Effective Actions, Lovelock Lagrangians and Bruno Zumino

Orlando Alvarez
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28 April 2015

Bruno Zumino Memorial Meeting at
My connection to Bruno
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We were colleagues for 12 years until I moved to Miami.
My paper with Bruno
Gravitational Anomalies and the Family's Index Theorem*

Orlando Alvarez¹ * *, I. M. Singer², and Bruno Zumino¹
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Abstract. We discuss the use of the family’s index theorem in the study of gravitational anomalies. The geometrical framework required to apply the family’s index theorem is presented and the relation to gravitational anomalies is discussed. We show how physics necessitates the introduction of the notion of local cohomology which is distinct from the ordinary topological cohomology. The recent results of Alvarez-Gaumé and Witten are derived by using the family’s index theorem.

I. Introduction

Alvarez-Gaumé and Witten [1] have calculated the gravitational anomalies of certain parity violating theories in $4k - 2$ dimensions. Their most striking result is that there is a unique minimal ten dimensional theory where the gravitational anomalies cancel. In this communication we reproduce their results in a different way by using the family’s index theorem [2] instead of Feynman diagram methods.

The relation of the family’s index theorem to anomalies has been discussed by Atiyah and one of the present authors in reference [3]. In that paper, the geometric setting for the family’s index theorem was presented and the relation to anomalies was discussed. The authors showed that the first characteristic class of the index bundle for the Dirac operator was related to anomalies. A number of papers have addressed the relationships among chiral anomalies, the geometry of the space of vector potentials, and the families of Dirac operators. We recommend the papers of Alvarez-Gaumé and Ginsparg [4], Lott [5], and Stora [6] to the reader. The first investigation of the behavior of the Dirac operator as a function of the metric is due to Hitchin [7].

* This work was supported in part by the National Science Foundation under Contracts PHY81-18547 and MCS80-23356, and by the Director, Office of High Energy and Nuclear Physics of the US Department of Energy under Contracts DE-AC03-76SF00098 and AT0380-ER10617

** Alfred P. Sloan Foundation Fellow
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Is Singer’s Birthday is 3 May.
He will be 91.
Zumino & Lovelock Lagrangians
Gravity Theories in more than Four Dimensions

Bruno Zumino

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1. Introduction

The study of gravity in more than 4 dimensions is motivated by the approach of Kaluza [1] and Klein [2] to the problem of unification of gravity with electromagnetism and the other elementary interactions. The Kaluza-Klein point of view has been revived recently [3] by the study of supergravity. The most promising approach in unification seems to be that based on string theories in the low energy limit [4].

The 4-dimensional gravity which emerges from superstring theories in the low energy limit contains in its action terms quadratic in the Kalb-Ramond tensor. Warren Siegel’s term emphasized that these terms give rise to gravity and Yang-Mills theory, while the string theory is entirely consistent. This puzzling contradiction has been resolved by Barton Zwiebach [7] who has pointed out that the action in 4-dimensional space-time

\[ \int \sqrt{g} R + \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \psi)^2 - \frac{1}{2} \phi \partial_R \psi \partial_R \psi + \frac{1}{8} F^2, \]

leads to ghost-free non-trivial gravitational interactions for \( n > 4 \). By explicit computations Zwiebach has shown that, if one expands (1.1) about Minkowski space, the terms quadratic in the gravitational field terms are proportional to the second derivative of the propagator, and the higher order terms are proportional to the Riemann tensor. In 4 dimensions the mass expression (1.1) is a total derivative and integrates to zero, so that (1.1) reduces to the usual Einstein-Hilbert action, which is renormalizable. In 4 dimensions the total derivative is renormalizable, and the terms proportional to the Riemann tensor are proportional to the Euler density. In 3 dimensions the Euler density vanishes and the terms proportional to the Riemann tensor are proportional to the Euler density. In 2 dimensions the Euler density vanishes and the terms proportional to the Riemann tensor are proportional to the Euler density. In 1 dimension the Euler density vanishes and the terms proportional to the Riemann tensor are proportional to the Euler density.

We shall see that this contradiction is rather easy to resolve, once the geometric meaning of the dimensionally regulated Euler densities are understood. As we shall see below, they form a particular class of solutions of the equations of motion.
$L_{0,0} = e_\alpha e^\alpha e_\beta e^\beta e_\gamma e^\gamma \epsilon^{abcdke}$,  \hspace{1cm} (3.7)

$L_{1,1} = R_{ab} e_\alpha e_\beta e_\gamma e^\delta e^\epsilon \epsilon^{abcdke}$,  \hspace{1cm} (3.8)

$L_{2,2} = R_{ab} R_{cd} e_\alpha e_\beta e_\gamma e^\delta e^\epsilon \epsilon^{abcdke}$,  \hspace{1cm} (3.9)

and

$L_{3,0} = R_{ab} R_{cd} R_{ef} e^{abcdke}$.  \hspace{1cm} (3.10)

Again, the first is a cosmological term, the second is proportional to the Einstein–Hilbert action and the last to the Euler invariant. Now we have the new possibility (3.9). Similarly for higher dimensions. Odd numbers of dimensions can be considered as well, but in this case the Euler invariant is absent, of course.

To be concrete, let us stay with 6 dimensions. Can one really have a term like (3.9) in the Lagrangian? At first sight one may think that such a term, which is quadratic in the Riemann tensor, will contribute to the bilinear part of the Lagrangian for the field $h$ which describes the deviation from Minkowski space

$$e^\alpha_m = \delta^\alpha_m + h^\alpha_m$$  \hspace{1cm} (3.11)

and thus spoil the particle interpretation by introducing ghosts [9]. However, one can see that this is not the case.

