Low-Energy Behavior of Massless Particles from Gauge Invariance

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Plan of the talk

1. Introduction
2. Scattering of a photon and $n$ scalar particles
3. Scattering of a graviton and $n$ scalar particles
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6. What about soft theorems in string theory?
7. Soft theorem for dilaton and $B_{\mu\nu}$
8. Soft theorem for $B_{\mu\nu}$
9. Comments on loop corrections: gauge theory
10. Comments on loop corrections: gravity
11. Conclusions and Outlook
In particle physics we deal with **three kinds of symmetries**. They all leave the action invariant, but have **different physical consequences**.

GLOBAL symmetries as isotopic spin (if $m_u = m_d$) in 2-flavor QCD.

Unique vacuum annihilated by the symmetry gener.: $Q_a |0\rangle = 0$

Particles are classified according to multiplets of this symmetry and all particles of a multiplet have the same mass.

If $m_u = m_d$, QCD is invariant under an $SU(2)_V$ flavor symmetry. and the proton and the neutron would have the same mass.

This is not the case because the quark mass matrix breaks explicitly $SU(2)$ ($m_u \neq m_d$).
Then, we have the **GLOBAL spontaneously broken symmetries**.

For zero quark mass, QCD with two flavors is invariant under $SU(2)_L \times SU(2)_R$.

Degenerate vacua: $Q_a |0\rangle = |0'\rangle$.

Not realized in the spectrum, but presence in the spectrum of massless particles, called **Goldstone bosons**.

They are the **pions** in QCD with 2 flavors.

This is one physical consequence of the spontaneous breaking.

Another one is the existence of low-energy theorems.

The $\pi \pi$ scattering amplitude is fixed at low energy.

Actually the scattering amplitude for massless pions is zero at low energy because Goldstone bosons interact with deriviative coupling (shift symmetry $\leftrightarrow$ Adler zeroes).

If one introduces a mass term, breaking explicitly chiral symmetry and giving a small mass to the pion, one gets the two Weinberg scattering lengths:

$$a_0 = \frac{7 m_\pi}{32 \pi F_\pi^2} \quad ; \quad a_2 = -\frac{m_\pi}{16 \pi F_\pi^2}$$
Finally, we have the LOCAL gauge symmetries for massless spin 1 and spin 2 particles.

Local gauge invariance is necessary to reconcile the theory of relativity with quantum mechanics.

It allows a fully relativistic description, eliminating, at the same time, the presence of negative norm states in the spectrum of physical states.

Although described by $A_\mu$ and $G_{\mu\nu}$, both photons and gravitons have only two physical degrees of freedom in $d=4$.

Another consequence of gauge invariance is charge conservation that, however, follows from the global part of the gauge group.

Yet another physical consequence of local gauge invariance is the existence of low-energy theorems for photons and gravitons: [F. Low, 1958; S. Weinberg, 1964]
Let us consider Compton scattering on spinless particles.

The scattering amplitude $M_{\mu\nu}$ is gauge invariant:

$$k_1^\mu M_{\mu\nu} = k_2^\nu M_{\mu\nu} = 0$$

The previous conditions determine the scattering amplitude for zero frequency photons and one gets the Thompson cross-section:

$$\sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2 = \frac{8\pi}{3} r_{cl}^2$$

where $r_{cl}$ is the classical radius of a point particle of mass $m$ and charge $e$. 
The interest on the soft theorems was recently revived by [Cachazo and Strominger, arXiv:1404.1491[hep-th]].

They study the behavior of the $n$-graviton amplitude when the momentum $q$ of one graviton becomes soft ($q \sim 0$).

The leading term $O(q^{-1})$ was shown to be universal by Weinberg in the sixties.

They suggest a universal formula for the subleading term $O(q^0)$.

They speculate that, as the leading term, it may be a consequence of BMS symmetry of asymptotically flat space-times.

In this seminar we show that the first three leading terms of order $q^{-1}, q^0, q$ are a direct consequence of gauge invariance.

This result is valid for an arbitrary space-time dimension $d$.

In the second part study soft theorems in string theory, in particular for dilaton and $B_{\mu\nu}$. 
The scattering amplitude $M_{\mu}(q; k_1 \ldots k_n)$, involving one photon and $n$ scalar particles, consists of two pieces:

$$
A_{\mu}^I(q; k_1, \ldots, k_n) = \sum_{i=1}^n e_i \frac{k_i^\mu}{k_i \cdot q} T_n(k_1, \ldots, k_i + q, \ldots, k_n)
$$

$$
+ N_{\mu}^I(q; k_1, \ldots, k_n).
$$

and must be gauge invariant for any value of $q$:

$$
q_\mu A_{\mu}^I = \sum_{i=1}^n e_i T_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_\mu N_{\mu}^I(q; k_1, \ldots, k_n) \equiv 0.
$$
Expanding around \( q = 0 \), we have

\[
0 = \sum_{i=1}^{n} e_i \left[ T_n(k_1, \ldots, k_i, \ldots, k_n) + q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_i, \ldots, k_n) \right]
\]

\[
+ q_\mu N_\mu^n (q = 0; k_1, \ldots, k_n) + \mathcal{O}(q^2)
\]

