Introduction to Cosmology

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Outline

- Isotropic and Homogeneous Universe
- GR in a Nutshell
- FLRW metric and light propagation
- Distances in Cosmology
- Dynamics for FRW metric
- Brief History of the Universe
- Horizons
- Problems of Standard Cosmological model
- Jeans Instability

S. Weinberg, Cosmology. Cambridge
E.W. Kolb, M.S. Turner; The Early Universe. Frontiers in Physics
V. Mukhanov; Physical Foundations of Cosmology. Cambridge
P. Patrick, J.-P. Uzan; Primordial Cosmology. Oxoford
D.S. Gorbunov, V.A. Rubakov; Intr. to the Theory of the Early Universe Vol. 1-2. World Scientific

Scales

Length parsec (pc) = 3.3 light year = 3.1 x10^{18} cm

= 2.1 x10^{5} a.u.

1 a.u. = Earth Sun distance ~ 3.1 x10^{24} cm

Pluto ~ 100 a.u.

Nearest star (Proxima Centauri) ~ 1.2 pc

center of Milky way ~ 8 kpc

Diameter of the Milky way ~ 30 kpc

nearest galaxy (Sagitarius dwarf in local group) ~ 30 kpc

nearest galaxy spiral (M31) ~ 800 kpc

size of the local cluster ~ 3 Mpc

distance from the nearest cluster (Virgo) ~ 15 Mpc

homogeneity scale ~ 100 Mpc

Hubble scale = c/H_0 ~ 4.2 Gpc

Observations (for instance Sloan Digital Sky Survey) tell us that there is a reference frame where Universe at scales ~ 100 Mpc appears

- homogeneous and Isotropic

Atlas of the Universe

At scales <<100 Mpc many structures can be distinguished

Filaments, Voids, Superclusters, Clusters, Galaxies ...

Hierarchical Approach:

Study first the Universe at large scales (> 100 Mpc)
deviation from homogeneity are small perturbations
(up to scales .5 Mpc)
at smaller scales nonlinear methods are required

Small deviation from primordial homogeneities
are amplified by gravitational instability:
gravitational instability is efficient when the Universe is non relativistic (matter dominated)

GR in a Nutshell

Cosmology needs gravity

GR: Dynamics of spacetime

Dynamical variable: the metric field $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

"Matters (EMT) tells spacetime (metric) how to curve" (Wheeler)

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = T_{\mu\nu}$$

Curvature = $[L^{-2}]$ ~ $\partial\left(g^{-1}\partial g\right) + \left(g^{-1}\partial g\right)\,\left(g^{-1}\partial g\right)$

Minkowski metric: flat space used in particle physics

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\
&= -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin\theta^2\, d\varphi^2\right)
\end{aligned}
$$

In flat space free particles follows straight lines

$$u^\nu \partial_\nu u^\mu = 0 \qquad u^\mu = \frac{dx^\mu}{d\tau}$$

$$u^\mu u^\nu \eta_{\mu\nu} = -1 \quad \text{massive particles} \qquad \mathrm{u}^\mu \mathrm{u}^\nu \eta_{\mu\nu} = 0 \quad \text{massless particles}$$

In curved space free particles follows geodesics

$$u^\nu \nabla_\nu u^\mu = 0 \qquad u^\mu u^\nu g_{\mu\nu} = -1 \qquad u^\mu = \frac{dx^\mu}{d\tau}$$

$$\nabla_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha}\, u^\alpha \qquad \Gamma^\mu_{\nu\alpha} = \frac{1}{2} g^{\mu\beta} \left(\partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\nu\beta} - \partial_\beta g_{\nu\alpha}\right)$$

for a scalar field:

$$\nabla_\mu \phi \equiv \partial_\nu \phi$$

Bianchi identities

$$\nabla^\nu \left(R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} \right) = \nabla^\nu\, G_{\mu\nu} = 0 \quad \to \nabla^\nu T_{\mu\nu} = 0$$

The EMT is covariantly conserved

EMT: response of matter action to a metric variation

$$\delta S_{\rm matter} = -\frac{1}{2} \int d^4x \sqrt{g}\, T_{\mu\nu}\, \delta g^{\mu\nu}$$

Examples

Scalar field

$$S = \int d^4x \sqrt{g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$$T_{\mu\nu} = \left[-\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi\, \partial_\beta \phi - V \right] g_{\mu\nu} + \partial_\mu \phi\, \partial_\nu \phi$$

Perfect Fluid

$$T_{\mu\nu} = p\, g_{\mu\nu} + (p + \rho)\, u_\mu\, u_\nu \qquad u^2 = u^\mu u^\nu g_{\mu\nu} = -1$$