Let us consider an infinitesimal variation of the connection and vielbein forms. The corresponding variation of $L_{2,2}$ is

$$\delta L_{2,2} = 2 R_{ab} R_{cd} e_\alpha e^\delta e^\epsilon \epsilon^{abcdke} + 2 R_{ab} R_{cd} \delta e_\alpha e^{abcdke}$$.  \hspace{1cm} (3.12)

Using (2.9), the first term on the right hand side is

$$2(D \delta \omega_{ab}) R_{cd} e_\alpha e^\delta e^\epsilon \epsilon^{abcdke}$.  \hspace{1cm} (3.13)

On the other hand, using the Bianchi identity (2.7) and the definition (2.4) of the torsion we have

$$2d(\delta \omega_{ab} R_{cd} e_\alpha e^\epsilon) = 2(D \delta \omega_{ab}) R_{cd} e_\alpha e^{abcdke} + 4 \delta \omega_{ab} R_{cd} T_{\alpha} e^{abcdke}$.  \hspace{1cm} (3.14)

Therefore, if the torsion vanishes, (3.12) can be written

$$\delta L_{2,2} = 2d(\delta \omega_{ab} R_{cd} e_\alpha e^\epsilon) + 2 R_{ab} R_{cd} \delta e_\alpha e^{abcdke}$.  \hspace{1cm} (3.15)

This equation tells us that, if we consider a power series expansion in $h$ starting from flat space, the terms in $L_{2,2}$ which are quadratic in $h$ appear under a derivative sign (first term on the right hand side in (3.15)); for a compact manifold or with suitable conditions at infinity, they drop out after integration. The first non-trivial term in the integrated action is cubic; it comes from the second term on the right hand side of (3.15) and can be immediately obtained from it. Clearly, the same result is true for $L_{2,}$ in
Lovelock Lagrangians in 6D

\( L_{0,6} = e_a^b e_b^c e_c^d e_d^e e_e^f e_f^g e^{abcdef} \),
\( L_{1,4} = R_{ab} e_a^b e_b^c e_c^d e_d^e e_e^f e^{abcdef} \),
\( L_{2,2} = R_{ab} R_{cd} e_a^b e_c^d e_d^e e^{abcdef} \),
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Bruno’s Favorite Technology
Bruno’s Favorite Technology
Effective Lagrangians and Lovelock Actions
An energy tube is a region of space where some mechanism changes the local energy relative to the vacuum. Typically you will need a field theory with a finite correlation length, and some type of boundary conditions. I do not include the traditional Casimir effect of here because you need a long range (massless) field.
Brane Neighborhood
DYNAMICS OF RELATIVISTIC VORTEX LINES AND THEIR RELATION TO DUAL THEORY

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Received 24 April 1974
What governs the dynamics of the Nielsen-Olsen vortex?

Nambu-Goto String Action for the dynamics of the core

$$S_{\text{dual}} = -\frac{1}{2\pi\alpha'} \int d^2\tau \sqrt{-g},$$
14 Years Later

FINITE-WIDTH CORRECTIONS
TO THE NAMBU ACTION FOR THE NIELSEN–OLESEN STRING

Kēi-ichi MAEDA
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EFFECTIVE ACTION FOR A COSMIC STRING

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Effective actions for bosonic topological defects

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(Received 2 August 1990)

We consider a gauge field theory which admits $p$-dimensional topological defects, expanding the equations of motion in powers of the defect thickness. In this way we derive an effective action and effective equation of motion for the defect in terms of the coordinates of the $p$-dimensional world surface defined by the history of the core of the defect.
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Clearly, upon integration, linear terms will disappear, leaving a contribution to the action of

$$S = \mu_0 \int \sqrt{-\gamma} \left[ 1 - \frac{\mu_1}{\mu_0} \epsilon^2 (p+1)R \right] d^{p+1}\sigma,$$  \hspace{1cm} (19)

where $\mu_0 = \int \mathcal{L}_0 d^m \xi$ is the energy per unit $p$ area of the defect, $\mu_1 = \int \xi^i (\mathcal{L}_0 d^m \xi^i / 2\epsilon^2)$ is a constant of order unity, and we have used the Gauss-Codazzi relations

$$\sum_i K_i^2 - K_{\mu\nu}^2 = - (p+1)R$$ \hspace{1cm} (20)

to write the action in terms of the Ricci curvature of the world surface.
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\]

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\[
\sum_i K_i^2 - K_{\mu\nu} = -(p+1) \mathcal{R}
\]

(20)

to write the action in terms of the Ricci curvature of the world surface.
There is a universal expression that describes the “mean field energy” (mean field action) of an energy tube.

There are corrections that I can describe to you after the talk.
What is a mathematical tube?
What is a tube?

Let $\Sigma^q \subset \mathbb{E}^n$ be an embedded submanifold without boundary, i.e. a closed submanifold. The *tube* $\mathcal{T}(\Sigma, \rho)$ of radius $\rho$ about $\Sigma$ is a subset of $\mathbb{E}^n$ with the following characterization: $x$ is in the tube if there exists a straight segment from $x$ to $\Sigma$ that intersects $\Sigma$ perpendicularly and the length of the segment is less than or equal to $\rho$. The tube $\mathcal{T}(\Sigma, \rho)$ is a fiber bundle over $\Sigma$ with fiber $B^l$, the $l$-dimensional ball (the solid $(l-1)$-sphere) where $n = q + l$. 
What is a tube?

