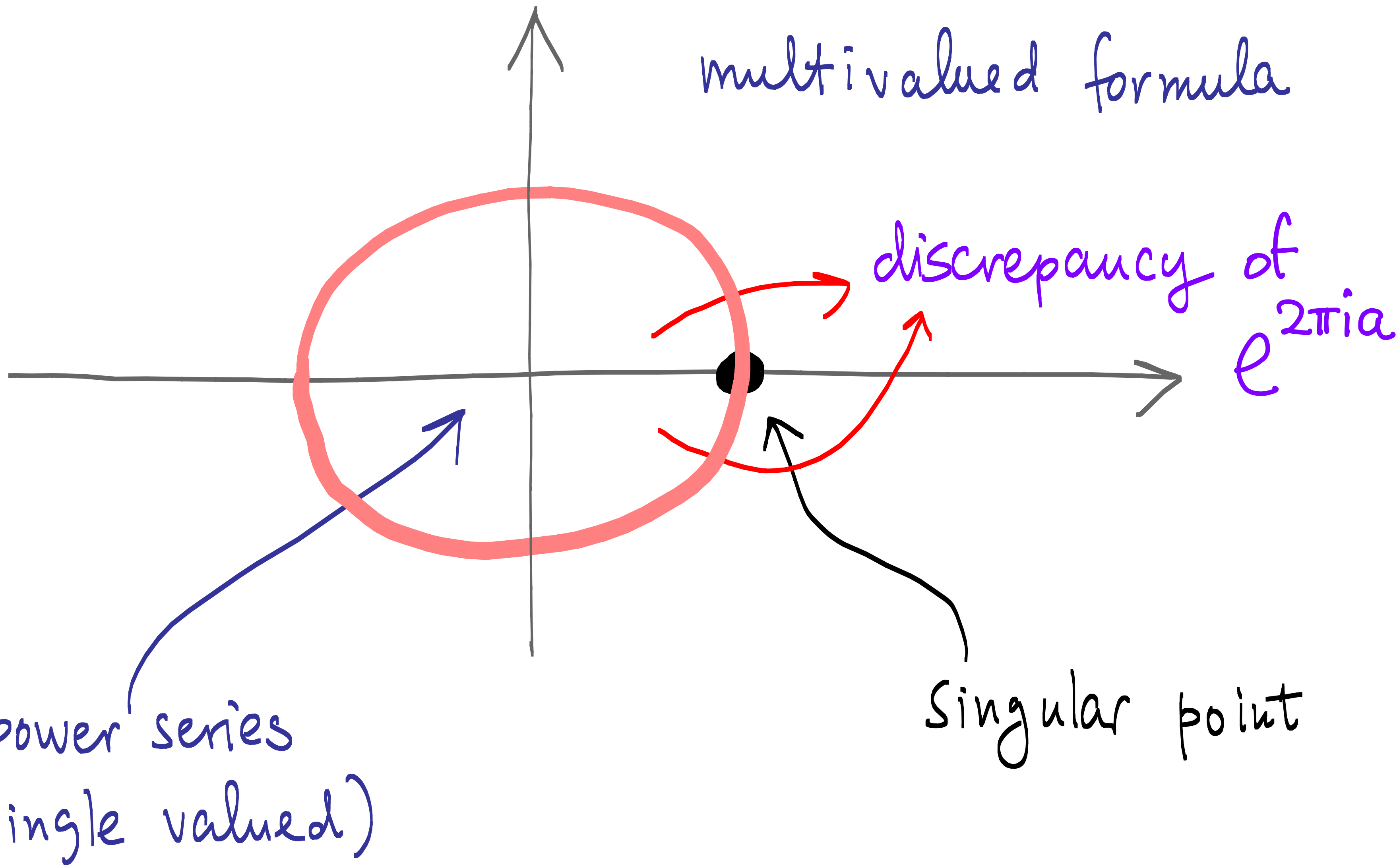
$$(1-z)^{-a} = 1 + az + \frac{a(a+1)}{2!} z^2 + \frac{a(a+1)(a+2)}{3!} z^3 + \dots$$

which has convergence radius $R = 1$, the distance to the singular point $z = 1$ $(a \neq 0, -1, -2, \dots)$

We can certainly compute

$$(1-z)^{-a} = \exp(-a \ln(1-z))$$

for any $z \neq 1$, except that is defined up to addition of $2\pi i$, so the function is defined up to a multiple of $\exp(2\pi i a)$



One says that the analytic continuation of the series $f(z) = 1 + az + \dots$ has monodromy $e^{2\pi i a}$ around the singular point $z = 1$

