An Analytical Benchmark for the Calculation of Current Distribution in Superconducting Cables

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Outline

- Motivation
- Historical Overview
- Model and assumptions
- Set of analytical formulae for current distribution in multistrand cables:
  - non uniform boundaries
  - localised quenches
  - localised driving voltages
- Conclusions

Why an analytical approach? (1)

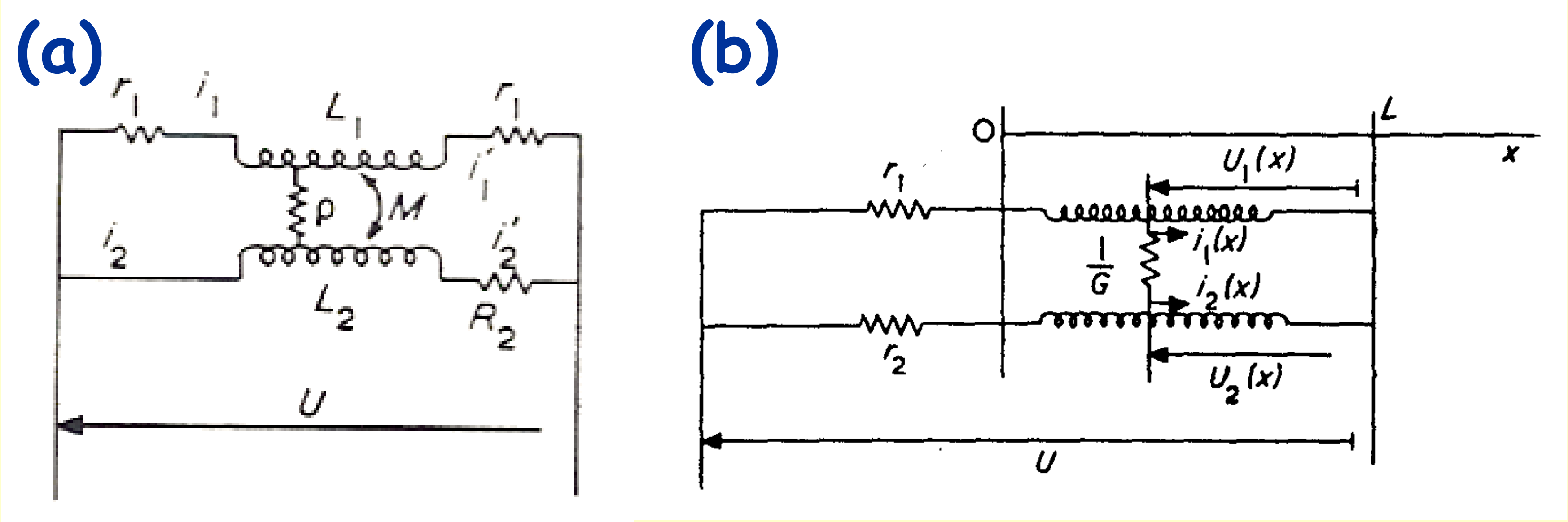
- Development of numerical codes including the calculation of current distribution
- Large effort in the experimental activity
- The experimental validation is essential, but can be affected by approximations of the electromagnetic model and uncertainty in the evaluation of the model parameters
- Analytical work can be used for a preliminary, independent code validation

Why an analytical approach? (2)

- Despite the large theoretical and experimental effort there is a lack of new design criteria taking into account uneven current distributions
- Analytical formulae for maximum currents induced in cables, time constants and redistribution lengths can be useful
- Previous analytical solutions are based on 2-strand cable models, a generalization to *N*-strand cables is interesting

Historical Overview (1)

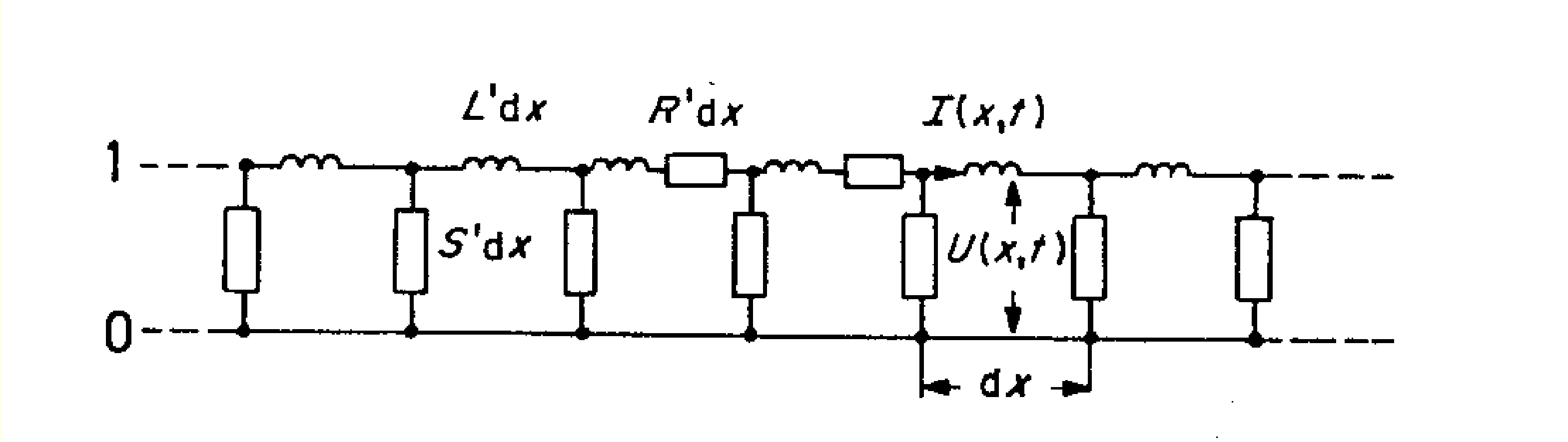
Turck (1974): analysis of 2-strand cables, both insulated (a) and non insulated (b)



- Axial propagation of current sharing imposed at the boundaries with a magnetic diffusivity D=D(G,L_1+L_2-2M)
- Faulty wires
- Short circuits between strands

Historical Overview (2)

Ries (1980): analysis of 2-strand cables



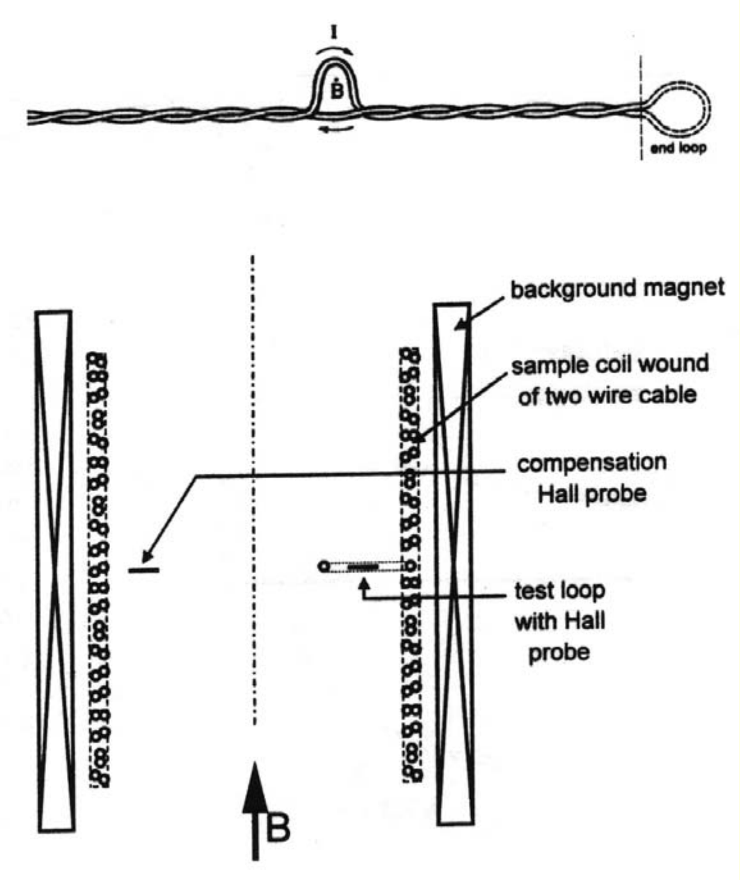
- Study of current sharing among quenching strands
- Determination of cable thermal stability through analytical calculation of power dissipated during transient
- Definition of a characteristic redistribution length and time constant:

$$\lambda = \frac{1}{\sqrt{R'S'}} \qquad \tau = \frac{L'}{\pi R'}$$

Historical Overview (3)