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\[ q = p + 1 \]
Why a short correlation length?
H. Weyl’s Formula (1939)
H. Weyl’s Formula (1939)

(1885-1955)
H. Weyl’s Formula (1939)
ON THE VOLUME OF TUBES

By H. Weyl

1. The problem. In a letter before the Mathematical Circle at Princeton, but in a letter before the Mathematical Circle at Princeton, and then formalized, there are two main problems: one on the theory of the volume of tubes, and the other on the theory of the volume of tubes.

Let \( C \) be the closed, convex, bounded set in \( \mathbb{R}^n \). The volume \( V(C) \) of \( C \) is to be determined. \( X \) and \( Y \) are two convex subsets of \( \mathbb{R}^n \).

For small values of \( s \), we will have in the same approximation:

\[ V(a) = 2\pi^{-\frac{1}{2}} s. \]

where \( a \) is the volume of the unit ball in \( \mathbb{R}^n \).

(1) \( a = \frac{\pi^{n/2}}{\Gamma(n/2)} \)

and \( b \) is the one of the unit ball in \( \mathbb{R}^n \). Professor Hankel showed that this formula is valid for \( n = 1 \) and for \( n = 2 \). A similar formula prevails in \( \mathbb{R}^n \) for \( n = 1 \), and it will not take the problem for higher dimensions.

The result for \( X \), is a formula resembling \( 1 + \frac{1}{n} \), at the following:

\[ V(C) = a \cdot \frac{\pi^{n/2}}{\Gamma(n/2)} \cdot \frac{t}{(1+t)^n} \]

where \( 0 \leq t \leq 1 \).

2. The fundamental formula for the volume of tubes. If an n-dimensional manifold \( M \) contains a curve or a surface, it is reduced to the problem of finding the volume of \( M \).

3. The problem of tubes in \( \mathbb{R}^n \). If an n-dimensional manifold \( M \) contains a curve or a surface, it is reduced to the problem of finding the volume of \( M \).

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H. Weyl’s Formula (1939)
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\[
\text{vol}(\mathcal{T}(\Sigma, \rho)) = V_l(B^l) \rho^l \, \text{vol}(\Sigma) + V_l(B^l) \frac{\rho^{l+2}}{2(l+2)} \int_\Sigma R \, \eta_\Sigma \\
+ V_l(B^l) \frac{\rho^{l+4}}{8(l+2)(l+4)} \int_\Sigma (R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \, \eta_\Sigma \\
+ O(\rho^{l+6}).
\]
H. Weyl’s Formula (1939)

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\[ + O(\rho^{l+6}). \]

Exact formula with a finite number of terms that only depend on the intrinsic geometry of the surface!
H. Weyl’s Formula (1939)

\[
\text{vol}(\mathcal{T}(\Sigma, \rho)) = V_l(B^l) \rho^l \text{ vol}(\Sigma) + V_l(B^l) \frac{\rho^{l+2}}{2(l + 2)} \int_{\Sigma} R \eta_{\Sigma} \\
+ V_l(B^l) \frac{\rho^{l+4}}{8(l + 2)(l + 4)} \int_{\Sigma} \left( R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \right) \eta_{\Sigma} \\
+ O(\rho^{l+6}).
\]

Exact formula with a finite number of terms that only depend on the intrinsic geometry of the surface!

The terms are precisely those that appear in Lovelock Theories of Gravity!
Thickened curve
Thickened curve

$$\text{vol}_3 \left( \mathcal{T} (\Sigma, \rho) \right) = \pi r^2 \cdot \text{vol}_1 (\Sigma)$$
Two dimensional surface
Two dimensional surface
Two dimensional surface

$$\text{vol}_3(\mathcal{T}(\Sigma, \rho)) = 2\rho \ \text{vol}_2(\Sigma) + \frac{4\pi}{3} \rho^3 \chi(\Sigma)$$
Two dimensional surface

$$\text{vol}_3(\mathcal{T}(\Sigma, \rho)) = 2\rho \text{ vol}_2(\Sigma) + \frac{4\pi}{3} \rho^3 \chi(\Sigma)$$

top and bottom
Thickened $S^2$

$$\text{vol}_3 (\mathcal{T}(S^2, \rho)) = \frac{4\pi}{3} (r + \rho)^3 - \frac{4\pi}{3} (r - \rho)^3$$

$$= 2\rho \cdot 4\pi r^2 + \frac{4\pi}{3} \rho^3 \cdot 2$$

Note that for a thickened torus the Euler characteristic term vanishes.
Thickened $S^2$

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$$\text{vol}_3(\mathcal{T}(\Sigma, \rho)) = 2\rho \text{ vol}_2(\Sigma) + \frac{4\pi}{3} \rho^3 \chi(\Sigma)$$

Note that for a thickened torus the Euler characteristic term vanishes.
Loveland Lagrangians

\[ I = \sum_{r=0}^{\lfloor q/2 \rfloor} \lambda_{2r} I_{2r} = \sum_{r=0}^{\lfloor q/2 \rfloor} \lambda_{2r} \int_{\Sigma} \mathcal{K}_{2r} \eta_{\Sigma}, \]

**The Einstein Tensor and Its Generalizations**

**David Lovelock**

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(Received 27 August 1970)