Already in this example, we see such basic facts as

Monodromy is trivial $\iff$ $e^{2\pi i a} = 1$

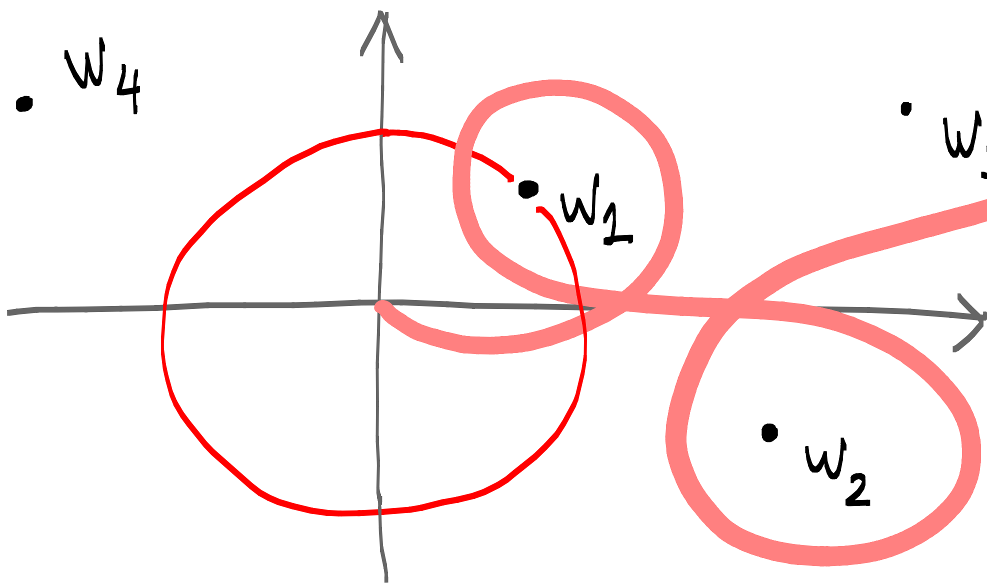
$\iff$ a is an integer

$\iff$ $f(z) = \dfrac{1}{(1-z)^a}$ is a rational function of z

Moreover, the monodromy determines $f(z)$ up to a rational function

Slightly more generally, the function

$$f(z) = \prod \left(1 - z/w_i\right)^{-a_i}$$



will pick up a factor of

$$e^{2\pi i a_i}$$

around each singularity $z = w_i$

A very special feature of this example is that the analytic continuation stays in a 1-dimensional space of functions and monodromies are 1×1 matrices

To see an example with 2×2 monodromy matrices, consider the hypergeometric function

$$F(z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{n! \, (c)_n} z^n ,$$

where

$$(a)_n = a(a+1)(a+2)\dots(a+n-1)$$

$$= \frac{\Gamma(a+n)}{\Gamma(a)}$$

Gamma function, it solves $a\Gamma(a) = \Gamma(a+1)$

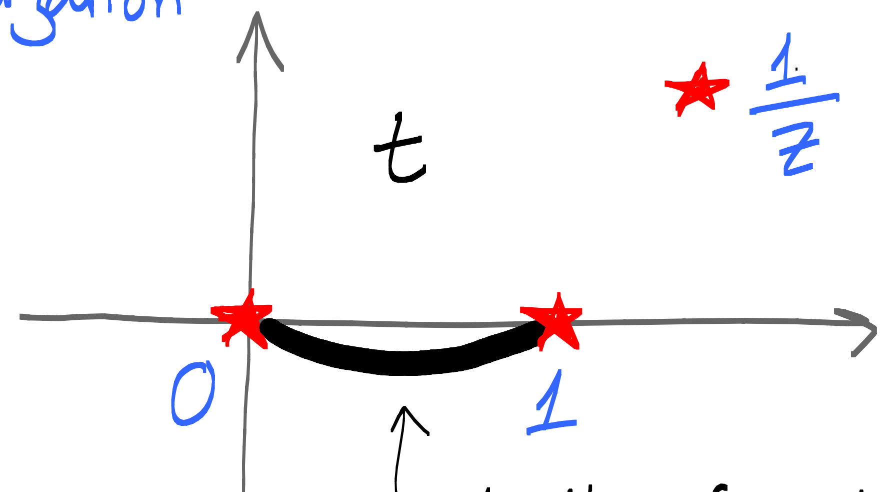
$\Gamma(n+1) = n!$

if $b=c$ get $(1-z)^{-a}$

The multivalued analytic continuation of $F(z)$ is given by the following integral

nice normalization

$$\frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\,F(z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\,dt$$