At leading order, this equation is

\[
\sum_{i=1}^{n} e_i = 0,
\]

which is simply a statement of charge conservation [Weinberg, 1964]

At the next order, we have

\[
q_\mu N_\mu^n (0; k_1, \ldots, k_n) = - \sum_{i=1}^{n} e_i q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n).
\]
This equation tells us that $N_n^{\mu}(0; k_1, \ldots, k_n)$ is entirely determined in terms of $T_n$ up to potential pieces that are separately gauge invariant.

It can be shown that they are of higher order in q.

We can therefore remove the $q_\mu$ leaving

$$N_n^{\mu}(0; k_1, \ldots, k_n) = - \sum_{i=1}^{n} e_i \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n),$$

thereby determining $N_n^{\mu}(0; k_1, \ldots, k_n)$ as a function of the amplitude without the photon.
Inserting this into the original expression yields

\[
A^\mu_n(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[ k_i^\mu - iq_\nu J_i^{\mu\nu} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q),
\]

where

\[
J_i^{\mu\nu} \equiv i \left( k_i^\mu \frac{\partial}{\partial k_i^\nu} - k_i^\nu \frac{\partial}{\partial k_i^\mu} \right)
\]

is the orbital angular-momentum operator.

The amplitude with a soft photon with momentum \( q \) is entirely determined, up to \( \mathcal{O}(q^0) \), in terms of the amplitude \( T_n(k_1, \ldots, k_n) \), involving \( n \) scalar particles and no photon.

This goes under the name of F. Low’s low-energy theorem.
Low’s theorem for photons is unchanged at loop level.

Can we get any further information at higher orders in the soft expansion?

One order further in the expansion, we find the extra condition,

\[
\frac{1}{2} \sum_{i=1}^{n} e_i q_{\mu} q_{\nu} \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + q_{\mu} q_{\nu} \frac{\partial N_{n\mu}^\mu}{\partial q_{\nu}} (0; k_1, \ldots, k_n) = 0.
\]

This implies

\[
\sum_{i=1}^{n} e_i \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + \left[ \frac{\partial N_{n\mu}^\mu}{\partial q_{\nu}} + \frac{\partial N_{n\nu}^\nu}{\partial q_{\mu}} \right] (0; k_1, \ldots, k_n) = 0,
\]
Gauge invariance determines only the symmetric part of the quantity \( \frac{\partial N_\mu^\nu}{\partial q_\mu}(0; k_1, \ldots, k_n) \).

The antisymmetric part is not fixed by gauge invariance.

Then, up to this order, we have

\[
A^\mu_n(q; k_1, \ldots, k_n) = \sum_{i=1}^n \frac{e_i}{k_i \cdot q} \left[ k_i^\mu - iq_\nu J^\mu\nu_i \left( 1 + \frac{1}{2} q_\rho \frac{\partial}{\partial k_i^\rho} \right) \right] T_n(k_1, \ldots, k_n)
\]

\[
+ \frac{1}{2} q_\nu \left[ \frac{\partial N_\mu^\mu_n}{\partial q_\nu} - \frac{\partial N_\nu^\nu_n}{\partial q_\mu} \right](0; k_1, \ldots, k_n) + O(q^2).
\]
Get an amplitude by contracting $A_{n}^{\mu}(q; k_{1}, \ldots, k_{n})$ with the photon polarization $\varepsilon_{q\mu}$.

Soft-photon limit:

$$A_{n}(q; k_{1}, \ldots, k_{n}) \rightarrow \left[ S^{(0)} + S^{(1)} \right] T_{n}(k_{1}, \ldots, k_{n}) + \mathcal{O}(q),$$

where

$$S^{(0)} \equiv \sum_{i=1}^{n} e_{i} \frac{k_{i} \cdot \varepsilon_{q}}{k_{i} \cdot q},$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} e_{i} \frac{\varepsilon_{q\mu} q_{\nu} J_{i}^{\mu\nu}}{k_{i} \cdot q},$$

where $J_{i}^{\mu\nu}$ is the angular momentum.
One graviton and $n$ scalar particles

- In the case of a graviton scattering on $n$ scalar particles, one can write

$$M_n^{\mu\nu}(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} T_n(k_1, \ldots, k_i + q, \ldots, k_n)$$

$$+ N_n^{\mu\nu}(q; k_1, \ldots, k_n),$$

- $N_n^{\mu\nu}(q; k_1, \ldots, k_n)$ is symmetric under the exchange of $\mu$ and $\nu$.
- On-shell gauge invariance implies

$$0 = q_\mu M_n^{\mu\nu}(q; k_1, \ldots, k_n)$$

$$= \sum_{i=1}^{n} k_i^\nu T_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_\mu N_n^{\mu\nu}(q; k_1, \ldots, k_n).$$