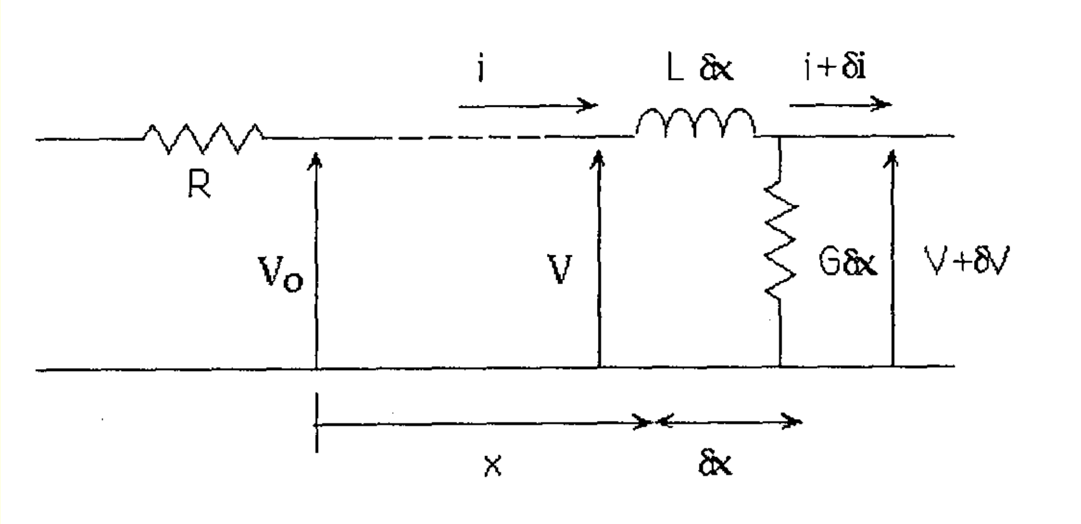
Krempasky-Schmidt (1995): "Theory of supercurrents" with a 2-strand cable model

- ➢ Study of longitudinal variations of dB/dt
- ➢ Two analytical solutions:
- • Field ramps (forced diffusion)
- • Constant field phases (free diffusion)
- ➢ Solution for a generic cycle is obtained through superposition due to linearity of the model



Historical Overview (4)

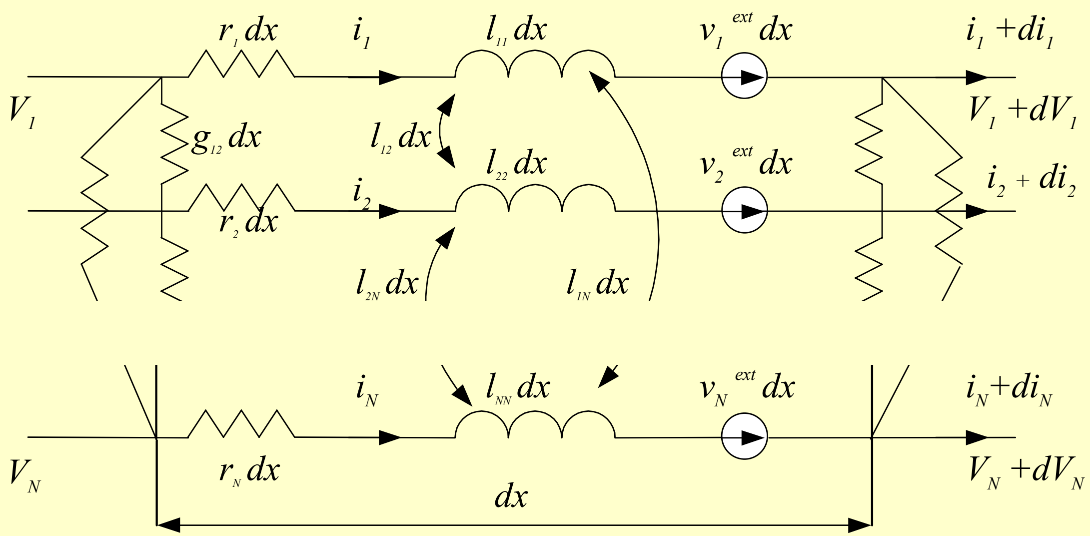
Mitchell (1999): analysis of 2-strand cables



- Study of current redistribution from a normal zone
- R is kept constant during the transient
- The redistribution region is considered to be in the superconducting state
- Development of a lumped circuit approximation to the solution

Model and assumptions

N-strand cable model



The model equations:

$$\mathbf{gl}\frac{\partial \mathbf{i}}{\partial t} + \frac{\partial^2 \mathbf{i}}{\partial x^2} + \mathbf{gri} - \mathbf{gv}^{ext} = 0$$

must be coupled with appropriate initial and boundary conditions

General solution

$$\mathbf{i}(x,t)=\frac{i_{op}(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\int_0^L d\xi\ \mathbf{K}^{(0)}(x,\xi,t)\ \mathbf{i}^{(0)}(\xi)+\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\ \mathbf{K}(x,\xi,t-\tau)\mathbf{v}^{ext}(\xi,\tau)$$

- General analytical solution applies to *circulant* **l** and **g** matrices: $l_{h,k} = l_{h-1,k-1}$ $h,k = 2,N$; $l_{1,k} = l_{N,K-1}$
- This condition is met:
  in Rutherford cables
  in CICCs wound in only one stage
  on average in multiple stage CICCs
- This solution involves:
  Intricate mathematical functions for kernels
  Numerical calculation of integrals

Assumptions for simplified solutions

1 Nil initial current distribution $i_h(x, 0) = 0$ $h = 1, N$
2 Nil longitudinal resistance $r_h(x, t) = 0$ $h = 1, N$
3 Simplified model matrices

$$l_{hk} = l \text{ if } h = k$$
$$l_{hk} = m \text{ if } h \neq k$$

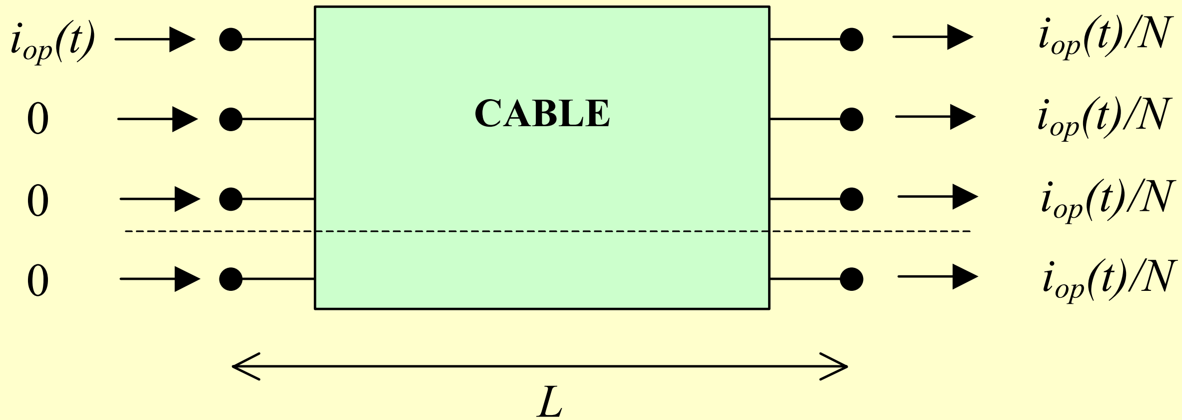
$$g_{hk} = g \text{ with } h \neq k$$

- These assumptions are strong, but:

they do not affect the validity of the benchmark when the solutions are compared to numerical codes

they allow an estimation of the mean behaviour of strand currents

1) Non uniform current distribution at the cable boundaries



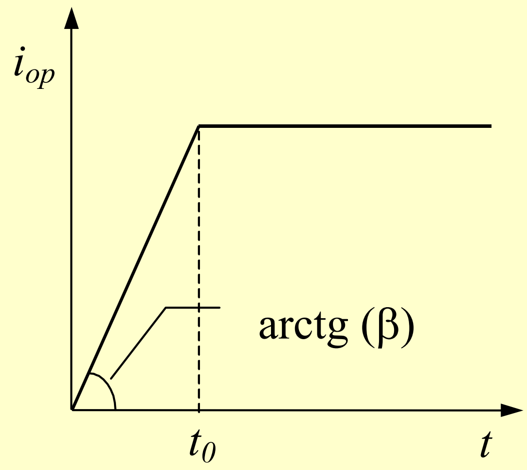
$$\begin{cases} \mathbf{g}\,\mathbf{l}\dfrac{\partial\,\mathbf{i}}{\partial t}(x,t)+\dfrac{\partial^2\mathbf{i}}{\partial x^2}(x,t)=0 \\ \mathbf{i}\,(x,t=0)=0 \\ i_1(x=0,\,t)=i_{op}(t), \qquad i_h(x=0,\,t)=0 \qquad h=2,N \\ i_h(x=L,t)=\dfrac{i_{op}(t)}{N} \qquad h=1,N \end{cases}$$

External voltage is neglected

Current cycle: ramp-up + plateau

$$i_{op}(t) = \beta\, t \qquad \text{for } t \le t_1$$

$$i_{op}(t) = \beta\, t_1 \qquad \text{for } t > t_1$$



Strand currents during the current ramp ($t \le t_1$)

Linear variation with x

Deviation from linearity

$$i_1(x,t) = i_{op}(t) - i_{op}(t)\frac{N-1}{N}\frac{x}{L} - \frac{N-1}{N}2\beta\sum_{n=1}^{\infty}\frac{\tau}{n^3\pi}\sin\left(\frac{n\pi x}{L}\right)\left[1-\exp\left(\frac{-t}{\tau_n}\right)\right]$$

$$i_h(x,t) = i_{op}(t)\frac{x}{NL} + \frac{1}{N}\beta\sum_{n=1}^{\infty}\frac{\tau}{n^3\pi}\sin\left(\frac{n\pi x}{L}\right)\left[1-\exp\left(\frac{-t}{\tau_n}\right)\right] \qquad h = 2, N$$

Strand currents during the current plateau ($t > t_1$)