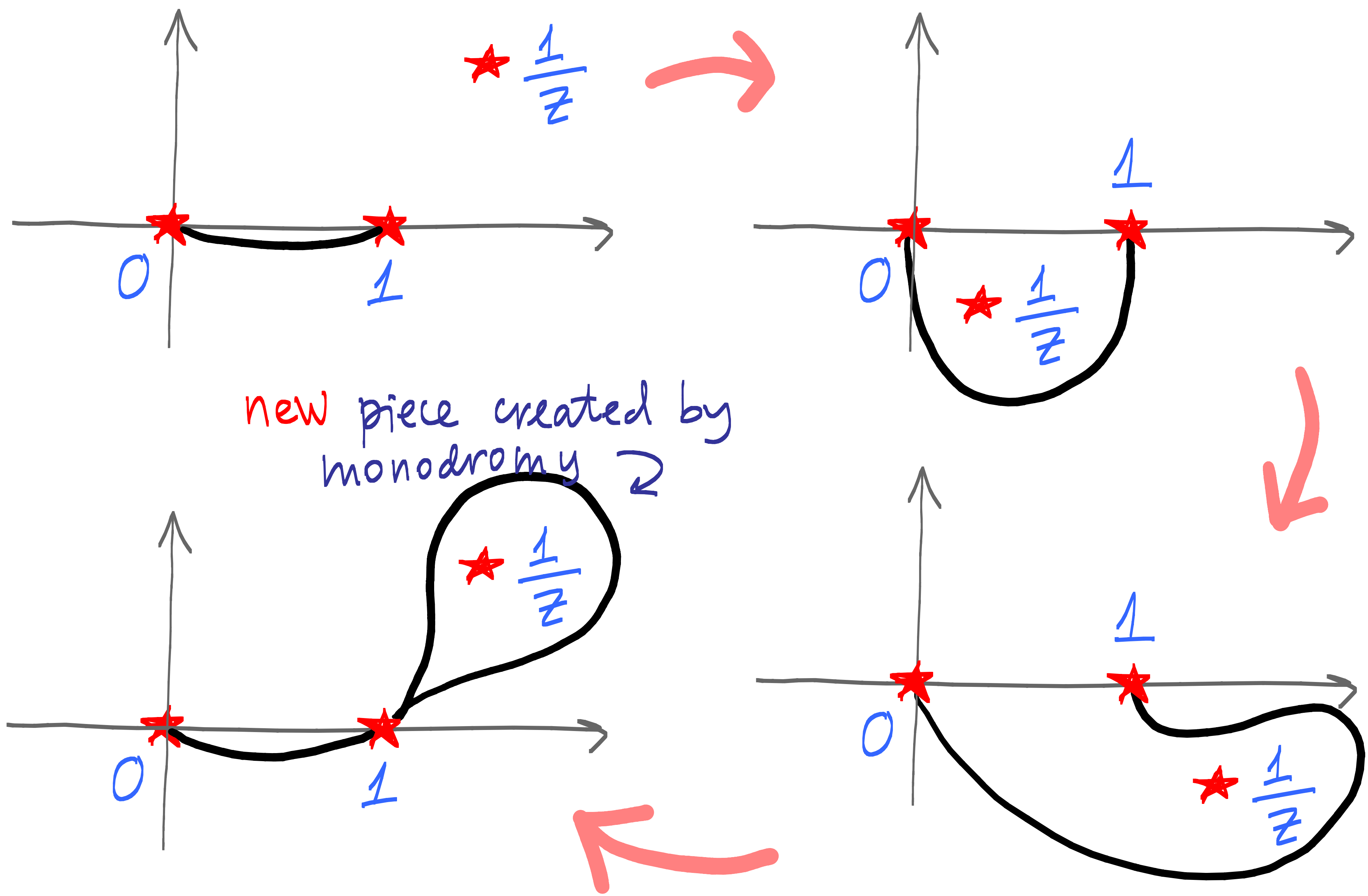
singularities of the integrand, plus the one at $t=\infty$

path of integration

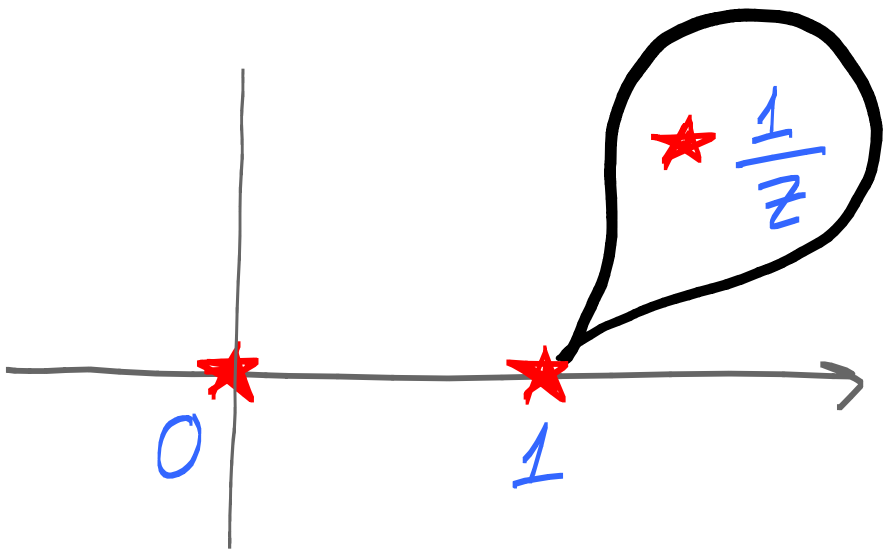
Monodromy of the integrand + choice of path $\Rightarrow$

$\Rightarrow$ monodromy of the integral

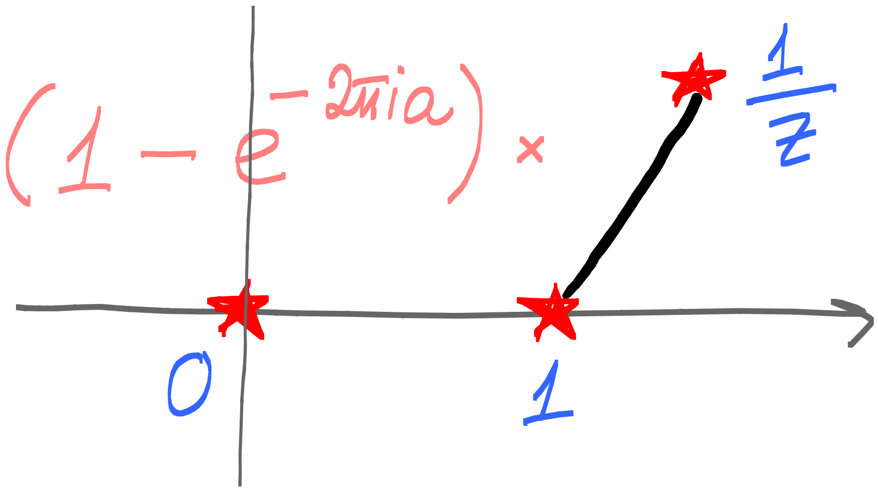
Now suppose z or, equivalently, $1/z$ goes around 1