- More precisely, gauge invariance imposes:

$$q_\mu M_n^{\mu\nu}(q, k_i) = f(q, k_i) q^\nu \implies q_\mu (M_n^{\mu\nu} - f(k_i) \eta^{\mu\nu}) = 0$$

but the extra term is irrelevant for gravitons: $\epsilon^G_{\mu\nu} \eta_{\mu\nu} = 0$. 
At leading order in $q$, we then have
\[ \sum_{i=1}^{n} k_i^\mu = 0 , \]

It is satisfied due to momentum conservation.

Different couplings to different particles would have prevented the leading term to vanish: Gravitons have universal coupling [Weinberg, 1964].

At first order in $q$, one gets
\[ \sum_{i=1}^{n} k_i^\nu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) + N_{\mu\nu}^n (0; k_1, \ldots, k_n) = 0 , \]

while at second order in $q$, it gives
\[ \sum_{i=1}^{n} k_i^\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\rho}} T_n(k_1, \ldots, k_n) + \left[ \frac{\partial N_{\mu\nu}^n}{\partial q_\rho} + \frac{\partial N_{\rho\nu}^n}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) = 0 . \]
As for the photon, this is true up to gauge-invariant contributions to $N_{\mu\nu}^n$.

However, the requirement of locality prevents us from writing any expression that is local in $q$ and not sufficiently suppressed in $q$.

Using the previous equations, we write the expression for a soft graviton as

$$M_{\mu\nu}^n(q; k_1, \ldots, k_n) = \sum_{i=1}^n \frac{k_i^\nu}{k_i \cdot q} \left[ k_i^\mu - iq_\rho J_i^{\mu\rho} \left( 1 + \frac{1}{2} q_\sigma \frac{\partial}{\partial k_i^\sigma} \right) \right] T_n(k_1, \ldots, k_n)$$

$$+ \frac{1}{2} q_\rho \left[ \frac{\partial N_{\mu\nu}^n}{\partial q_\rho} - \frac{\partial N_{\rho\nu}^n}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) + O(q^2).$$

This is essentially the same as for the photon except that there is a second Lorentz index in the graviton case.

Unlike the case of the photon, the antisymmetric quantity in the last line of the previous equation can also be determined from the amplitude $T_n(k_1, \ldots, k_n)$ without the graviton.
From the equation above (implied by gauge invariance) and remembering that $N_{\mu\nu}^n$ is a symmetric matrix, one gets the following relation:

$$-i \sum_{i=1}^{n} J_i^{\mu\rho} \frac{\partial}{\partial k_{i\nu}} T_n(k_1, \ldots, k_n) = \left[ \frac{\partial N_{\rho\nu}^n}{\partial q_{\mu}} - \frac{\partial N_{\mu\nu}^n}{\partial q_{\rho}} \right] (0; k_1, \ldots, k_n),$$

which fixes the antisymmetric part of the derivative of $N_{\mu\nu}^n$ in terms of the amplitude $T_n(k_1, \ldots, k_n)$ without the graviton.
Using the previous equation, we can then rewrite the terms of $O(q)$ as follows:

\[
M_{\mu\nu}^n(q; k_1, \ldots, k_n) \big|_{O(q)} = -\frac{i}{2} \sum_{i=1}^{n} \frac{q_\rho q_\sigma}{k_i \cdot q} \left[ k_i^{\nu} J_i^{\mu\rho} \frac{\partial}{\partial k_i^{\sigma}} - k_i^{\sigma} J_i^{\mu\rho} \frac{\partial}{\partial k_i^{\nu}} \right] T_n(k_1, \ldots, k_n)
\]

\[
= -\frac{i}{2} \sum_{i=1}^{n} \frac{q_\rho q_\sigma}{k_i \cdot q} \left[ J_i^{\mu\rho} k_i^{\nu} \frac{\partial}{\partial k_i^{\sigma}} - (J_i^{\mu\rho} k_i^{\sigma}) \frac{\partial}{\partial k_i^{\nu}} \right] T_n(k_1, \ldots, k_n)
\]

\[
- J_i^{\mu\rho} k_i^{\sigma} \frac{\partial}{\partial k_i^{\nu}} + (J_i^{\mu\rho} k_i^{\sigma}) \frac{\partial}{\partial k_i^{\nu}} \right] T_n(k_1, \ldots, k_n)
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \frac{1}{k_i \cdot q} \left[ \left( (k_i \cdot q)(\eta^{\mu\nu} q^{\sigma} - q^{\mu} \eta^{\nu\sigma}) - k_i^{\mu} q^{\nu} q^{\sigma} \right) \frac{\partial}{\partial k_i^{\sigma}}
\]

\[- q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma} \right] T_n(k_1, \ldots, k_n).
\]

Finally, we contract with the physical polarization tensor of the soft graviton, $\varepsilon_{q\mu\nu}$.
We see that the physical-state conditions
\[ q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = 0 ; \quad \eta^{\mu\nu} \epsilon_{\mu\nu} = 0 \]
set to zero the terms that are proportional to \( \eta^{\mu\nu} \), \( q^\mu \) and \( q^\nu \).