Linear variation with x

Deviation from linearity

$$i_1(x,t) = i_{op}(t) - i_{op}(t)\frac{N-1}{N}\frac{x}{L} - \frac{N-1}{N}2\beta\sum_{n=1}^{\infty}\frac{\tau}{n^3\pi}\sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-t}{\tau_n}\right)\left[\exp\left(\frac{t_1 n^2}{\tau}\right)-1\right]$$

$$i_h(x,t) = i_{op}(t)\frac{x}{NL} + \frac{1}{N}2\beta\sum_{n=1}^{\infty}\frac{\tau}{n^3\pi}\sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-t}{\tau_n}\right)\left[\exp\left(\frac{t_1 n^2}{\tau}\right)-1\right] \qquad h = 2, N$$

Cable main time constant $\quad \tau = N\,(l-m)\,g\left(\frac{L}{\pi}\right)^2$

Cable time constants $\quad \tau_n = \left(\frac{\tau}{n^2}\right)$

At times much longer than the time constant ($t \gg \tau$)

Current ramp: the non linear term becomes negligible with respect to the linear term

$$\lim_{t\to\infty}\sum_{n=1}^{\infty}\frac{\tau}{n^3\pi}\sin\left(\frac{n\pi x}{L}\right)\left[1-\exp\left(\frac{-tn^2}{\tau}\right)\right]=\frac{\pi^2}{12}\tau\frac{x}{L}\left(1-\frac{x}{L}\right)\left(2-\frac{x}{L}\right)$$

$$i_{op}(t)\frac{N-1}{N}\frac{x}{L}$$ the linear term increases in time

Current plateau: the non linear term becomes negligible with respect to the linear term

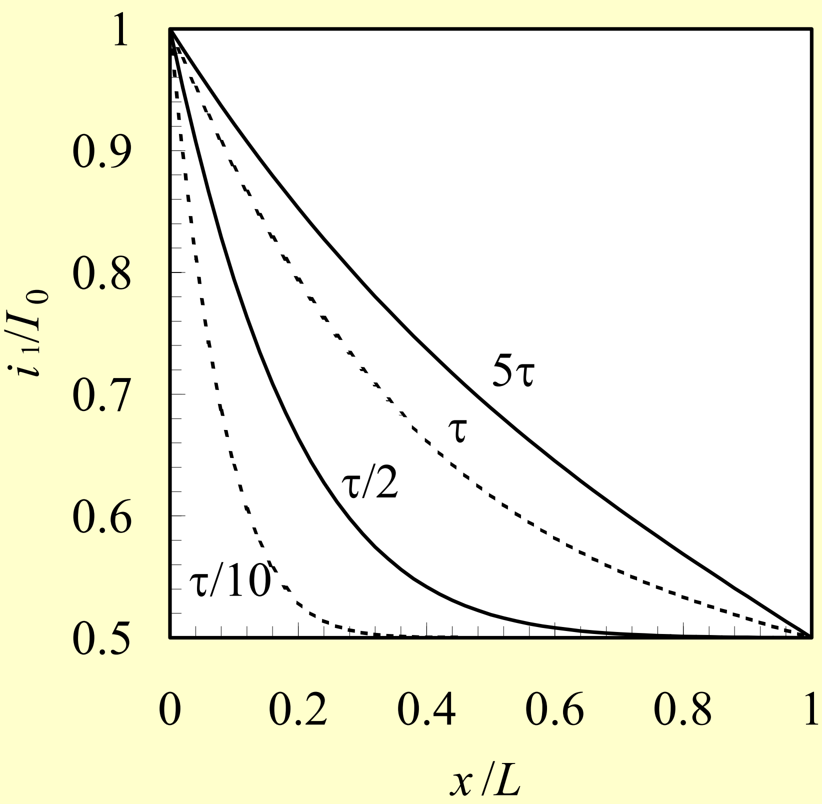
$$\lim_{t\to\infty}\sum_{n=1}^{\infty}\frac{\tau}{n^3\pi}\sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-tn^2}{\tau}\right)\left[\exp\left(\frac{t_1 n^2}{\tau}\right)-1\right]=0$$

$$i_{op}(t)\frac{N-1}{N}\frac{x}{L}$$ the linear term is constant

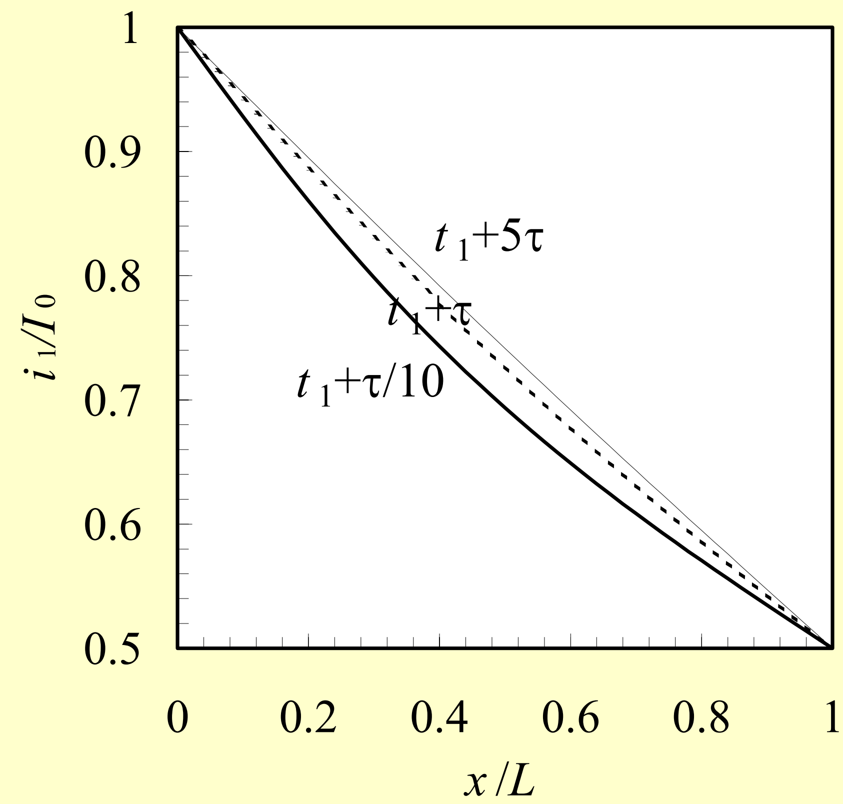
Calculations: space dependance

2-strand cable: L = 2.3 m, l = 5.0 10^{-6} H/m, m = 2.5 10^{-6} H/m, g =7.463 10^{6} S/m, τ = 2 s, β = 60 A/s, t_1 = 5 τ