By the monodromy properties of the integrand

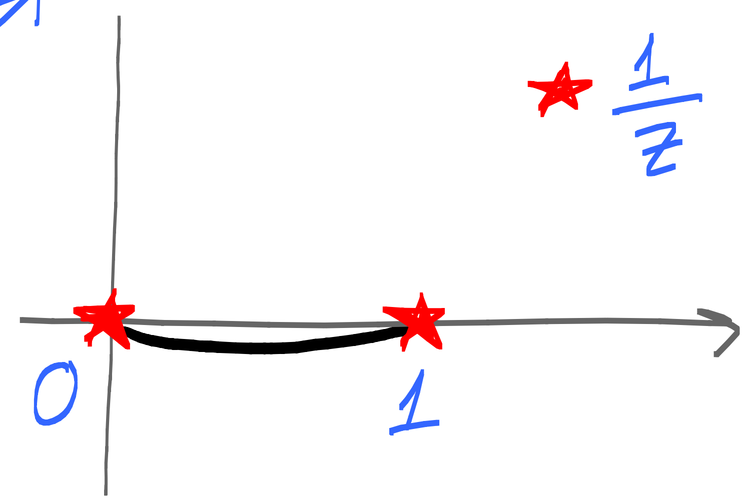


$$= \left(1 - e^{-2\pi i a}\right) \times$$



And, more generally, in the basis of these two integrals the monodromy matrices are Laurent polynomials in

$$e^{2\pi i a},\ e^{2\pi i b},\ e^{2\pi i c}$$



There are vast generalizations of the hypergeometric function given by much more convoluted, yet similar in spirit multivariate sums or integrals

The monodromy of any one of them may, in principle, be computed as a collection of matrices depending polynomially on $A_k = e^{2\pi i a_k}$

parameters like a, b, c

heavy stuff, in need of synthetic understanding

In situations of great practical interest, such synthetic understanding of monodromy comes from the idea of

categorification !

This is a deep, old, and powerful idea, which one can try to explain as follows

Let R be an algebra over a field $\mathbb{k}$, such as

(1) $R = \mathbb{k}$ itself or $\mathrm{Mat}(n \times n, \mathbb{k})$, or

(2) $R = \mathbb{k}S(n)$ group algebra of permutation group, or

(3) $R = \mathbb{k}[x_1, \dots, x_n]$ poly's in n variables, or

(★) $R = \mathbb{k}S(n) \ltimes \mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_n]$

An R-module M is, by definition, a $\mathbb{k}$-vector space with a map

$$R \to \mathrm{Mat}(V)$$

May form a complicated category (many different modules, many maps between them). Lots of structure!

Simpler $\rightarrow$ Grothendieck group K

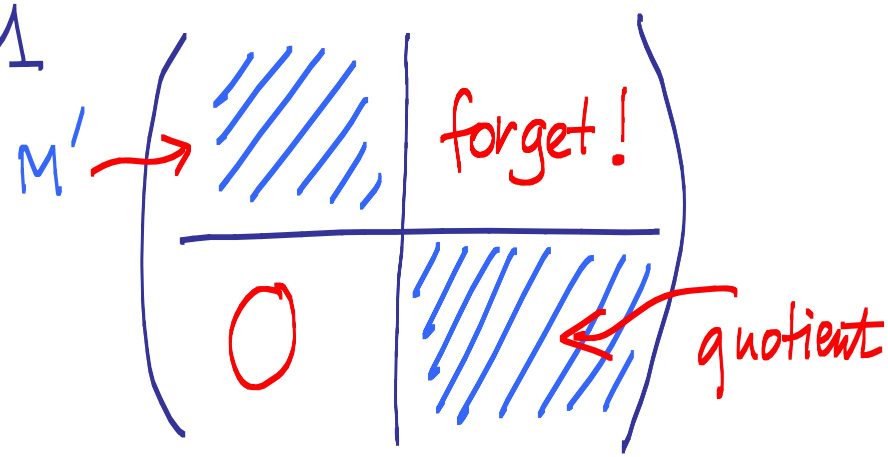
= abelian group generated by symbols $[M]$ where M is a module