We are then left with the following expression for the graviton soft limit of a single-graviton, \( n \)-scalar amplitude:
\[ M_n(q; k_1, \ldots, k_n) \to \left[ S^{(0)} + S^{(1)} + S^{(2)} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q^2), \]

where
\[ S^{(0)} \equiv \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} k_i^{\mu} k_i^{\nu}}{k_i \cdot q}, \]
\[ S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} k_i^{\mu} q_{\rho} J_i^{\nu\rho}}{k_i \cdot q}, \]
\[ S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu} q_{\rho} J_i^{\mu\rho} q_{\sigma} J_i^{\nu\sigma}}{k_i \cdot q}. \]

These soft factors follow entirely from gauge invariance.
We consider a tree-level color-ordered amplitude where gluon \((n + 1)\) becomes soft with \(q \equiv k_{n+1}\).

Being the amplitude color-ordered, we have to consider only the two poles with the soft particle attached to the two adjacent legs.

We proceed as before.
Contract with external polarization vectors:

\[ A_{n+1}(q; k_1, \ldots, k_n) \rightarrow \left[ S^{(0)} + S^{(1)} \right] A_n(k_1, \ldots, k_n) + \mathcal{O}(q), \]

where

\[ S^{(0)} \equiv \frac{k_1 \cdot \varepsilon_q}{\sqrt{2}(k_1 \cdot q)} - \frac{k_n \cdot \varepsilon_q}{\sqrt{2}(k_n \cdot q)}, \]
\[ S^{(1)} \equiv -i\varepsilon_{q\mu} q_{\sigma} \left( \frac{J^\mu_1_{\sigma}}{\sqrt{2}(k_1 \cdot q)} - \frac{J^\mu_n_{\sigma}}{\sqrt{2}(k_n \cdot q)} \right). \]

Here

\[ J^\mu_\sigma_i \equiv L^\mu_\sigma_i + S^\mu_\sigma_i, \]

where

\[ L^\mu_\sigma_i \equiv i \left( k^\mu_i \frac{\partial}{\partial k_{i\nu}} - k^\nu_i \frac{\partial}{\partial k_{i\mu}} \right), \quad S^\mu_\sigma_i \equiv i \left( \varepsilon^\mu_i \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon^\sigma_i \frac{\partial}{\partial \varepsilon_{i\mu}} \right). \]
Soft limit of \((n + 1)\)-graviton amplitude

\[
M_{n+1}(q; k_1, \ldots, k_n) = \left[ S^{(0)} + S^{(1)} + S^{(2)} \right] M_n(k_1, \ldots, k_n) + \mathcal{O}(q^2),
\]

\[
S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q},
\]

\[
S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_{\rho} J_i^{\nu\rho}}{k_i \cdot q},
\]

\[
S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_{\rho} J_i^{\mu\rho} q_{\sigma} J_i^{\nu\sigma}}{k_i \cdot q}.
\]

where \(J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + S_i^{\mu\sigma}\) and \((\varepsilon_{i}^{\mu\nu} \equiv \varepsilon_i^\mu \varepsilon_i^\nu)\)

\[
L_i^{\mu\sigma} \equiv i \left( k_i^\mu \frac{\partial}{\partial k_i^\sigma} - k_i^\sigma \frac{\partial}{\partial k_i^\mu} \right), \quad S_i^{\mu\sigma} \equiv i \left( \varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_i^\sigma} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_i^\mu} \right).
\]

These soft factors follow from gauge invariance and agree with those computed by Cachazo and Strominger.
What about soft theorems in string theory?