Current ramp: strand 1



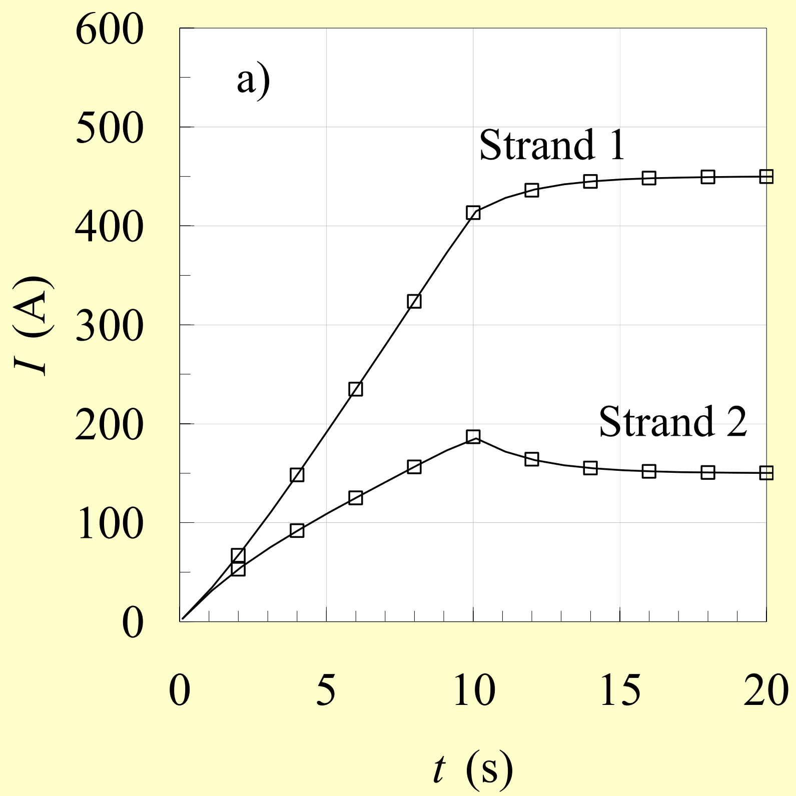
Current plateau: strand 1



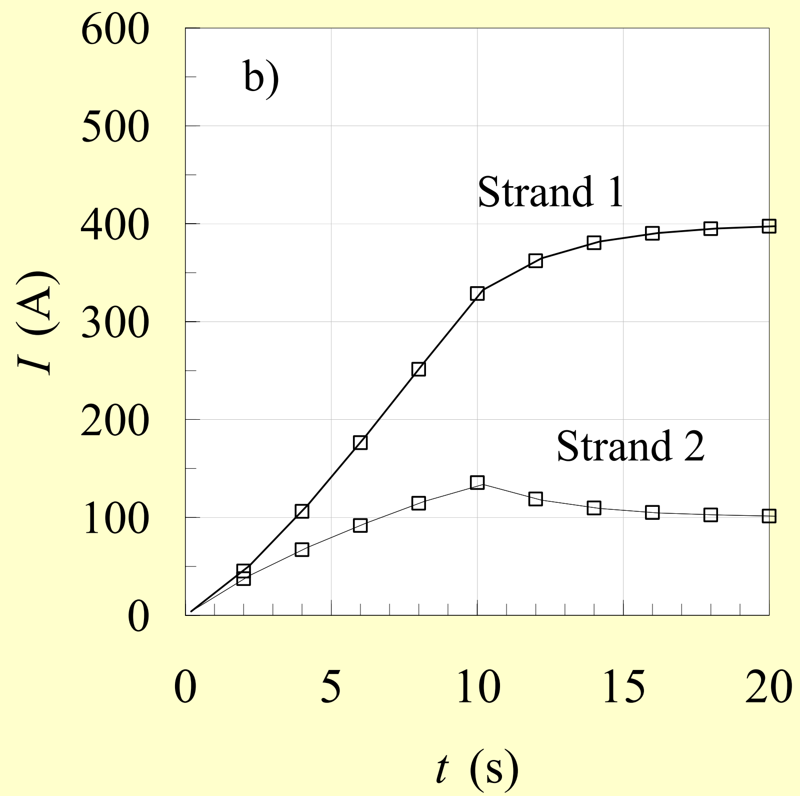
Calculations: time dependance

2 and 3-strand cable: analytical solution (symbols) vs numerical simulation (lines)

2-strand cable



3-strand cable

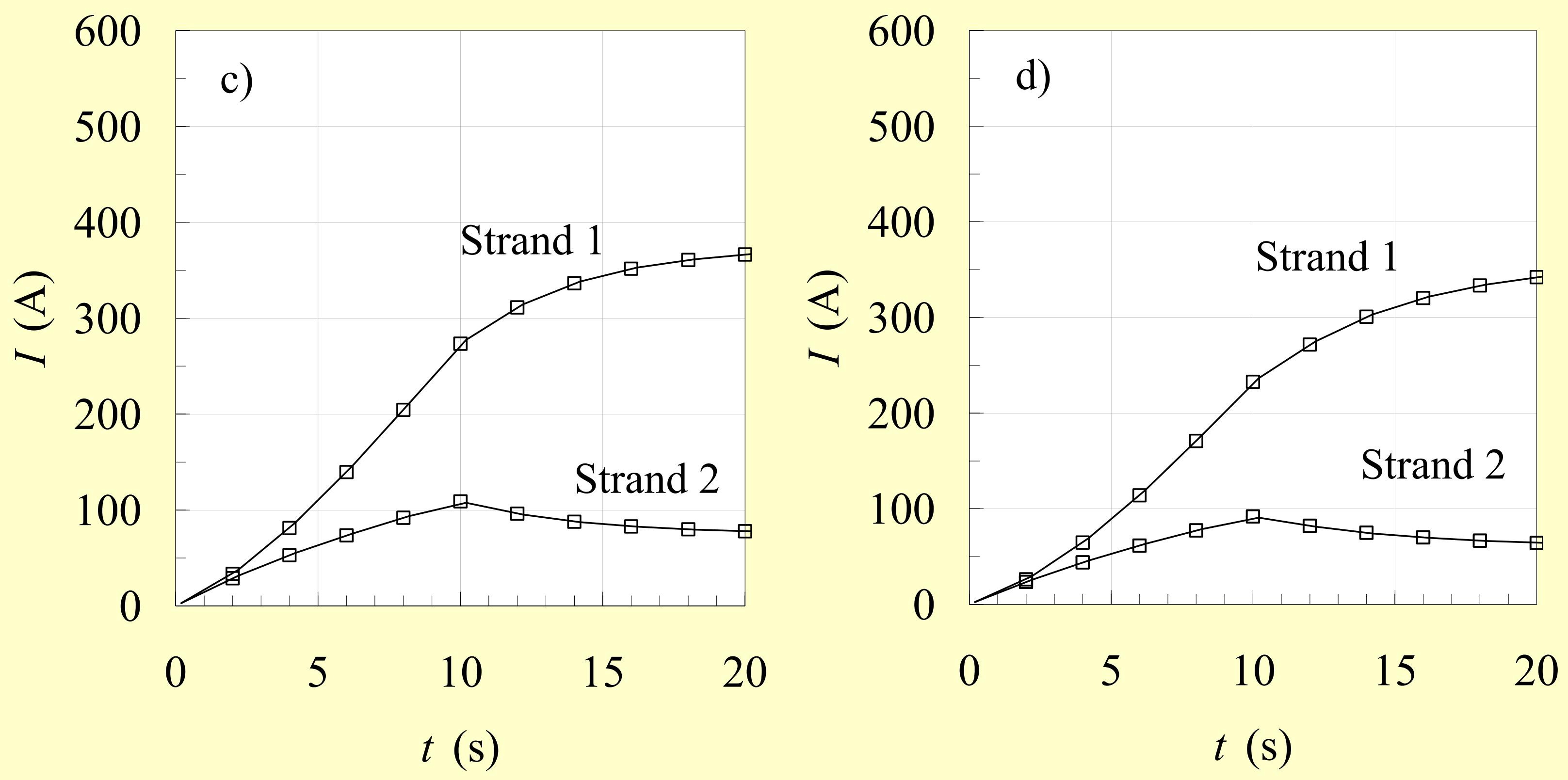


Calculations: time dependance

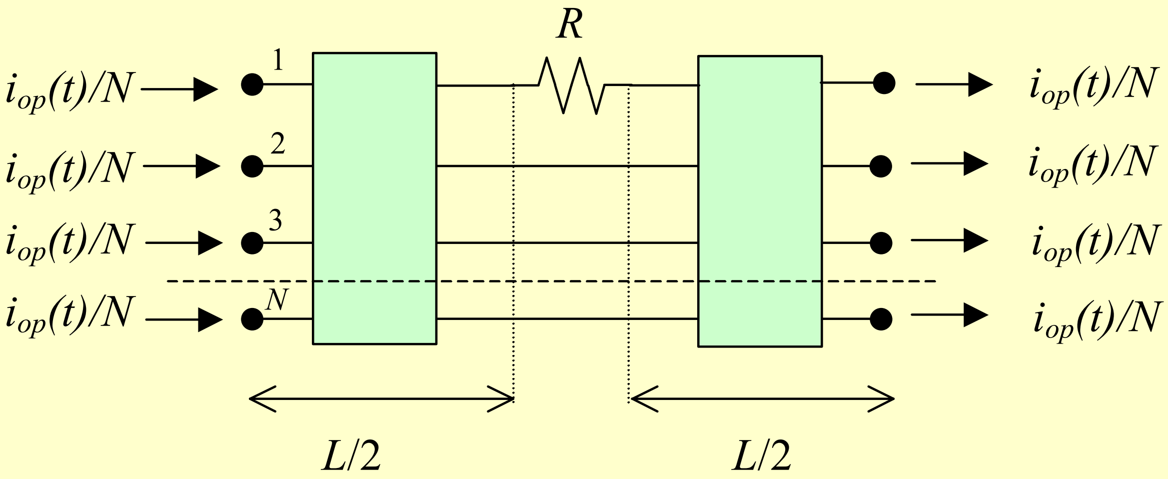
4 and 5-strand cable: analytical solution (symbols) vs numerical simulation (lines)

4-strand cable

5-strand cable



2) Quench in one strand



- Uniform current distribution at the boundaries
- External voltage is neglected
- A lumped resistance in the middle of the cable is representative of the first phases of quench in strand #1
- Extension of the solution already available for 2-strand cables (Turck, Mitchell)

Strand currents during a current ramp with ramp rate β ($t \le t_1$) in $0 \le x \le L/2$

Linear variation with x

Deviation from linearity

$$i_1(x,t) = \frac{i_{op}(t)}{N} + \frac{i_{op}(t)}{N}\frac{2x}{L}\frac{\omega}{1-\omega} + \frac{\beta}{N} A\left(x,t,\omega\right)$$

$$i_h(x,t) = \frac{i_{op}(t)}{N} - \frac{i_{op}(t)}{N}\frac{1}{N-1}\frac{2x}{L}\frac{\omega}{1-\omega} - \frac{\beta}{N(N-1)} A(x,t,\omega) \qquad h = 2, N$$

Strand currents during the current plateau ($t > t_1$) in $0 \le x \le L/2$

$$i_1(x,t) = \frac{i_{op}(t)}{N} + \frac{i_{op}(t)}{N}\frac{2x}{L}\frac{\omega}{1-\omega} + \frac{\beta}{N}\left[A\left(x,t,\omega\right) - A\left(x,t-t_1,\omega\right)\right]$$

$$i_h(x,t) = \frac{i_{op}(t)}{N} - \frac{i_{op}(t)}{N}\frac{1}{N-1}\frac{2x}{L}\frac{\omega}{1-\omega} - \frac{\beta}{N(N-1)}\left[A\left(x,t,\omega\right) - A\left(x,t-t_1,\omega\right)\right]$$

$$h = 2, N$$

Linear term (current in the quenched strand)

$$\frac{i_{op}(t)}{N} + \frac{i_{op}(t)}{N}\frac{\omega}{1-\omega}\frac{2x}{L} = \begin{cases} \dfrac{i_{op}(t)}{N} \text{ in } x = 0 \\ \dfrac{i_{op}(t)}{N} + \dfrac{i_{op}(t)}{N}\dfrac{\omega}{1-\omega} \quad \text{in } x = L/2 \end{cases}$$

$$\omega = -R\ g\ L\ (N-1)/4$$

If $R \to \infty$ or $g \to \infty$ then $\omega \to \infty$ and the current in the normal zone $x=L/2$ goes to zero

$$\frac{i_{\mathrm{op}}(\mathrm{t})}{N} + \frac{i_{\mathrm{op}}(\mathrm{t})}{N}\frac{\omega}{1-\omega} = 0$$