with the relation

$$[M] = [M'] + [M/M']$$

for any submodule of M

i.e. R acts by matrices of the form



If R is graded, that is $R = \bigoplus_{d \in \mathbb{Z}^m} R_d$

$R_d R_{d'} \subset R_{d+d'}$

then we can similarly

require $M = \bigoplus M_d$ $\qquad R_d M_{d'} \subset M_{d+d'}$

Shifts of grading $\qquad M(d')_d = M_{d'+d}$

make the K-group of graded modules a module

over $\mathbb{Z}[A_1^{\pm 1}, \dots, A_m^{\pm 1}]$

shifts the grading by $(0, 0, \dots, \pm 1)$

In our examples, nontrivial gradings are

(1) $R = \mathrm{Mat}(n \times n, \Bbbk)$ none

(2) $R = \Bbbk S(n)$ none

(3) $R = \Bbbk[x_1, \dots, x_n]$ $\mathbb{Z}^n$ by degree in each x_i

(★) $R = \Bbbk S(n) \ltimes \Bbbk[x_1, .., x_n, y_1, \dots, y_n]$

$\mathbb{Z}^2$ by total degree in x and total degree in y

And the K-groups of graded finite-dimensional modules are

(1) $R = \mathrm{Mat}(n \times n, \mathbb{k})$ $\qquad \mathbb{Z}$

(2) $R = \mathbb{k}S(n)$ $\qquad \mathbb{Z}^{\#\text{ of partitions of } n}$

char $p \gg 0$

(3) $R = \mathbb{k}[x_1, \ldots, x_n]$ $\qquad \mathbb{Z}[A_1^{\pm 1}, \ldots, A_n^{\pm 1}]$

(★) $R = \mathbb{k}S(n) \ltimes \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$

a massive loss of information here $\longrightarrow$ $\mathbb{Z}[A_1^{\pm 1}, A_2^{\pm 1}]^{\#\text{ of partitions}}$

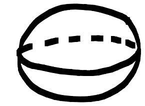
categorical statements much richer than their image here

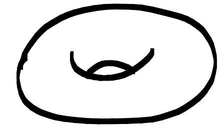
By definition, categorification is a lift of the kind

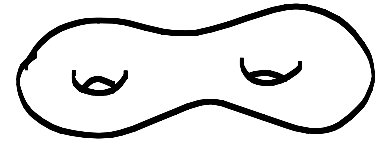
$$
\begin{tikzcd}
& \text{Monodromy} \arrow[dl] \arrow[dr] & \\
\mathrm{Mat}\left(\begin{array}{c}\#\text{ of} \\ \text{partitions}\end{array}, \mathbb{Z}[A_1^{\pm 1}, A_2^{\pm 1}]\right)^{\times} & & \begin{array}{l}\text{(derived)} \\ \text{automorphisms} \\ \text{of the category of} \\ R^{\star}\text{-modules}\end{array} \arrow[ll]
\end{tikzcd}
$$

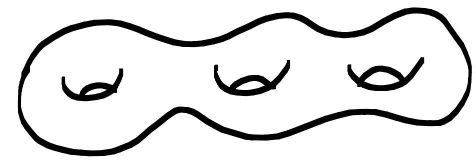
lots more structure here

In particular, in string theory

RESULT = contribution of 

+ g^2 contribution of 

+ g^4 contribution of 

+ g^6 contribution of 

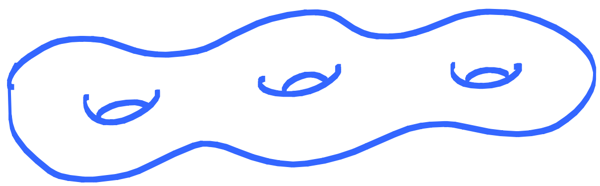
+ ... ,

where g is a parameter (coupling constant)

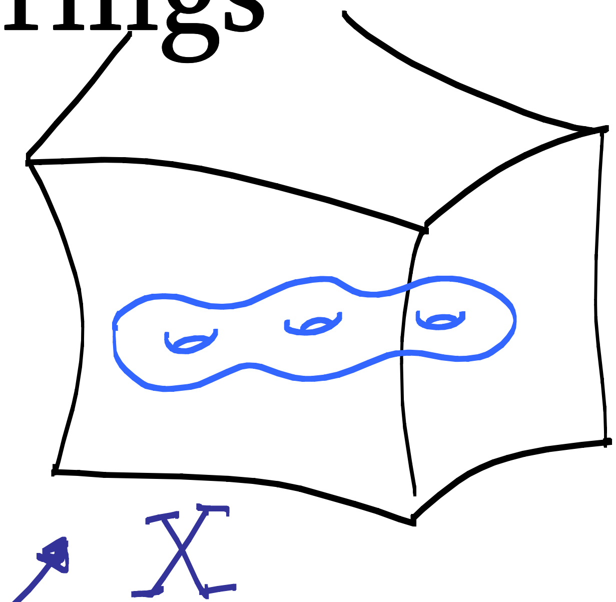
Now, how can this be useful in string theory?