The Einstein tensor \( G^{ij} \) is symmetric, divergence free, and a concomitant of the metric tensor \( g_{\alpha\beta} \) together with its first two derivatives. In this paper all tensors of valency two with these properties are displayed explicitly. The number of independent tensors of this type depends crucially on the dimension of the space, and, in the four dimensional case, the only tensors with these properties are the metric and the Einstein tensors.
The terms that appear to all orders in the radius in Weyl's tube volume formula are the "dimensional continuations" of the Euler densities. From the physics viewpoint this is astonishing. Gravitational theories defined by lagrangians containing those terms were discussed by Lovelock in the early 1970s who was interested constructing generalizations of the Einstein tensor. He required his tensors to be symmetric, rank two, divergence free and that they contained at most the first two derivatives of the metric (canonical formulation for gravity). The appearance of Lovelock lagrangians in string theory was first observed by Zwiebach (1985) who noted that compatibility of a ghost free theory with the presence of curvature squared terms in the gravitational lagrangian required a special combination that reduced to the Euler density in four dimensions. By studying the 3-graviton on shell vertex in string theory he verified that this curvature squared combination appears. Zumino (1986) generalized Zwiebach’s results and showed that gravitational theories containing higher powers of the curvature were ghost free if the additional terms in the lagrangian were "dimensional continuations" of Euler densities in the appropriate dimensionality, i.e., Lovelock type lagrangians.
I do not believe that any of us at Berkeley at the time were aware of Lovelock’s results. They are not mentioned in any of the papers nor do I recollect any allusion to them at that time.
Weyl’s Volume Element Formula

\[ x = X(\sigma) + \nu \]
\[ \nu = \nu^i \hat{n}_i \]

\[ T_\sigma E^\sigma = T_\sigma \Sigma \oplus (T_\sigma \Sigma)^\bot \]

Locally choose an orthonormal frame \((\hat{e}_1, \ldots, \hat{e}_q)\) for \(T\Sigma\) and an orthonormal frame \((\hat{n}_{q+1}, \ldots, \hat{n}_n)\) for \((T\Sigma)^\bot\).

Let \((\sigma^1, \ldots, \sigma^q)\) be local coordinates on \(\Sigma\), then \((\sigma^1, \ldots, \sigma^q, \nu^{q+1}, \ldots, \nu^n)\) are local coordinates for the tubular neighborhood.

\[ dX = \hat{e}_a \theta^a \]

\[ d\hat{e}_a = \hat{e}_b \omega^b_{\ a} - \hat{n}_j K^b_{\ aj} \theta^b \]

\[ d\hat{n}_i = \hat{n}_j \omega^j_{\ i} + \hat{e}_a K^j_{\ ai} \theta^b , \]
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\[ dX = \hat{e}_a \theta^a \]

tangential to surface

\[ d\hat{e}_a = \hat{e}_b \omega_{ba} - \hat{n}_j K_{abj} \theta^b \]
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Locally choose an orthonormal frame \((\hat{e}_1, \ldots, \hat{e}_q)\) for \(T\Sigma\) and an orthonormal frame \((\hat{n}_{q+1}, \ldots, \hat{n}_n)\) for \((T\Sigma)_\perp\).

Let \((\sigma^1, \ldots, \sigma^q)\) be local coordinates on \(\Sigma\), then \((\sigma^1, \ldots, \sigma^q, \nu^{q+1}, \ldots, \nu^n)\) are local coordinates for the tubular neighborhood.

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\((\hat{e}, \hat{n})\) are defined on \(\Sigma\) and thus their differentials are only defined on \(\Sigma\). This means that the objects \(\theta^a, \omega_{ab}, \omega_{ij}, K^i_{ab}\) only depend on \(\sigma\) and \(d\sigma\).
\[ x = X(\sigma) + \nu \]
Volume Element Formula (continued)

\[ x = X(\sigma) + \nu \]

\[ dx = \hat{E}_\mu \, dx^\mu = \hat{e}_a \left( \delta_{ab} + \nu^i K_{abi} \right) \theta^b + \hat{n}_i \, D\nu^i, \quad D\nu^i = d\nu^i + \omega_{ij} \nu^j \]
Volume Element Formula (continued)

\[ \mathbf{x} = \mathbf{X}(\sigma) + \mathbf{\nu} \]

\[ d\mathbf{x} = \hat{\mathbf{E}}_\mu \, d\mathbf{x}^\mu = \hat{\mathbf{e}}_a \left( \delta_{ab} + \mathbf{\nu}^i K_{abi} \right) \theta^b + \hat{\mathbf{n}}_i \, D\mathbf{\nu}^i, \quad D\mathbf{\nu}^i = d\mathbf{\nu}^i + \omega_{ij} \mathbf{\nu}^j \]
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\[ x = X(\sigma) + \nu \]

\[ dx = \hat{E}_\mu \, d\xi^\mu = \hat{e}_a \left( \delta_{ab} + \nu^i K_{abi} \right) \theta^b + \hat{n}_i \, D\nu^i, \quad D\nu^i = d\nu^i + \omega_{ij} \nu^j \]

An orthonormal coframe at \( x \) is given by \( \left( (\delta_{ab} + \nu^i K_{abi}) \theta^b, D\nu^i \right) \).

\[ ds^2 = dx \cdot dx \]
Weyl’s volume element

\[ d^nx = \det(I + \mathbf{v} \cdot \mathbf{K}) \, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^q \wedge D\nu^{q+1} \wedge \cdots \wedge D\nu^n, \]

\[ = \det(I + \mathbf{v} \cdot \mathbf{K}) \, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^q \wedge d\nu^{q+1} \wedge d\nu^{q+2} \wedge \cdots \wedge d\nu^n, \]

\[ = \det(I + \mathbf{v} \cdot \mathbf{K}) \, \eta_\Sigma \wedge d\nu^{q+1} \wedge d\nu^{q+2} \wedge \cdots \wedge d\nu^n. \]