- Soft theorems for gluons and gravitons are of course satisfied, as one can check computing explicitly the amplitude.
- Study the soft theorems for other massless particles as the dilaton and the $B_{\mu\nu}$. 
The field theory action for the dilaton and $B_{\mu\nu}$:

$$S_{\text{string}} = \frac{1}{2\hat{\kappa}^2} \int d^d x \sqrt{-G} \ e^{-2\phi} \left[ R + 4G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2 \cdot 3!} H_{\mu\nu\rho}^2 \right]$$

- There is no gauge invariance for the dilaton as for the graviton.
- Therefore, we don’t expect low-energy theorems for the dilaton.
- No long range force associated with the $B_{\mu\nu}$ (no term of $\mathcal{O}(q^{-1})$).
- We cannot use its gauge invariance as for gravitons.
- On the other hand, the soft dilaton behavior in string theory goes back to the 70s [Ademollo et al., 1975] and [Shapiro, 1975].
In string theory the scattering amplitudes involving a graviton or a dilaton or a Kalb-Ramond field are all obtained from the same two-index tensor $M^{\mu \nu}(q; k_i)$ by saturating it with a polarization tensor satisfying respectively the following conditions:

- **Graviton** ($g_{\mu \nu}$)  \[ \epsilon^{\mu \nu}_g = \epsilon^{\nu \mu}_g \quad ; \quad \eta_{\mu \nu} \epsilon^{\mu \nu}_g = 0 \]
- **Dilaton** ($\phi$)  \[ \epsilon^{\mu \nu}_d = \eta^{\mu \nu} - q^\mu \bar{q}^\nu - q^\nu \bar{q}^\mu \]
- **Kalb-Ramond** ($B_{\mu \nu}$)  \[ \epsilon^{\mu \nu}_B = - \epsilon^{\nu \mu}_B \]

where $\bar{q}$ is, similarly to $q$, a lightlike vector such that $q \cdot \bar{q} = 1$.

The soft theorem for a dilaton can, in principle, be computed starting from the expression that we obtained for the graviton.

But now we cannot neglect extra terms proportional to $\eta^{\mu \nu}$ as we did in the case of a graviton.
Imposing $q^\mu M_{\mu\nu} = q^\nu M_{\mu\nu} = 0$, for the graviton we got:

$$M_n^{\mu\nu}(q; k_1 \ldots k_n) = \sum_{i=1}^{n} \frac{k_i^\nu}{k_i \cdot q} \left[ k_i^\mu - i q_\rho J_i^{\mu\rho} \right] T_n(k_1, \ldots, k_n)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \frac{1}{k_i \cdot q} \left[ \left( (k_i \cdot q)(\eta^{\mu\nu} q^{\sigma} - q^\mu \eta^{\nu\sigma}) - k_i^\mu q^{\nu} q^{\sigma} \right) \frac{\partial}{\partial k_i^\sigma} - q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma} \right] T_n(k_1, \ldots, k_n).$$

and we have neglected the terms in the third line because the graviton polarization satisfies the identities:

$$q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = \eta^{\mu\nu} \epsilon_{\mu\nu} = 0$$

More precisely, gauge invariance imposes:

$$q_\mu M_n^{\mu\nu}(q, k_i) = f(q, k_i) q^\nu \implies q_\mu (M_n^{\mu\nu} - f(k_i) \eta^{\mu\nu}) = 0$$

The extra term with $\eta^{\mu\nu}$ is irrelevant for the graviton, but not for the dilaton.
Because of this we cannot in general get low-energy theorems for the dilaton.

But, let us forget for a moment this problem, and compute the amplitude with a massless closed string state and $n$ closed string tachyons:

$$M_{\mu\nu}^n \sim \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}} \prod_{i<j} |z_i - z_j|^{\alpha' k_i k_j} \int d^2 z \prod_{i=1}^n |z - z_i|^{\alpha' k_i q}$$

$$\times \alpha' \sum_{i=1}^n \frac{k_i^\mu}{z - z_i} \sum_{i=1}^n \frac{k_i^\nu}{\bar{z} - \bar{z}_i}$$

We have explicitly computed the first three terms of order $q^{-1}$, $q^0$ and $q^1$. 
The calculation is rather long and at the end we get the following expression:

\[ M_n^{\mu\nu}(q; k_1 \ldots k_n) = \kappa_d \left\{ \sum_{i=1}^{n} \frac{k_{i\mu}k_{i\nu}}{k_i q} + \sum_{i=1}^{n} \frac{k_{i\nu}q^\rho}{k_i q} \left( k_{i\mu} \frac{\partial}{\partial k_{i\rho}} - k_{i\rho} \frac{\partial}{\partial k_{i\mu}} \right) \right. \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \frac{q^\rho q^\sigma}{k_i q} \left[ k_{i\nu} \left( k_{i\mu} \frac{\partial}{\partial k_{i\rho}} - k_{i\rho} \frac{\partial}{\partial k_{i\mu}} \right) \frac{\partial}{\partial k_{i\sigma}} \right. \]

\[ \left. \left. - k_{i\sigma} \left( k_{i\mu} \frac{\partial}{\partial k_{i\rho}} - k_{i\rho} \frac{\partial}{\partial k_{i\mu}} \right) \frac{\partial}{\partial k_{i\nu}} \right] \right\} T_n \]

where

\[ T_n = \frac{8\pi}{\alpha'} \left( \frac{\kappa_d}{2\pi} \right)^{n-2} \int \frac{d^2 \mathbf{z}_i}{dV_{abc}} \prod_{i \neq j} |\mathbf{z}_i - \mathbf{z}_j|^{-\frac{\alpha'}{2}} k_ik_j \]

is the correctly normalized \( n \)-tachyon amplitude.