... in topological strings

One integrates over



holomorphic maps $\longrightarrow$

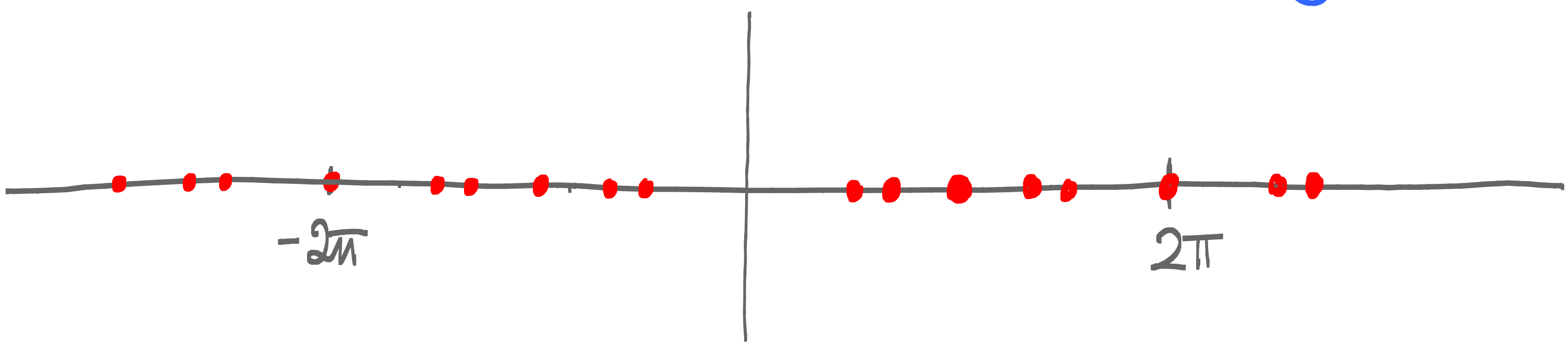


algebraic variety, need not be Calabi-Yau

These form a finite-dimensional family for fixed genus and degree $\approx$ area = energy

We will sum over all genera for fixed/bounded degree

General conjectures (known in very many cases) imply that the singularities in the coupling constant g

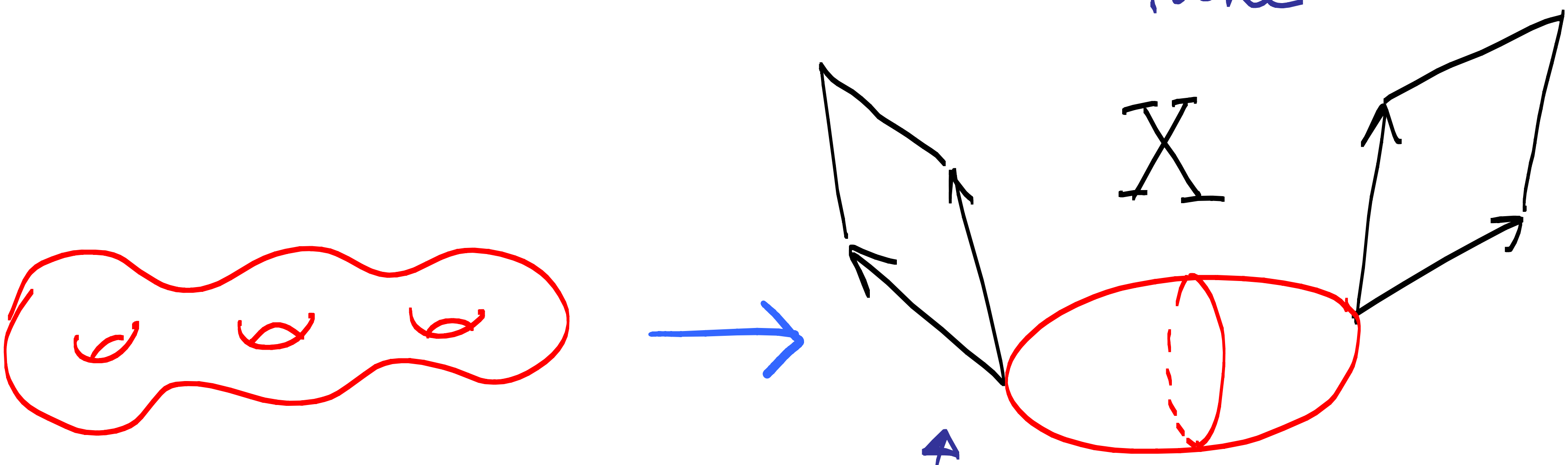


are at $\quad g = 2\pi \frac{a}{b} \quad , \quad a, b \in \mathbb{Z}$

where b is bounded in terms of the degree

Further, there is no monodromy (!) around these singularities, the whole thing is a rational function of $\quad z = -\exp(ig)$

While the whole thing is a rational function, its natural parts have the same singularities <u>and</u> interesting monodromy around them. For example, take