By linearizing the determinant: \( d \det(I + \mathbf{v} \cdot \mathbf{K})|_{\mathbf{v}=0} = d\mathbf{v} \cdot \text{Tr}(\mathbf{K}) \)

you see that an extremal surface has vanishing mean curvature vector \( \delta^{ab} K^i_{\ ab} \hat{n}_i. \) The mean curvature vector points in the direction of fastest increase in local volume of the surface.
Weyl’s volume element

\[ d^n x = \det(I + \mathbf{v} \cdot \mathbf{K}) \, \theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^q \wedge D\nu^{q+1} \wedge \ldots \wedge D\nu^n , \]
\[ = \det(I + \mathbf{v} \cdot \mathbf{K}) \, \theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^q \wedge d\nu^{q+1} \wedge d\nu^{q+2} \wedge \ldots \wedge d\nu^n , \]
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Energy Tubes

Explain what is an energy tube and energy formula

\[(\Delta E)_{\text{eff}}(X) = \int_{\mathbb{E}^n} u(x, X) \, d^n x\]

\[E = \int_{\Sigma} \eta_{\Sigma}(\sigma) \left( \int_{(T_{\sigma} \Sigma)_{\perp}} u(\sigma, \nu) \, \det(I + \nu \cdot K) \, d^l \nu \right).\]
Spherically Symmetric Energy Density

\[ E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{[q/2]} C_{2r} \int_\Sigma \mathcal{K}_{2r}(\Sigma) \eta_\Sigma \int_0^\infty d\nu \nu^{2r+l-1} u^{(0)}(\sigma, \nu) \]

\[ C_0 = 1, \quad C_{2r} = \prod_{k=0}^{r-1} \frac{1}{l+2k} \]

\[ \mathcal{K}_0(\Sigma) = 1, \]
\[ \mathcal{K}_1(\Sigma) = \frac{1}{2} R, \]
\[ \mathcal{K}_2(\Sigma) = \frac{1}{8} \left( R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \right). \]
Spherically Symmetric Energy Density

\[ E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{[q/2]} C_{2r} \int_{\Sigma} \mathcal{K}_{2r}(\Sigma) \eta_\Sigma \int_0^\infty d\nu \, \nu^{2r + l - 1} u^{(0)}(\sigma, \nu) \]

Where did the $K_{ab}^i$ go?

\[
\mathcal{K}_0(\Sigma) = 1, \\
\mathcal{K}_1(\Sigma) = \frac{1}{2} R, \\
\mathcal{K}_2(\Sigma) = \frac{1}{8} \left( R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \right).
\]

\[ C_0 = 1, \quad C_{2r} = \prod_{k=0}^{r-1} \frac{1}{l + 2k} \]
Spherically Symmetric Energy Density

\[ E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_{\Sigma} \mathcal{K}_{2r}(\Sigma) \eta_{\Sigma} \int_{0}^{\infty} d\nu \, \nu^{2r+l-1} u^{(0)}(\sigma, \nu) \]

Where did the \( K_{ab} \) go?

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Spherically Symmetric Energy Density

\[ E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{[q/2]} C_{2r} \int_\Sigma \mathcal{K}_{2r}(\Sigma) \eta_\Sigma \]

Where did the \( K_{ab}^i \) go?

\[ \mathcal{K}_0(\Sigma) = 1, \]
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radial moments

\[ \int_0^\infty d\nu \, \nu^{2r+l-1} u^{(0)}(\sigma, \nu) \]

\[ C_0 = 1, \quad C_{2r} = \prod_{k=0}^{r-1} \frac{1}{l + 2k} \]
Convert extrinsic geometry to intrinsic geometry by using the Gauss equation:

\[ R_{abcd} = K_{ac}^i K_{bd}^i - K_{ad}^i K_{bc}^i \]

condition for an isometric embedding
Induced Scalar Fields

\[ E^{(0)} = \sum_{r=0}^{[q/2]} C_{2r} \int_{\Sigma} \mu_{2r}^{(0)}(\sigma) K_{2r}(\Sigma) \eta_{\Sigma}, \]

\[ \mu_{2r}^{(0)}(\sigma) = \int_{(T_{\sigma} \Sigma)^\perp} \|\nu\|^{2r} u^{(0)}(\sigma, \|\nu\|) \, d^l \nu = V_{l-1}(S^{l-1}) \int_{0}^{\infty} d\nu \, \nu^{2r+l-1} u^{(0)}(\sigma, \nu). \]
Induced Scalar Fields

\[ E^{(0)} = \sum_{r=0}^{[q/2]} C_{2r} \int_{\Sigma} \mu_{2r}^{(0)}(\sigma) K_{2r}(\Sigma) \eta_{\Sigma}, \]

Induced Scalar Fields
(live on \( \Sigma \))

\[ \mu_{2r}^{(0)}(\sigma) = \int_{(T_\sigma \Sigma)} \|\nu\|^{2r} u^{(0)}(\sigma, \|\nu\|) \, d^l\nu = \nu_{l-1}(S^{l-1}) \int_0^\infty d\nu \nu^{2r+l-1} u^{(0)}(\sigma, \nu). \]
Induced Scalar Fields

\[ E^{(0)} = \sum_{r=0}^{[q/2]} C_{2r} \int_{\Sigma} \mu_{2r}^{(0)}(\sigma) K_{2r}(\Sigma) \eta_{\Sigma}, \]

\[ \mu_{2r}^{(0)}(\sigma) = \int_{(T_\sigma \Sigma)'} \|\mathbf{v}\|^{2r} u^{(0)}(\sigma, \|\mathbf{v}\|) \, d^l\mathbf{v} = V_{l-1}(S^{l-1}) \int_0^\infty d\nu \nu^{2r+l-1} u^{(0)}(\sigma, \nu). \]
Induced Scalar Fields

\[ E^{(0)} = \sum_{r=0}^{[q/2]} C_{2r} \int_{\Sigma} \mu_{2r}^{(0)}(\sigma) K_{2r}(\Sigma) \eta_{\Sigma}, \]