This is precisely the expression obtained from gauge invariance with the general argument imposing the conditions:

\[ q_\mu M^{\mu\nu} = q_\nu M^{\mu\nu} = 0. \]
By saturating it with the graviton polarization one gets, of course, the previous general expression.

By saturating it with the dilaton “polarization”

\[ \epsilon_{\mu\nu} = (\eta_{\mu\nu} - q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu) \quad ; \quad q^2 = \bar{q}^2 = 0 \quad ; \quad q\bar{q} = 1 \]

one gets \((m_i^2 = -\frac{4}{\alpha'})\)

\[
S^{(0)} + S^{(1)} + S^{(2)} = - \sum_{i=1}^{n} m_i^2 \left( 1 + q_\rho \frac{\partial}{\partial k_{i\rho}} + \frac{1}{2} q_\rho q^\sigma \frac{\partial^2}{\partial k_{i\rho} \partial k_{i\sigma}} \right) \left. \frac{\partial}{\partial k_{i\mu}} \right|_{k_i q} \\
- \sum_{i=1}^{n} k_{i\mu} \frac{d}{dk_{i\mu}} + 2 \\
+ \sum_{i=1}^{n} \left( -k_{i\mu} q_\sigma \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\sigma}} + \frac{1}{2} (k_i q) \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\mu}} \right) \\
+ \mathcal{O}(q^2)
\]

\[ B_{\mu\nu} \] not coupled to \(n\) tachyons \((\text{invariance under w.s. parity } \Omega)\).
Soft behavior of a massless closed string in an amplitude involving an arbitrary number of other massless closed strings (bosonic+superstring).

In this case we have performed the calculation up to the $O(q^0)$.

For the symmetric part of $M_{\mu\nu}$ we get:

$$M_{\mu\nu}^S(q; k_i, \epsilon_i) = \kappa_d \sum_{i=1}^{n} \left( \frac{k_i^\mu k_i^\nu - \frac{i}{2} k_i^\nu q_{\rho} J_i^{\mu\rho} - \frac{i}{2} k_i^\mu q_{\rho} J_i^{\nu\rho}}{q k_i} \right) M_n(k_i, \epsilon_i)$$

where $M_n(k_i, \epsilon_i)$ is the amplitude with $n$ massless states,

$$J_i^{\mu\nu} = \mathcal{L}_i^{\mu\nu} + \mathcal{S}_i^{\mu\nu} + \bar{\mathcal{S}}_i^{\mu\nu},$$

$$L_i^{\mu\nu} = i \left( k_i^\mu \frac{\partial}{\partial k_i^\nu} - k_i^\nu \frac{\partial}{\partial k_i^\mu} \right), \quad \mathcal{S}_i^{\mu\nu} = i \left( \epsilon_i^\mu \frac{\partial}{\partial \epsilon_i^\nu} - \epsilon_i^\nu \frac{\partial}{\partial \epsilon_i^\mu} \right),$$

$$\bar{\mathcal{S}}_i^{\mu\nu} = i \left( \bar{\epsilon}_i^\mu \frac{\partial}{\partial \bar{\epsilon}_i^\nu} - \bar{\epsilon}_i^\nu \frac{\partial}{\partial \bar{\epsilon}_i^\mu} \right); \quad \epsilon_i^{\mu\nu} \equiv \epsilon_i^\mu \bar{\epsilon}_i^\nu.$$
By saturating with the polarization of the graviton, one gets (of course) the soft behavior obtained from gauge invariance.

If we instead saturate it with the polarization of the dilaton we get:

\[ M_{n+1} = \kappa_d \left[ 2 - \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} \right] M_n + \mathcal{O}(q), \]

It can be written in a more suggestive way by observing that, in general, \( M_n \) has the following form:

\[ M_n = \frac{8\pi}{\alpha'} \left( \frac{\kappa_d}{2\pi} \right)^{n-2} F_n(\sqrt{\alpha'} k_i), \quad \kappa_d = \frac{1}{2^{d-10}} \frac{g_s}{\sqrt{2}} \left( \frac{d-3}{2} \right) \left( \sqrt{\alpha'} \right)^{\frac{d-2}{2}}, \]

where \( F_n \) is dimensionless and obviously satisfies the equation:

\[ \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} F_n = \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} F_n. \]

One gets:

\[ M_{n+1} = \kappa_d \left[ -\sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + \frac{d-2}{2} g_s \frac{\partial}{\partial g_s} \right] M_n + \mathcal{O}(q). \]
Same result if we include massless open strings (on a Dp-brane).

No extra term proportional to $\eta^{\mu\nu}$ is needed to reproduce the previous amplitude.