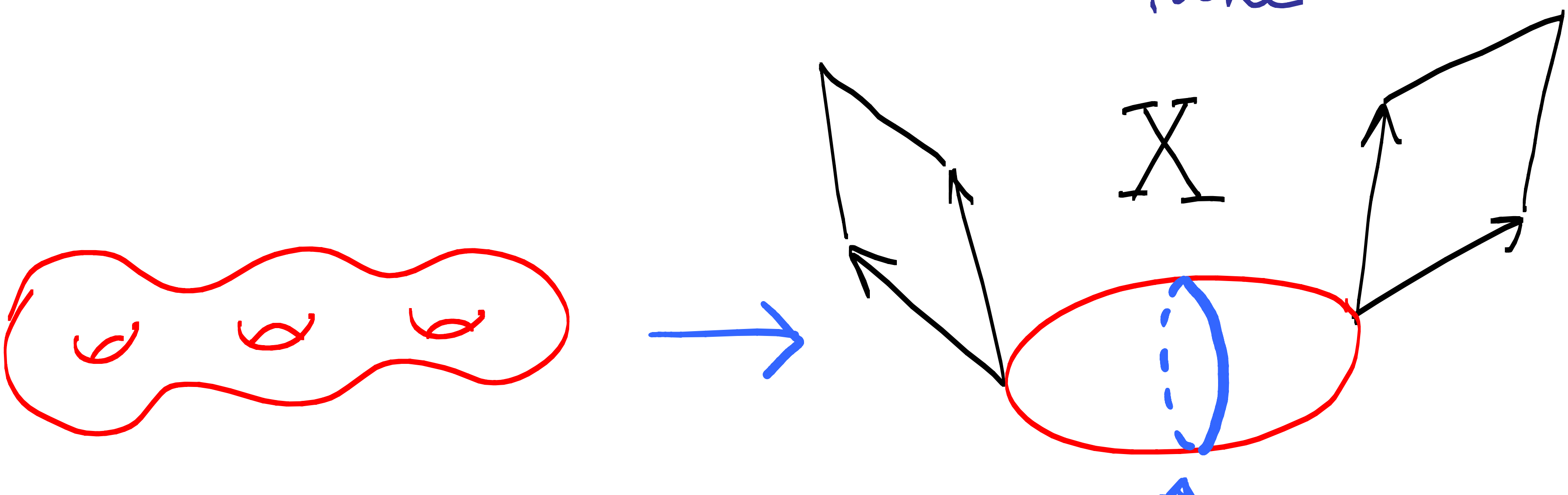


rank 2 vector bundle over $\mathbb{P}^1$

$\mathrm{rk}\,\mathrm{Aut}(X) = 3$

and one can do equivariant counts that depend nontrivially on 2 variables

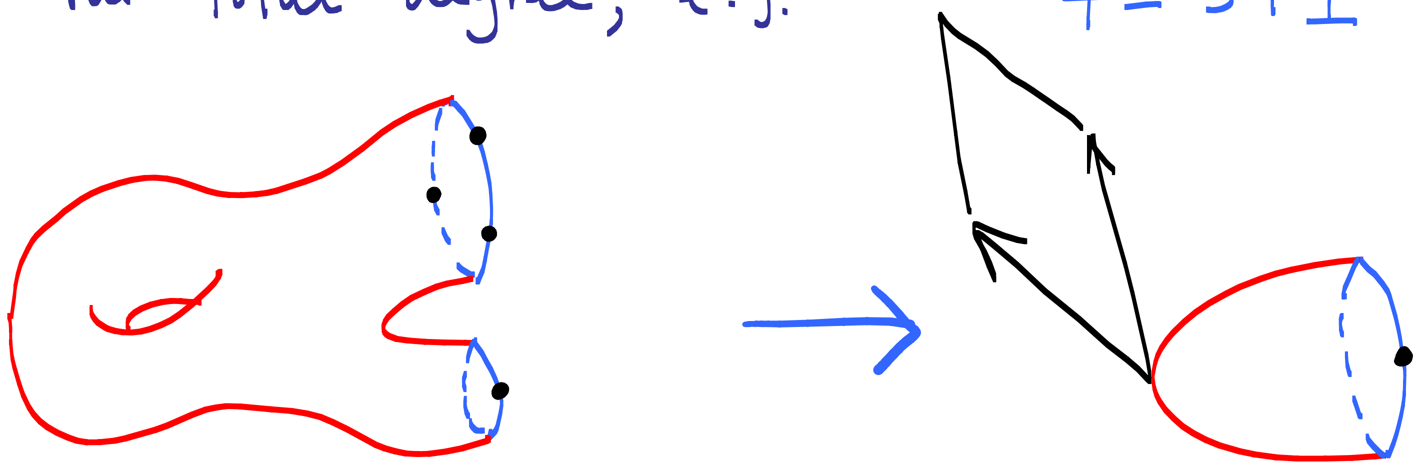
While the whole thing is a rational function, its natural parts have the same singularities and interesting monodromy around them. For example, take



This geometry we can cut along the equator of $\mathbb{P}^1$ to get two pieces that give ...

Equivariant counts of surfaces wrapping the disk.

The degrees of the boundary maps give a partition of the total degree, e.g. $4 = 3+1$



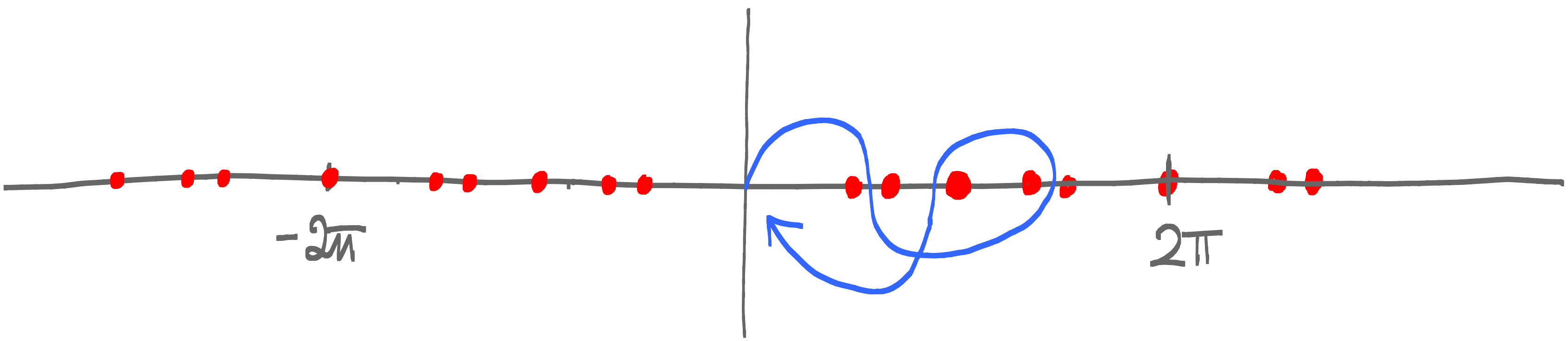
and hence a set of fun ions of g labelled by partitions

As a part of the GW=DT package (known here) these are

hypergeometric functions of $z = -e^{ig}$

associated to the Hilbert scheme of points in $\mathbb{C}^2$

These functions transform among themselves as we loop around the singularities in the g plane



and this monodromy has a beautiful categorification!