Non linear term (current in the quenched strand)

Space dependance

Time dependance

$$A(x,t,\omega) = 2\sum_{n=1}^{\infty} \frac{\cos(\xi_n(\omega))\sin(\xi_n(\omega)\cdot 2x/L)}{\cos(\xi_n(\omega))\sin(\xi_n(\omega)) - \xi_n(\omega)} (-\tau)\left(\frac{\pi/2}{\xi_n(\omega)}\right)^2 \left(\exp\left(-\frac{t}{\tau_n}\right) - 1\right)$$

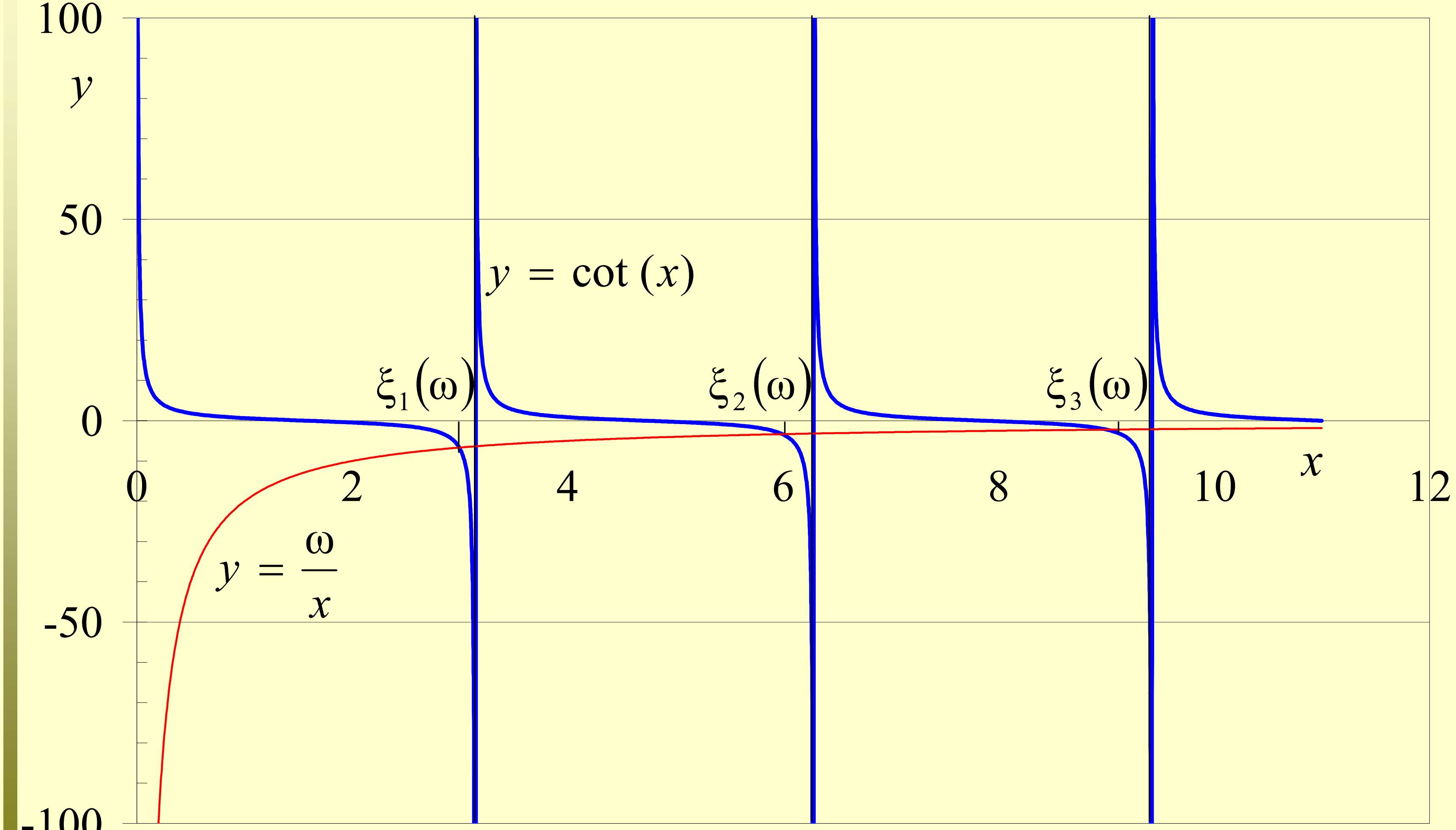
Cable main time constant $\tau = N\,(l - m)\,g\left(\frac{L}{\pi}\right)^2$

Cable time constants $\tau_n = \frac{\tau}{\left(\frac{\xi_n(\omega)}{\pi/2}\right)^2}$

At times much longer than the time constant ($t \gg \tau$)

The non linear term becomes negligible with respect to the linear term, both during the current ramp and the current plateau

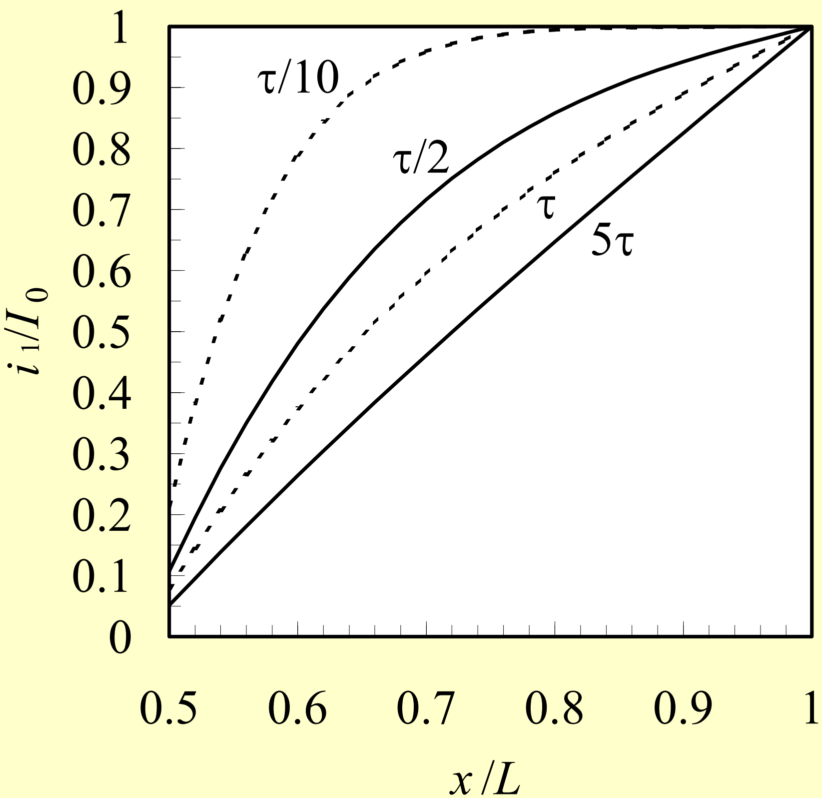
$\xi_n(\omega)$ is the solution of $\cot x = \dfrac{\omega}{x}$



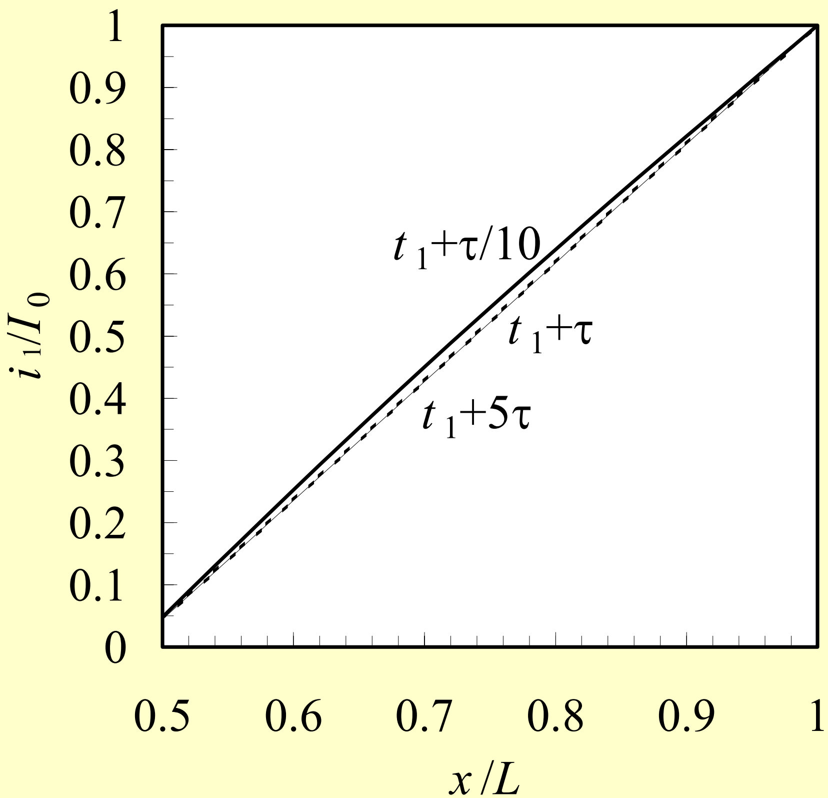
Calculations: space dependance

2-strand cable: L = 2.3 m, l = 5.0 10^{-6} H/m, m = 2.5 10^{-6} H/m, g =7.463 10^{6} S/m, τ = 2 s, β = 60 A/s, t_1 = 5 τ