The amplitude of a soft dilaton is obtained from the amplitude without a dilaton by a simultaneous rescaling of the Regge slope $\alpha'$ and the string coupling constant $g_s$.

Same rescaling that leaves Newton’s constant invariant:

$$\left[-\sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + \frac{d-2}{2} g_s \frac{\partial}{\partial g_s}\right] \kappa_d = 0$$

No fundamental dimensionless constant in string theory.

From it we can rewrite the soft dilaton theorem:

$$M_{n+1} = \kappa_d \frac{d-2}{2} \frac{d}{d\phi_0} M_n + \mathcal{O}(q) ; \quad g_s \equiv e^{\phi_0}$$
Apply to the case $n = 5$ with 5 dilatons:

$$M_5 = \kappa_d \left( 2 - \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} \right) M_4 + \mathcal{O}(q)$$

where

$$M_4 = 2\kappa_d^2 \left( \frac{tu}{s} + \frac{su}{t} + \frac{st}{u} \right) \frac{\Gamma(1 - \frac{\alpha' s}{4})\Gamma(1 - \frac{\alpha' u}{4})\Gamma(1 - \frac{\alpha' t}{4})}{\Gamma(1 + \frac{\alpha' s}{4})\Gamma(1 + \frac{\alpha' u}{4})\Gamma(1 + \frac{\alpha' t}{4})}$$

In the field theory limit ($\alpha' \to 0$), one gets zero because one has a homogenous function of degree 2.

In string theory one gets a non-trivial right-hand-side.
Soft theorem for $B_{\mu\nu}$

- In order to formulate a soft theorem for the antisymmetric tensor we have to make a distinction between the momentum of the holomorphic part, which we call $k_i$, from that of the anti-holomorphic part, which we call $\bar{k}_i$.
- This means that the amplitude $M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)$, on which the soft operator acts, is a function of both $k_i$ and $\bar{k}_i$.
- Together with the operators $L_i$, $S_i$ and $\bar{S}_i$, we then also introduce:

$$L_i^{\mu\nu} = i \left( k_i^{\mu} \frac{\partial}{\partial \bar{k}_{i\nu}} - k_i^{\nu} \frac{\partial}{\partial \bar{k}_{i\mu}} \right).$$

- In terms of these operators, the soft behavior for $B_{\mu\nu}$ reads:

$$M_{n+1} = -i \epsilon q_{\mu\nu} \kappa_d \sum_{i=1}^{n} \left[ k_i^{\nu} q_{\rho} (L_i + S_i)^{\mu\rho} \frac{k_i^{\nu} q_{\rho}}{q k_i} - k_i^{\nu} q_{\rho} (\bar{L}_i + \bar{S}_i)^{\mu\rho} \frac{k_i^{\nu} q_{\rho}}{q k_i} \right]$$

$$M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \bigg|_{k=\bar{k}} + O(q)$$
It is equal to

\[ M_{n+1} = -i \epsilon_q B_{\mu \nu} \kappa d \sum_{i=1}^{n} \left[ \frac{1}{2} (L_i - \bar{L}_i)^{\mu \nu} + \frac{k_i^\nu q_\rho}{k_i q} (S_i - \bar{S}_i)^{\mu \rho} \right] \]

\[ \times M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \bigg|_{k=\bar{k}} + \mathcal{O}(q) \, . \]

As expected from Weinberg’s general argument, we do not get any term of \( \mathcal{O}(q^{-1}) \), corresponding to a long range force, but there are several terms of \( \mathcal{O}(q^0) \).

It is not clear how to get the soft operator of the antisymmetric field by directly using its own gauge symmetry, as it has been done for the graviton.

It is not really a soft theorem because the amplitude \( M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \) is not a physical amplitude before we act with the soft operators.

It is nevertheless easy to show that it is gauge invariant.
Under a gauge transformation for the Kalb-Ramond field, 
\[ \epsilon^B_{q \mu \nu} \rightarrow \epsilon^B_{q \mu \nu} + q^\mu \chi_\nu - q^\nu \chi_\mu, \]
the amplitude changes as follows
\[
\hat{S}^{(1)} M_n \rightarrow \hat{S}^{(1)} M_n + i q^\rho \chi_\mu \sum_{i=1}^n \left[ (L_i + S_i)^{\mu \rho} - (\bar{L}_i + \bar{S}_i)^{\mu \rho} \right] \\
\times M_n (k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \bigg|_{k=\bar{k}}.
\]

The extra term vanishes as a consequence of the identity
\[
\sum_{i=1}^n (L_i + S_i)^{\mu \rho} M_n (k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \bigg|_{k=\bar{k}} = \sum_{i=1}^n (\bar{L}_i + \bar{S}_i)^{\mu \rho} M_n (k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \bigg|_{k=\bar{k}}
\]
which can be proved by a direct calculation, ensuring gauge invariance of the amplitude.
Comments on loop corrections: gauge theory

- At one-loop the amplitude will have in general IR and UV divergences.
- We are not giving here a complete study of them.
- The one-loop contributions have been classified into the factorizing ones and the non-factorizing ones.
- We will concentrate here to the factorizing ones.
- They modify the vertex present in the pole term.
- For the gauge theory they are of the type shown in the figure.