Momentum density measured by an observer with four velocity v^μ

$$-T_{\mu\nu} v^\nu$$

Energy density measured by an observer with four velocity v^μ

$$T_{\mu\nu} v^\nu v^\nu$$

Energy density seen by an observer co-moving with fluid

$$\rho = T_{\mu\nu} u^\nu u^\nu$$

Pressure part leaves in space orthogonal the fluid velocity

$$T_{\mu\nu} = \rho\, u_\mu\, u_\nu + p\, (g_{\mu\nu} + u_\mu\, u_\nu)$$

$$(g_{\mu\nu} + u_\mu\, u_\nu)\, u^\nu = 0$$

Perfect Fluid in flat space

Take a homogeneous configuration (no space dependence)

co-moving coordinates: $u^\mu = (1,0,0,0)$

$T_{00} = \rho$ $\qquad T_{0i} = 0$ $\qquad T_{ij} = p\,\delta_{ij}$

Exercises

For the scalar field

- in flat space compute the EMT and check that it is conserved
- do the same in the case the metric is generic
- compute the EMT for a configuration of the form $\phi(t)$ and check that is perfect fluid, find then p and ρ

Friedmann-Lemaitre-Robertson-Walkwer (FLRW)

Observed Homogeneity and Isotropy ➡
at any given time t, the hypersurface (3dim.)
t=const. has no preferred direction and is
translational invariant ➡
maximally symmetric 3d geometry ➡
3d curvature is constant

$$ds^2 = -dt^2 + a(t)^2 d\Sigma^2 \qquad\qquad d\Sigma^2 = \gamma_{ij}\, dx^i dx^i$$

Observers with 4-velocity u^μ=(1,0,0,0) sees the
Universe homogenous and isotropic

$$d\Sigma^2 = \gamma_{ij}\, dx^i dx^i = \frac{dr^2}{1-\kappa\, r^2} + r^2\left[d\theta^2 + \sin(\theta)^2\, d\varphi^2\right]$$

κ=0 flat 3d geometry

κ=1 positive curvature
(spatially closed Universe)

κ=-1 negative curvature
(spatially open Universe)

Observers with 4-velocity $u^\mu=(1,0,0,0)$ follow geodesics

Exercise: check that at least in the case case $\kappa=0$

Overwhelming evidence that:
our Universe is spatially flat ($\kappa=0$)

For spatially flat ($\kappa=0$) and open ($\kappa<0$) Universe
the spatial volume is infinite
For a spatially closed ($\kappa>0$) Universe
the spatial volume is finite

The distance d(t,r) between the points:
(t,0,0,0) and (t,r,0,0) is not constant

$$d(t,r) = \int ds = a(t) \int_0^r \frac{dx}{(1-\kappa x^2)^{1/2}} = a(t) f_k(r)$$

$$f_0(r) = r \qquad f_{\kappa>0}(r) = \sin^{-1}(k^{1/2}\, r) \qquad f_{\kappa<0}(r) = \sinh^{-1}(|k|^{1/2}\, r)$$

Hubble Parameter

$$\frac{\dot{d}(t,r)}{d(t,r)} = \frac{\dot{a}\, f_k(r)}{a\, f_k(r)} = \frac{\dot{a}}{a} = H(t)$$

Hubble constant =H today = H_0 = h 100 km s^{-1} Mpc^{-1}

h = 0.673(12)

H >0 the Universe is expanding

Propagation of light in a FLRW Universe

4-velocity of photon (radial motion) $u^\mu = \frac{dx^\mu}{d\ell}$ $\quad u^\mu u^\nu g_{\mu\nu} = u^2 = 0$

$$\Rightarrow ds^2 = 0 = -dt^2 + a^2 \frac{dr^2}{(1-\kappa\, r^2)} \Rightarrow \frac{dt}{a} = \pm \frac{dr^2}{(1-\kappa\, r^2)^{1/2}}$$

emission event (t_E, r_E,0,0) event
absorption event: now and here (t_0,0,0,0)

$$\int_{t_E}^{t_0} \frac{dt}{a(t)} = \int_0^{r_E} \frac{dr}{(1-\kappa\, r^2)^{1/2}} \equiv f_k(r_E)$$

for a second photon traveling in the way

$$\int_{t_E+\delta t_E}^{t_0+\delta t_0} \frac{dt}{a} = f_k(r_E) \quad \Rightarrow \quad \int_{t_E}^{t_0} \frac{dt}{a(t)} = \int_{t_E+\delta t_E}^{t_0+\delta t_0} \frac{dt}{a}$$

$$\int_{t_E}^{t_E+\delta t_e} \frac{dt}{a(t)} = \int_{t_0+\delta t_0}^{t_0} \frac{dt}{a} \quad \Rightarrow \quad \frac{\delta t_E}{a(t_E)} = \frac{\delta t_0}{a(t_0)}$$

$$\frac{\delta t_E}{\delta t_0} = \frac{a_E}{a_0} \quad \Rightarrow \quad \frac{\nu_0}{\nu_E} = \frac{a_E}{a_0}$$