Effective Lovelock Action

\[ \mu_{2r}^{(0)}(\sigma) = \int_{(T_{\sigma} \Sigma)^{\perp}} \|\nu\|^{2r} u^{(0)}(\sigma, \|\nu\|) \, d^l \nu = V_{l-1}(S^{l-1}) \int_0^{\infty} dv \, v^{2r+l-1} u^{(0)}(\sigma, \nu). \]
Weyl’s Volume of a Tube

For the volume of a tube you have:

\[ u(\sigma, \nu) = \begin{cases} 
1 & \text{if } \|\nu\| < \rho, \\
0 & \text{if } \|\nu\| > \rho,
\end{cases} \]

\[
\text{vol}_n(\mathcal{T}(\Sigma, \rho)) = V_l(B^l)\rho^l \text{ vol}_q(\Sigma) + V_l(B^l) \sum_{r=1}^{[q/2]} \frac{\rho^{l+2r}}{\prod_{k=1}^r (l + 2k)} \int_{\Sigma} K_{2r}(\Sigma) \eta_{\Sigma}.
\]

This formula is exact!
\[\delta_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_m} = \det \begin{pmatrix}
\delta_{i_1 i_1}^{j_1} & \delta_{i_1 i_2}^{j_1} & \cdots & \delta_{i_1 i_m}^{j_1} \\
\delta_{i_2 i_1}^{j_2} & \delta_{i_2 i_2}^{j_2} & \cdots & \delta_{i_2 i_m}^{j_2} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{i_m i_1}^{j_m} & \delta_{i_m i_2}^{j_m} & \cdots & \delta_{i_m i_m}^{j_m}
\end{pmatrix}
\]

\[= \frac{1}{(n-m)!} \epsilon_{i_1 i_2 \cdots i_m k_{m+1} k_{m+2} \cdots k_n} \epsilon^{j_1 j_2 \cdots j_m k_{m+1} k_{m+2} \cdots k_n}.
\]

\[\det(I + tS) = \sum_{m=0}^{n} \frac{t^m}{m!} \delta_{i_1 \cdots i_m}^{j_1 \cdots j_m} S_{i_1}^{i_1} S_{i_2}^{i_2} \cdots S_{i_m}^{i_m} j_m \text{ if } S \text{ is symmetric}\]
Multilinear Algebra

\[ \eta_{i_1i_2\ldots i_m} = \star (\theta^{i_1} \wedge \cdots \wedge \theta^{i_m}) = \frac{1}{(n-m)!} \varepsilon_{i_1i_2\ldots i_mj_{m+1}\ldots j_n} \theta^{j_{m+1}} \wedge \cdots \wedge \theta^{j_n} . \]

\[ \Omega_{ab} = \frac{1}{2} R_{abcd} \theta^c \wedge \theta^d \]
\[ \mathcal{K}_{2r}(\Sigma) \eta_{\Sigma} = \frac{1}{4^r r!} \delta^{b_1 \ldots b_{2r}}_{a_1 \ldots a_{2r}} R_{a_1 a_2 b_1 b_2} \cdots R_{a_{2r-1} a_{2r} b_{2r-1} b_{2r}} \eta_{\Sigma}, \]

\[ = \frac{1}{2^r r!} \eta^{a_1 a_2 \ldots a_{2r-1} a_{2r}} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2r-1} a_{2r}}. \]

If \( \dim \Sigma = q = 2r \) is even then the differential form above of maximal degree is

\[ \mathcal{K}_q(\Sigma) \eta_{\Sigma} = \frac{1}{2^r r!} \epsilon^{a_1 a_2 \ldots a_{2r-1} a_{2r}} \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2r-1} a_{2r}}, \]

where \( \mathcal{K}_q(\Sigma) = \text{pf}(\Omega) \) is the pfaffian of the “antisymmetric matrix valued 2-form” \( \Omega_{ab} \). The Euler characteristic is \( \chi(\Sigma) = (1/2\pi)^{q/2} \int_{\Sigma} \text{pf}(\Omega) \eta_{\Sigma} \) by the generalized Gauss-Bonnet Theorem.
\[ \int_{\Sigma} \eta_{\Sigma} \int_{(T_{e} \Sigma)^{\perp}} u(\sigma, \nu) \det(I + \nu \cdot K) \ d^l \nu \]

\[ = \int_{\Sigma} \eta_{\Sigma} \sum_{r=0}^{q} \int_{\nu=0}^{\infty} \int_{S^{l-1}} u^{(0)}(\sigma, \nu) \frac{\nu^r}{r!} \delta_{a_{1} \ldots a_{r}}^{b_{1} \ldots b_{r}} K^{a_{1}b_{1}i_{1}} K^{a_{2}b_{2}i_{2}} \cdots K^{a_{r}b_{r}i_{r}} \times \hat{\nu}^{i_{1}} \hat{\nu}^{i_{2}} \cdots \hat{\nu}^{i_{r}} \cdot \nu^{l-1} \ d\nu \ d\text{vol}_{S^{l-1}}, \]