![Diagram of the gauge theory](image)
They have been computed in QCD and are given by:

\[
D^{\mu, \text{fact}} = \frac{i}{\sqrt{2}} \frac{1}{3} \frac{1}{(4\pi)^2} \left( 1 - \frac{n_f}{N_c} + \frac{n_s}{N_c} \right) (q - k_a)^\mu \left[ (\epsilon_n \cdot \epsilon_a) - \frac{(q \cdot \epsilon_a)(k_a \cdot \epsilon_n)}{(k_a \cdot q)} \right]
\]

[Z. Bern, V. Del Duca, C.R. Schmidt, 1998]
[Z. Bern, V. Del Duca, W.B. Kilgore, C.R. Schmidt, 1999]

- It is both IR and UV finite and the limit \( \epsilon \rightarrow 0 \) has been taken.
- It is non-local because of the pole in \((qk_a)\).
- It is gauge invariant under the substitution \( \epsilon_n \rightarrow q \).
- It does not contribute to the leading soft behavior.
Attaching to it the rest of the amplitude

\[ D_{\mu}^{\text{fact}} \frac{-i}{2q \cdot k_a} J^\mu, \]

\[ J^\mu \] is a conserved current:

\[ (q + k_a)_\mu J^\mu = 0, \]

assuming that all the remaining legs are contracted with on-shell polarizations.

We can trade \( k_a \) with \( q \) and we get immediately:

\[ D_{\mu}^{\text{fact}} \frac{-i}{2q \cdot k_a} J^\mu = \mathcal{O}(q^0), \]

No leading \( \mathcal{O}(\frac{1}{q}) \) correction from the factorizing contribution to the one-loop soft functions.
A similar calculation can be done for the gravity case.

We consider only the case in which scalar fields circulate in the loop.

The result of this calculation is:

\[
\mathcal{D}^{\mu\nu,\text{fact},s} = \frac{i}{(4\pi)^2} \left( \frac{\kappa}{2} \right)^3 \frac{1}{30} \left[ (\varepsilon_n \cdot \varepsilon_a) - \frac{(q \cdot \varepsilon_a)(k_a \cdot \varepsilon_n)}{(q \cdot k_a)} \right] \\
\times \left( (q \cdot \varepsilon_a)(k_a \cdot \varepsilon_n) - (\varepsilon_n \cdot \varepsilon_a)(q \cdot k_a) \right) k_a^{\mu} k_a^{\nu} + \mathcal{O}(q^2),
\]
As in the gauge-theory case, the diagrams \( D_{\mu\nu,\text{fact},s} \) contract into a conserved current:

\[
(k_a + q)^\mu J_{\mu\nu} = f(k_i, \epsilon_i) (k_a + q)_\nu, \quad (k_a + q)^\nu J_{\mu\nu} = f(k_i, \epsilon_i) (k_a + q)_\mu.
\]

This means

\[
k_a^\mu k_a^\nu J_{\mu\nu} = (k_a + q)^\mu (k_a + q)^\nu J_{\mu\nu} + O(q)
= f(k_i, \epsilon_i) (k_a + q)^2 + O(q) = 2f(k_i, \epsilon_i) q \cdot k_a + O(q) = O(q)
\]

We therefore have

\[
D_{\mu\nu,\text{fact},s} \frac{i}{2q \cdot k_a} J_{\mu\nu} = O(q).
\]

No modification of the two first leading terms.

As in QCD, we expect that the contribution of other particles circulating in the loop will not modify this result.
Conclusions

- We have extended Low’s proof of the universality of sub-leading behavior of photons to non-abelian gauge theory and to gravity.
- On-shell gauge invariance fully determines the first sub-leading soft-gluon and the first two sub-leading soft-graviton behavior at tree level.
- Factorizing one-loop contributions preserve the leading behavior in gauge theories and the first two leading behaviors in gravity.
- One computes the low-energy behavior of $M_{\mu\nu}$ by imposing the Eqs. $q^\mu M_{\mu\nu} = q^\nu M_{\mu\nu} = 0$.
- Saturating $M_{\mu\nu}$ with the polarization of Graviton/Dilaton, one gets automatically their soft behavior.
- This is the result for all amplitudes we have looked at: BCJ/KLT?
- We get also a kind of soft theorem for $B_{\mu\nu}$.
- Extend our considerations to one-loop diagrams.
- Study the double-soft behavior both in field and string theory.