Expanding universe: $a_E < a_0$ then $\nu_0 < \nu_E$ Cosmological redshift

$$z = \frac{\nu_E - \nu_0}{\nu_0} = -1 + \frac{a_0}{a_E} \quad \Rightarrow \quad 1 + z = \frac{a_0}{a_E}$$

Setting t as the emission time $a_E = a(t)$ and $a_0 = 1$
we can replace time with redshift t(z)

$$\frac{dz}{dt} = -(1+z)\, H$$

Hubble Law

At small z $$a(t) = a_0 - (t_0 - t)\dot{a}_0 + \cdots = a_0\left[1 - H_0(t_0 - t)\right] + \cdots$$

$$(1+z)a(t) = a_0 \quad \Rightarrow \quad (1+z)\left[1 - H_0(t_0 - t)\right] = 1 + \cdots$$

$$z = H_0\,(t_0 - t)c^{-1} + \cdots = H_0\,d + \cdots$$

In general galaxies have peculiar velocities due to local gravitational field

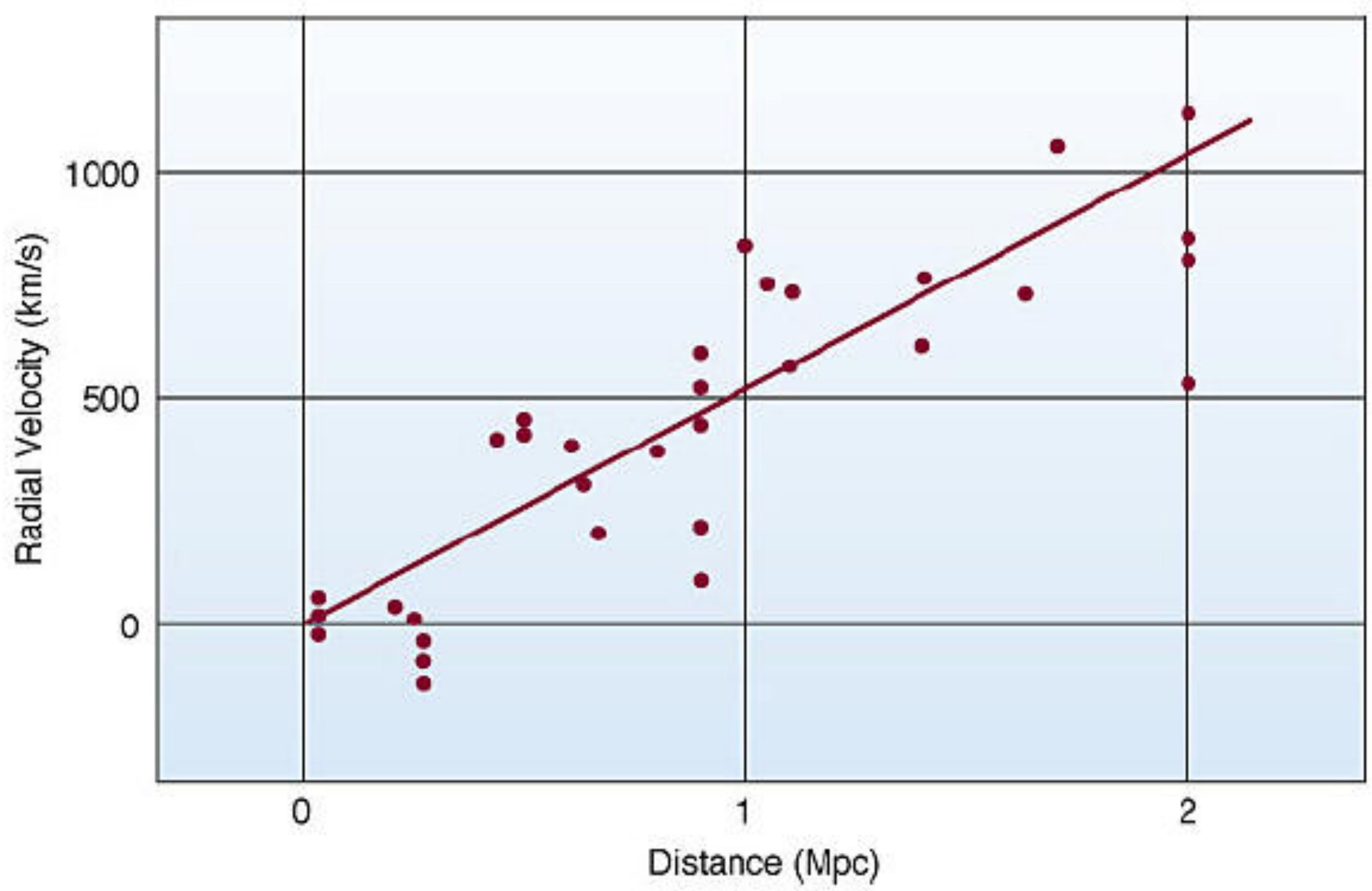
$v = v_{cosm} + v_{peculiar}$

To overcome pec. velocities ~ 100 Km/s one needs zc >> 100 Km/s

$d = c\,z/H_0 = h\,3\times10^3$ Mpc z

known galaxies (Hubble telescope) of z~11.9 13 bilion years old