\[ = \int_{\Sigma} \eta_{\Sigma} \sum_{r=0}^{[q/2]} \frac{1}{(2r)!} \int_{0}^{\infty} d\nu \ \nu^{2r+l-1} u^{(0)}(\sigma, \nu) \times V_{l-1}(S^{l-1}) \ (2r - 1)!! C_{2r} \delta_{a_{1} \ldots a_{2r}}^{b_{1} \ldots b_{2r}} K^{a_{1}b_{1}i_{1}} K^{a_{2}b_{2}i_{1}} \cdots K^{a_{2r-1}b_{2r-1}i_{r}} K^{a_{2r}b_{2r}i_{r}}. \]

\[ R_{abcd} = K_{ac}^{i} K_{bd}^{\ i} - K_{ad}^{\ i} K_{bc}^{i} \]

\[ E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{[q/2]} C_{2r} \int_{\Sigma} K_{2r}(\Sigma) \eta_{\Sigma} \int_{0}^{\infty} d\nu \ \nu^{2r+l-1} u^{(0)}(\sigma, \nu) \]
General Formula

\[ E = \int_{\Sigma} \eta_{\Sigma}(\sigma) \left( \int_{(T_{\sigma} \Sigma)^\perp} u(\sigma, \nu) \det(I + \nu \cdot K) \, d^l \nu \right). \]

Spherical multipole expansion for SO(l)

\[ u(\sigma, \nu) = \sum_{j=0}^{\infty} \sum_{M=1}^{\dim W^j} u^{(j)}_M (\sigma, ||\nu||) Y^j_M (\hat{\nu}). \]
Faux Cartesian Spherical Harmonics $\mathcal{Y}_{i_1 i_2 \ldots i_j}^j (\hat{\mathcal{V}})$

The faux cartesian spherical harmonics are not a basis but an over complete set for the irreducible representation.

\[
\begin{align*}
\mathcal{Y}^0 (\hat{\mathcal{V}}) &= 1, \\
\mathcal{Y}^1_i (\hat{\mathcal{V}}) &= \hat{\mathcal{V}}^i, \\
\mathcal{Y}^2_{ii'} (\hat{\mathcal{V}}) &= \hat{\mathcal{V}}^i \hat{\mathcal{V}}^{i'} - \frac{1}{l} \delta^{ii'}, \\
\mathcal{Y}^3_{i_1 i_2 i_3} (\hat{\mathcal{V}}) &= \hat{\mathcal{V}}^{i_1} \hat{\mathcal{V}}^{i_2} \hat{\mathcal{V}}^{i_3} - \frac{1}{l + 2} \left[ \delta^{i_1 i_2} \hat{\mathcal{V}}^{i_3} + \delta^{i_2 i_3} \hat{\mathcal{V}}^{i_1} + \delta^{i_3 i_1} \hat{\mathcal{V}}^{i_2} \right].
\end{align*}
\]
The faux cartesian spherical harmonics are not a basis but an over complete set for the irreducible representation.

\[
\mathcal{Y}^0(\hat{\mathbf{v}}) = 1, \\
\mathcal{Y}^1_i(\hat{\mathbf{v}}) = \hat{v}^i, \\
\mathcal{Y}^2_{ii'}(\hat{\mathbf{v}}) = \hat{v}^i \hat{v}^{i'} - \frac{1}{l} \delta^{ii'}, \\
\mathcal{Y}^3_{i_1i_2i_3}(\hat{\mathbf{v}}) = \hat{v}^{i_1} \hat{v}^{i_2} \hat{v}^{i_3} - \frac{1}{l+2} \left[ \delta^{i_1i_2} \hat{v}^{i_3} + \delta^{i_2i_3} \hat{v}^{i_1} + \delta^{i_3i_1} \hat{v}^{i_2} \right].
\]
For $l > 1$ are uniquely specified by

1. $\mathcal{Y}^j_{i_1 i_2 \ldots i_j} (\hat{\nu})$ is totally symmetric under any permutation of $i_1, i_2, \ldots, i_j$.

2. $\mathcal{Y}^j_{i_1 i_2 \ldots i_j} (\hat{\nu})$ is traceless with respect to contraction on any pair of indices. Because the harmonic is totally symmetric this reduces to $\mathcal{Y}^j_{i_1 i_2 \ldots i_j} (\hat{\nu}) = 0$.

3. The parity of $\mathcal{Y}^j$ is $(-1)^j$.

4. $\mathcal{Y}^j_{i_1 i_2 \ldots i_j} (\hat{\nu})$ is an inhomogeneous polynomial of degree $j$ in the $\hat{\nu}^i$ with normalization determined by

$$\mathcal{Y}^j_{i_1 i_2 \ldots i_j} (\hat{\nu}) = \hat{\nu}^{i_1} \hat{\nu}^{i_2} \cdots \hat{\nu}^{i_j} + (\text{polynomial of degree } j - 2).$$
For $l > 1$ are uniquely specified by

1. $\mathcal{Y}_{i_1 \ldots i_j}^j (\hat{\nu})$ is totally symmetric under any permutation of $i_1, i_2, \ldots, i_j$.