Current ramp: strand 1



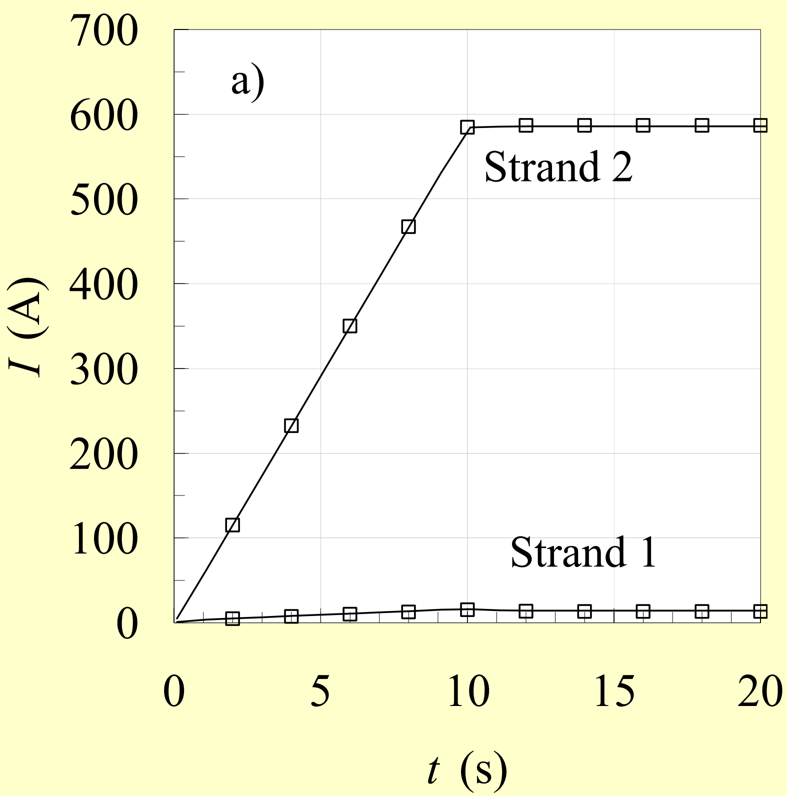
Current plateau: strand 1



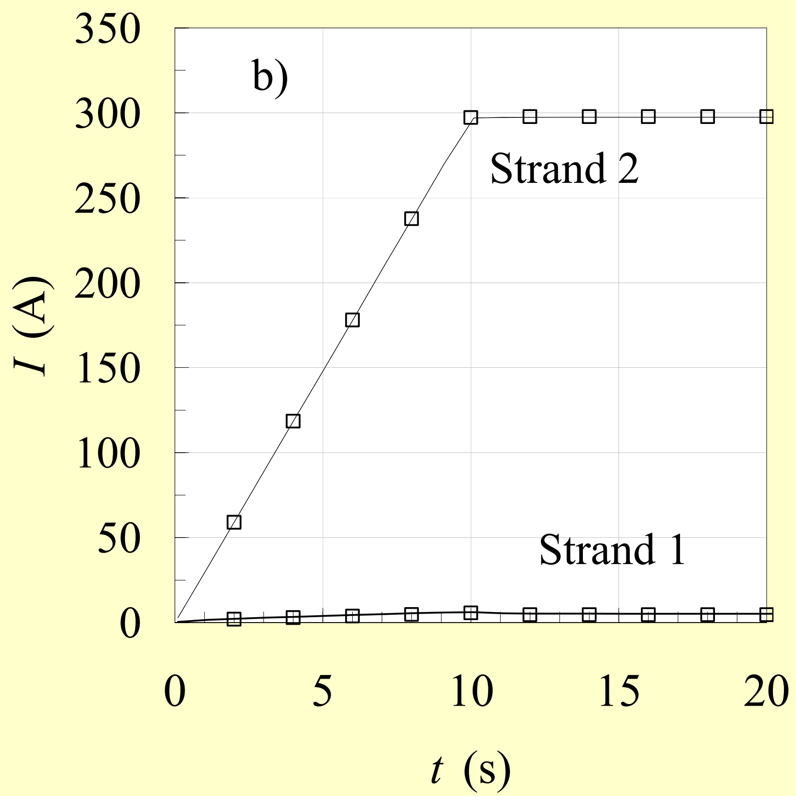
Calculations: time dependance

2 and 3-strand cable: analytical solution (symbols) vs numerical simulation (lines)

2-strand cable



3-strand cable

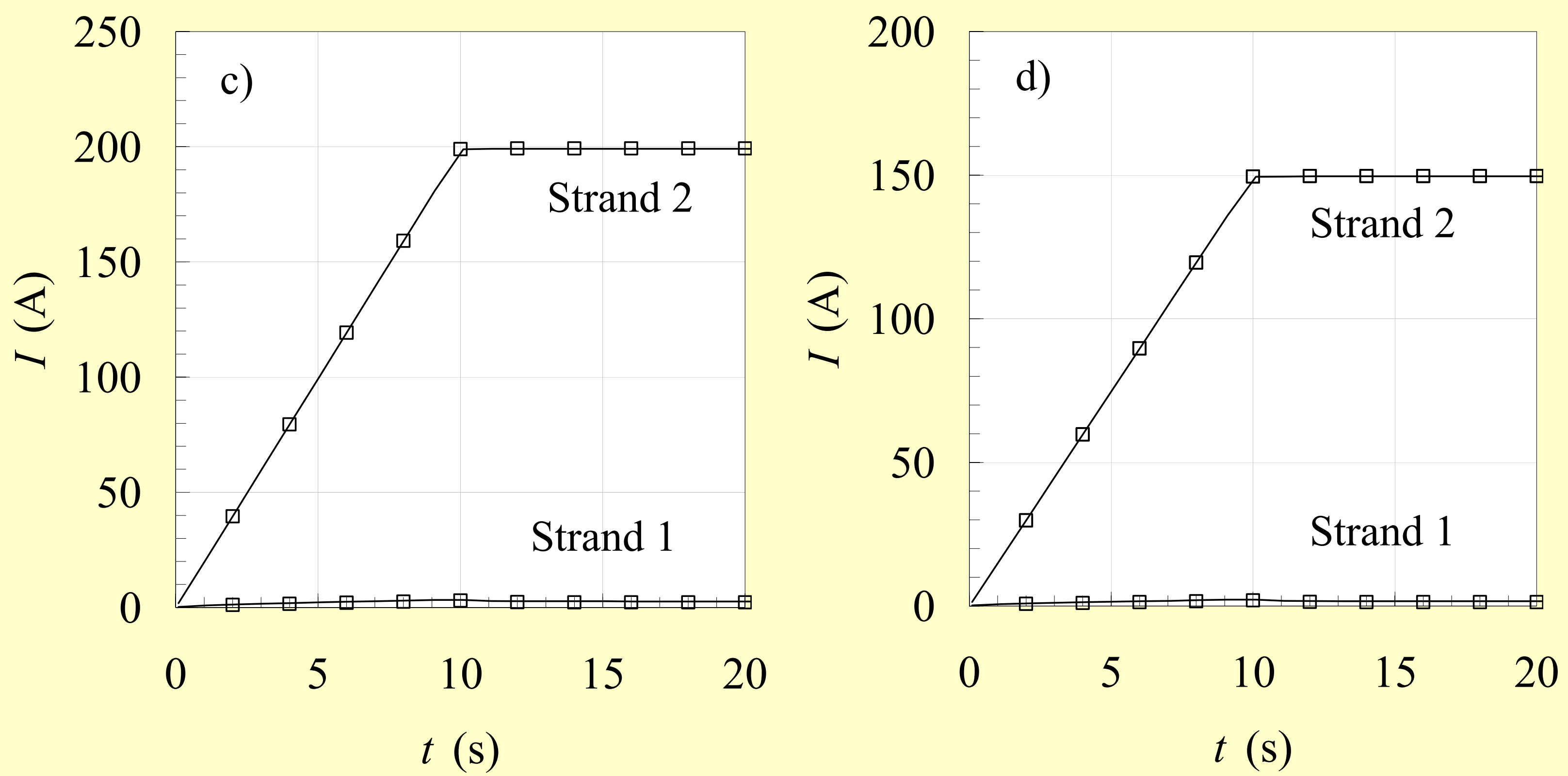


Calculations: time dependance

4 and 5-strand cable: analytical solution (symbols) vs numerical simulation (lines)