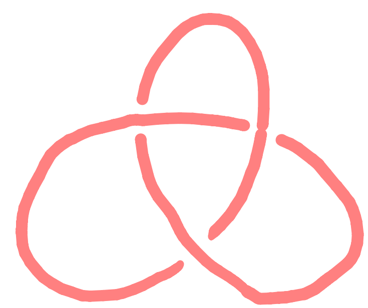
Namely ...

Building on very deep insights of

- Kontsevich
- Bridgeland
- Bezrukavnikov

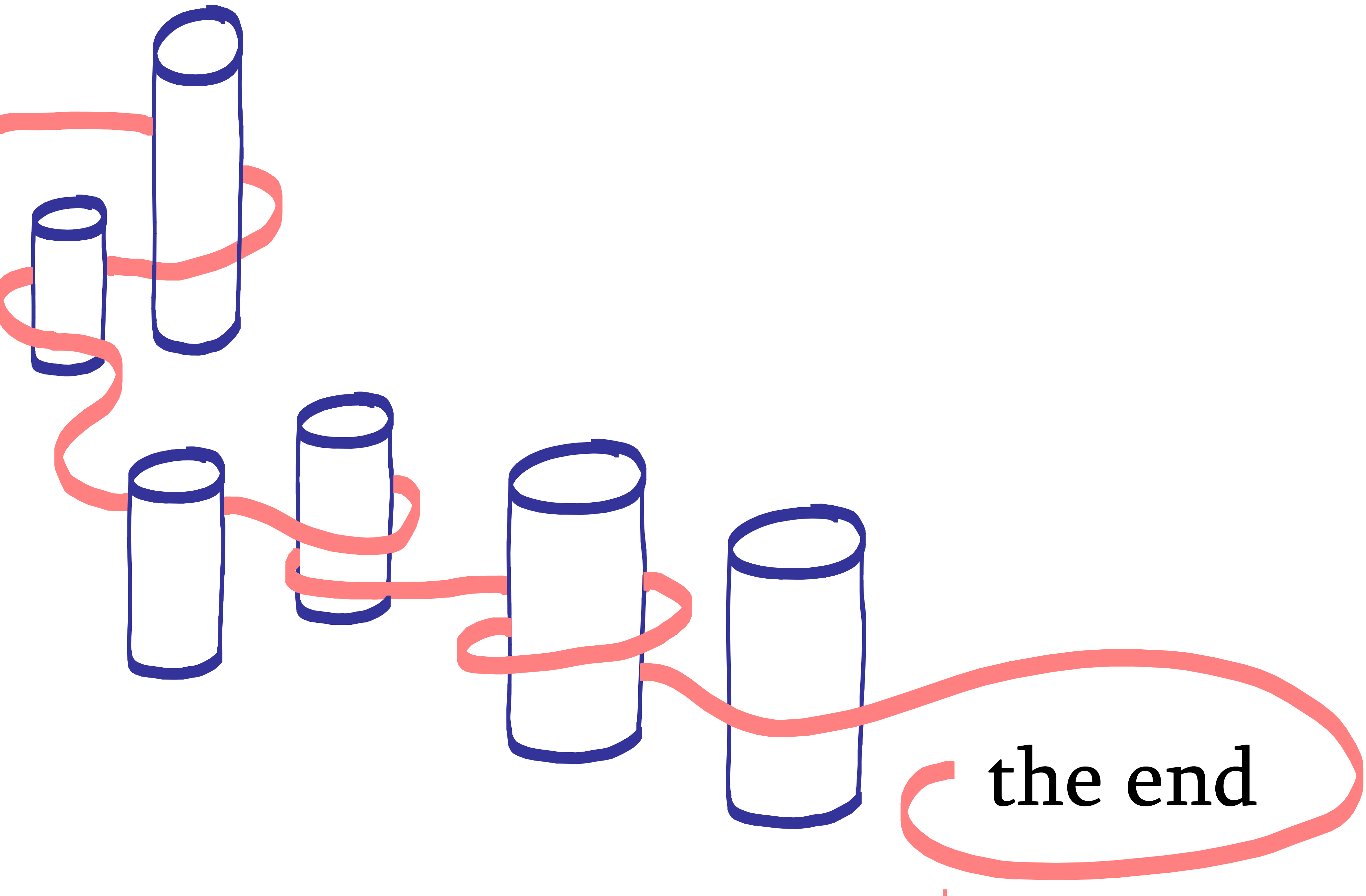
one could (conjecturally) identify it with specific derived equivalences of $R^{\star}$-modules constructed by Bezrukavnikov via representation theory of Cherednik algebras in prime characteristic ...

includes reflections in



Mathematically rather challenging....

However, after years of thinking about it and various technical advances (e.g. stable envelopes) filling the right pieces of the puzzle, I believe Bezrukavnikov and I have a proof of this (and many similar) statement....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