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Our Multipole Expansion

\[ u(\sigma, \nu) = \sum_{j=0}^{\infty} \sum_{i_1, \ldots, i_j} u_{i_1 \ldots i_j}^{(j)}(\sigma, \|\nu\|) \mathcal{Y}_{i_1 \ldots i_j}^j(\hat{\nu}). \]

\[ u_{i_1 \ldots i_j}^{(j)} \] is totally symmetric and traceless in the indices \( i_1 \cdots i_j \) and is the \( 2^j \)-pole

\[ \mu_{k_1 \ldots k_j, 2s+j}^{(j)}(\sigma) = \int_{(T_\sigma \Sigma)\perp} \|\nu\|^{2s+j} u_{k_1 \ldots k_j}^{(j)}(\sigma, \|\nu\|) \, d^l \nu, \]

\[ = V_{l-1}(S^{l-1}) \int_0^\infty d\nu \, \nu^{2s+j+l-1} u_{k_1 \ldots k_j}^{(j)}(\sigma, \nu). \]

radial moments
General Formula

\[
E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s s!} \times \int \Sigma \mu_{k_1 \ldots k_j, 2s+j}^{(j)}(\sigma) \kappa_{b_1}^{k_1} \wedge \cdots \wedge \kappa_{b_j}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2s-1} a_{2s}} \wedge \eta_{b_1 \ldots b_j a_1 \ldots a_{2s}}.
\]

\[
\kappa_a^k = K_{ab}^k \theta^b
\]

Note that the Gauss equation may be written as \( \Omega_{ab} = \kappa_a^k \wedge \kappa_b^k \) and the SO(l)-curvature 2-form of the normal bundle is \( F_{ij} = \kappa_a^i \wedge \kappa_a^j \). Since the cartesian multipole moments \( \mu_{k_1 \ldots k_j}^{(j)} \) are traceless in the \( k \) indices we see that the \( \kappa \) terms above cannot be transformed into terms involving the intrinsic curvature \( R_{abcd} \) of the surface.
General Formula

\[ E = \sum_{j=0}^{q} \sum_{s=0}^{[(q-j)/2]} \frac{C_{2j+2s}}{2^s s!} \]

\[ \times \int \mu_{k_1 \ldots k_j, 2s+j}(\sigma) \kappa_{b_1}^{k_1} \wedge \cdots \wedge \kappa_{b_j}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2s-1} a_{2s}} \wedge \eta^{b_1 \ldots b_j a_1 \cdots a_{2s}}. \]

\[ \kappa_{a}^{k} = K_{ab}^{k} \theta^{b} \]
General Formula

\[ E = \sum_{j=0}^{q \lfloor (q-j)/2 \rfloor} \sum_{s=0}^{C_{2j+2s}} \frac{C_{2j+2s}}{2^s s!} \]

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\[ \kappa_{a}^{k} = K_{ab}^{k} \theta^{b} \]

There are only finite number of terms in the expansion. There are roughly \( q^4/4 \).
Generalizes to Constant Curvature Spaces

\[ \Omega^{\mu\nu} = k \theta^\mu \wedge \theta^\nu \]

\[ R_{\mu\nu\rho\sigma} = k (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \]

\[ E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s s!} \]

\times \int_{\Sigma} \mu_{k_1 \ldots k_j, 2s+j \ (\sigma)}^{(j)} \kappa_{b_1}^{k_1} \wedge \cdots \wedge \kappa_{b_j}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2s-1} a_{2s}} \wedge \eta^{b_1 \ldots b_j a_1 \ldots a_{2s}}. \]
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\[
E = \sum_{j=0}^{q} \sum_{s=0}^{[(q-j)/2]} \frac{\binom{2j+2s}{2s} s!}{C_{2j+2s}}
\]

\[
\times \int_{\Sigma} \mu_{(j)}^{k_1 \ldots k_j, 2s+j} (\sigma) \kappa_{b_1}^{k_1} \wedge \ldots \wedge \kappa_{b_j}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \ldots \wedge \Omega_{a_{2s-1} a_{2s}} \wedge \eta_{b_1 \ldots b_j a_1 \ldots a_{2s}}.
\]

Case \( k < 0 \)

\[
\mu_{(j)}^{k_1 \ldots k_j, 2s+j} (\sigma) = V_{l-1}(S^{l-1}) \int_0^\infty d\nu \left( \cosh |k|^{1/2} \nu \right)^{q-j-2s} \left( \frac{\sinh |k|^{1/2} \nu}{|k|^{1/2}} \right)^{2s+j+l-1} u_{k_1 \ldots k_j}^{(j)} (\sigma, \nu).
\]
Generalizes to Constant Curvature Spaces

$$\Omega^{\mu\nu} = k \theta^\mu \wedge \theta^\nu$$

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$$E = \sum_{j=0}^q \sum_{s=0}^{[(q-j)/2]} \frac{C_{2j+2s}}{2^s s!} \times \int_{\Sigma} \mu_{k_1 \ldots k_j, 2s+j}^{(j)}(\sigma) \kappa_{b_1}^{k_1} \wedge \ldots \wedge \kappa_{b_j}^{k_j} \wedge \Omega_{a_1a_2} \wedge \ldots \wedge \Omega_{a_{2s-1}a_{2s}} \wedge \eta^{b_1 \ldots b_j a_1 \ldots a_{2s}}.$$  

Case $k < 0$

$$\mu_{k_1 \ldots k_j, 2s+j}^{(j)}(\sigma) = V_{l-1}(S^{l-1}) \int_0^\infty d\nu \left( \cosh |k|^{1/2} \nu \right)^{q-j-2s} \left( \frac{\sinh |k|^{1/2} \nu}{|k|^{1/2}} \right)^{2s+j+l-1} u_{k_1 \ldots k_j}^{(j)}(\sigma, \nu).$$

For $k > 0$ replace the hyperbolic functions by the corresponding trigonometric functions.
Emergent Gravity?
Emergent Gravity?

For another talk…
Emergent Gravity?
Emergent Gravity?

THE END