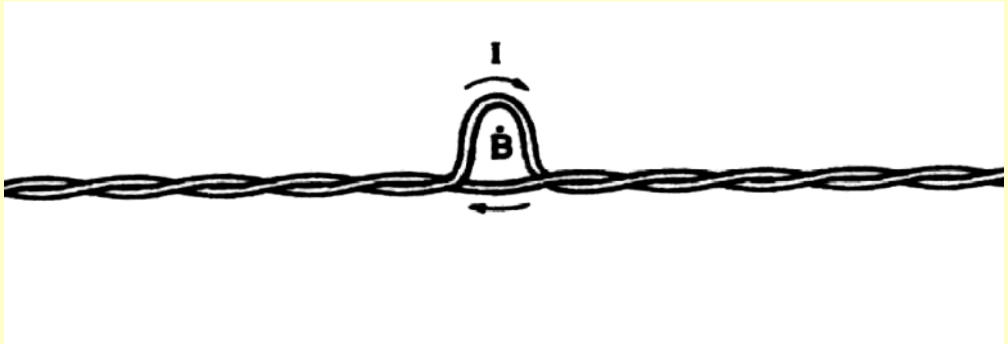
4-strand cable

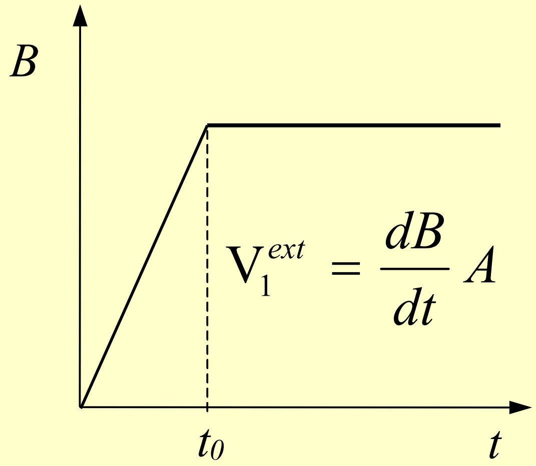
5-strand cable



3) Driving voltage excitation

- Extension of the solution already available for 2-strand cables (Krempasky-Schmidt)





$$\begin{cases} \mathbf{g}\,\mathbf{l}\frac{\partial\,\mathbf{i}}{\partial t}(x,t) + \frac{\partial^2 \mathbf{i}}{\partial x^2}(x,t) - \mathbf{g}\mathbf{v}^{ext}(x,t) = 0 \\ \mathbf{i}\,(x, t=0) = 0 \\ i_{op}(t) = 0 \\ v_1^{ext} = \frac{V_1^{ext}}{\delta} \text{ for } \quad x \in \left[\frac{L-\delta}{2}, \frac{L+\delta}{2}\right],\; v_1^{ext} = 0 \quad \text{for } x \in \left[0, \frac{L-\delta}{2}\right[\text{ and } x \in \left]\frac{L+\delta}{2}, L\right] \\ v_h^{ext} = 0 \quad \text{for } x \in [0, L] \quad \text{with } h = 2, N \end{cases}$$

Strand currents during the field ramp

Time dependance

Space dependance

$$i_h(x,t) = \frac{4}{\pi\alpha} I_h \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \left[\frac{1}{n^2} \left(1 - e^{-\frac{t}{\tau_n}}\right) \sin\left(\frac{n\,\alpha\,x}{w}\right) \sin\left(n\,\alpha\right) \right]$$

Strand currents during the constant field phase

Exponential decay

$$i_h(x,t) = \frac{4}{\pi\alpha} I_h \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \left[\frac{1}{n^2} \left(1 - e^{-\frac{t_1}{\tau_n}}\right) e^{-\frac{(t-t_1)}{\tau_n}} \sin\left(\frac{n\,\alpha\,x}{w}\right) \sin\left(n\,\alpha\right) \right]$$

$$\alpha = \pi \frac{L - \delta}{2L}$$

Cable main time constant $\tau = N\,(l - m)\,g \left(\frac{L}{\pi}\right)^2$

$$w = \frac{L - \delta}{2}$$

Cable time constants $\tau_n = \frac{\tau}{n^2}$

Maximum currents

$$I_1 = (N-1)\frac{wgV^{ext}}{2}$$

$$I_h = -\frac{wgV^{ext}}{2} \quad \text{with } h = 2, N$$

Redistribution length

$$\vartheta = \frac{t}{\tau}$$ Adimensional time

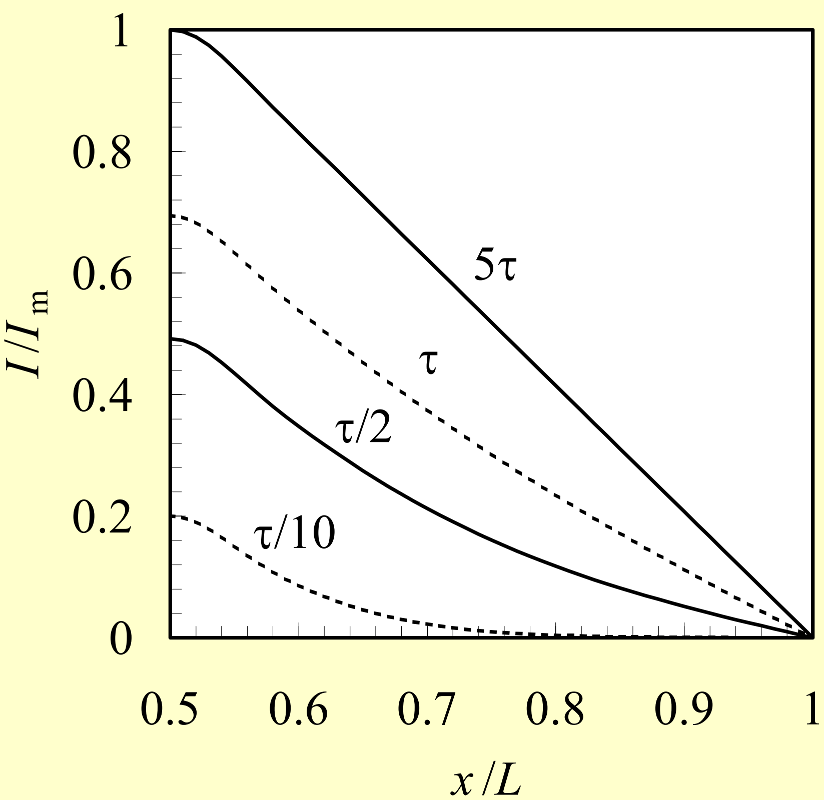
$$\eta_h = \frac{i_h}{I_h}$$ Adimensional current

$$\eta_h(x,t) = \frac{4}{\pi\alpha}\sum_{\substack{n=1 \\ n\ odd}}^{\infty}\left[\frac{1}{n^2}\left(1-e^{-\vartheta\, n^2}\right)\sin\left(\frac{n\,\alpha\,x}{w}\right)\sin(n\,\alpha)\right]$$

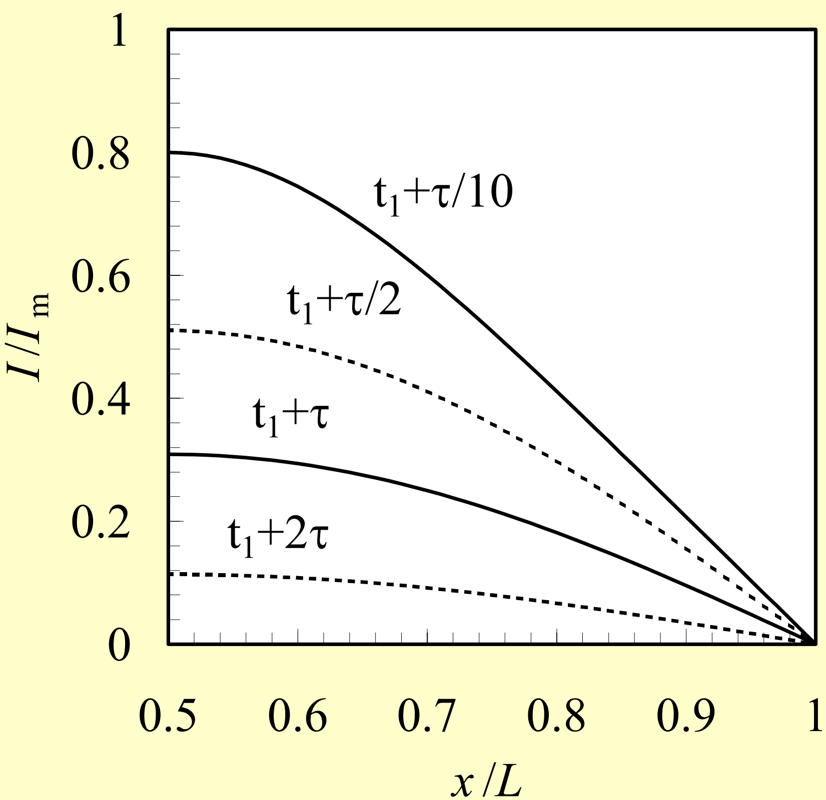
Calculations: space dependance

2-strand cable: L = 2.3 m, l = 5.0 10^{-6} H/m, m = 2.5 10^{-6} H/m, g =7.463 10^{6} S/m, τ = 2 s, $t_1 = 5\ \tau$

Field ramp: strand 1

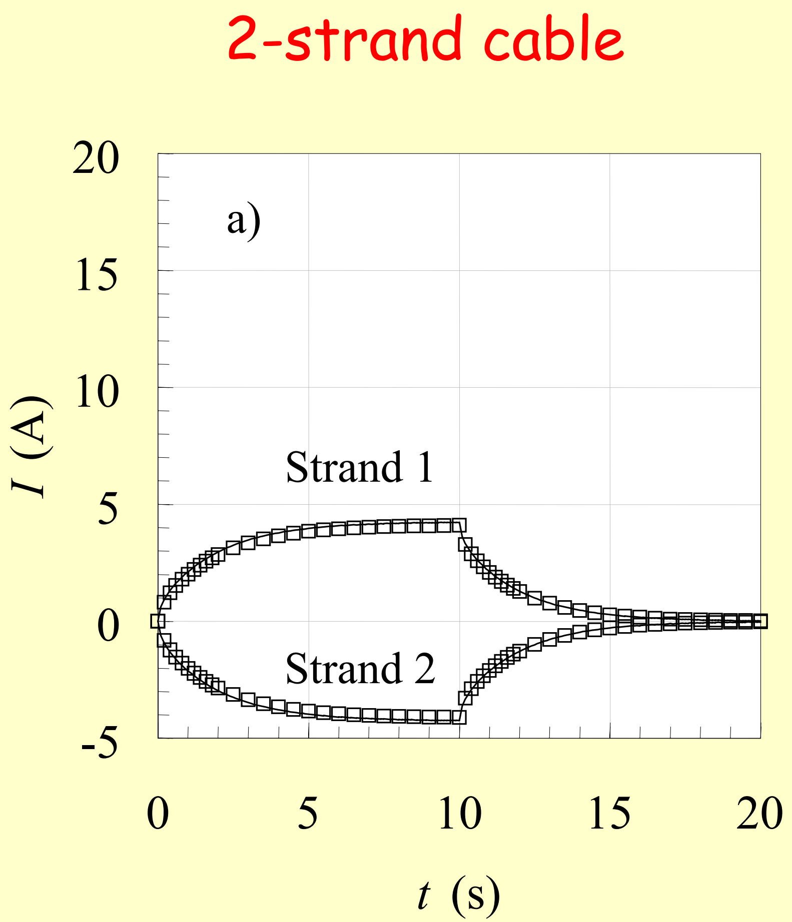


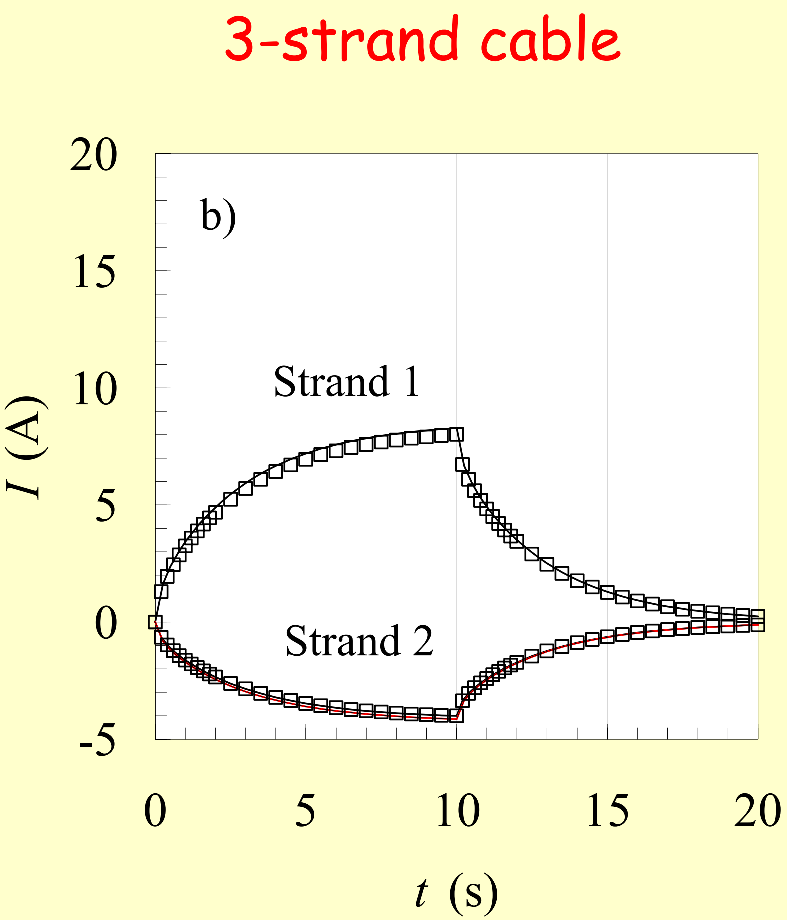
Field plateau: strand 1



Calculations: time dependance

2 and 3-strand cable: analytical solution (symbols) vs numerical simulation (lines)

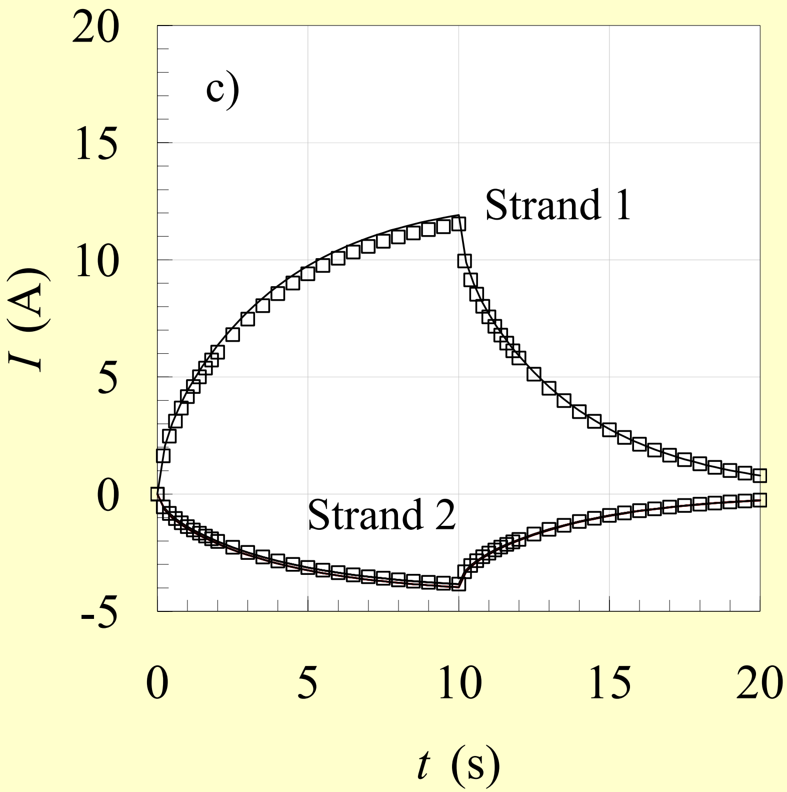




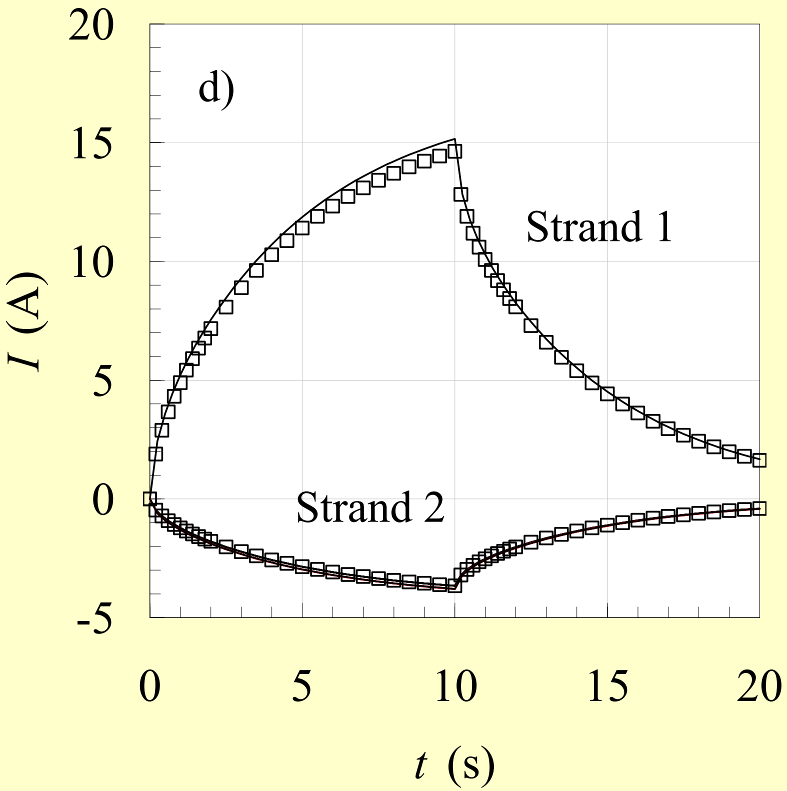
Calculations: time dependance

4 and 5-strand cable: analytical solution (symbols) vs numerical simulation (lines)

4-strand cable



5-strand cable



Conclusions (1)

Set of analytical formulae for current distribution in superconducting cables for:

- Preliminary benchmarking of numerical codes based on distributed parameters models (CDCABLE, NUCCIC, THEA, ...)
- Estimation of strand currents mean behaviour in the presence of:
  - non uniform boundaries
  - localised quenches (Turck, Mitchell for 2-strand cables)
  - localised driving voltages (Krempasky-Schmidt for 2-strand cables)

Conclusions (2)

- Quick Estimation of:
  - Time constants
  - Redistribution lengths
  - Maximum currents due to driving voltages
- Non-linear current-voltage characteristics of strands is not taken into account
- Substitution of numerical codes is not possible