Original Hubble Diagram



FLRW Dynamics

Solve Einstein equations with matter represented by a perfect fluid

$$T_{00} = \rho \qquad T_{0i} = 0 \qquad T_{ij} = p\,\gamma_{ij}$$

EMT conservation $\nabla^\nu T_{\mu\nu} = 0$ gives (check !!)

$$\dot\rho + 3\,H\,(\rho + p) = \dot\rho + 3\,H\,(1+w)\,\rho = 0 \qquad p = w\,\rho$$

p = w ρ is called fluid equation of state

EMT conservation ⇔ 1st principle of thermodynamics

$$\partial_t(\rho\, a^3) = -p\partial_t(a^3) \qquad dU = \delta L + dS$$

adiabatic process

EMT conservation can be integrated for w constant

$$\rho(t) = \frac{\rho_0}{a^{3(w+1)}}$$

NB: $t=t_0$ with $a(t_0)=1$

ρ_0 present density

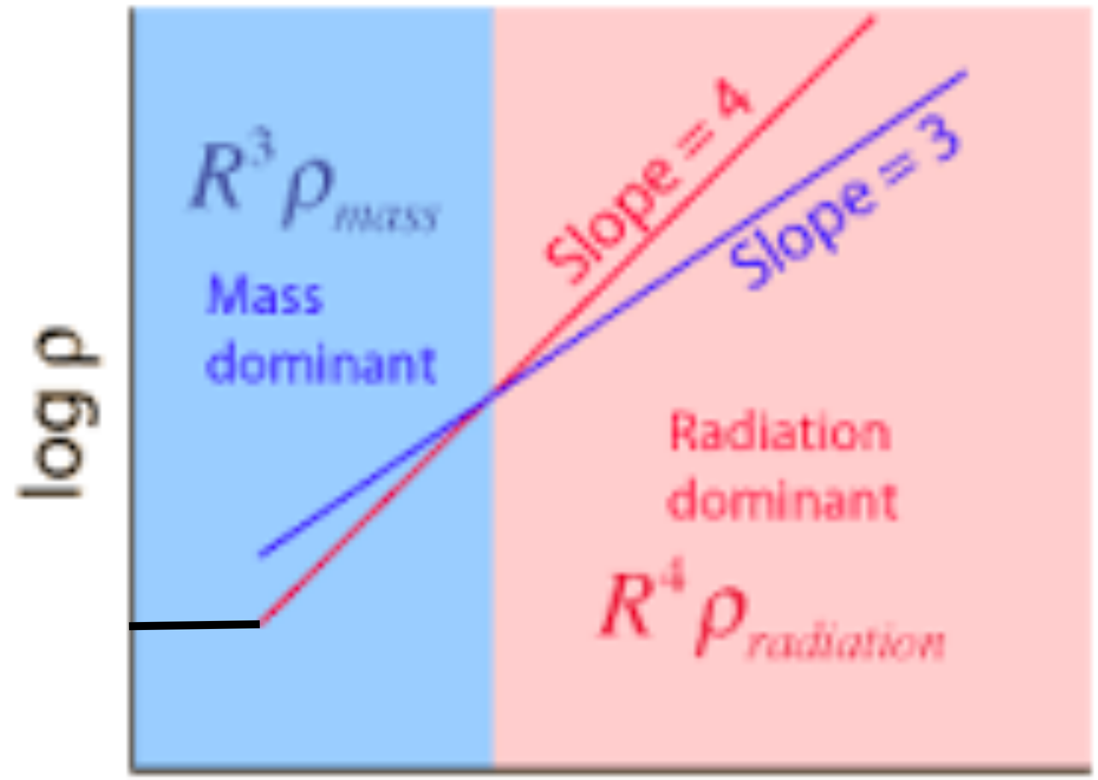
w =0, p = 0 non-relativistic matter $\qquad \rho = \frac{\rho_0}{a^3}$

w =1/3 relativistic matter $\qquad \rho = \frac{\rho_0}{a^4}$

w = -1 Cosmological Constant $\qquad \rho = \Lambda$

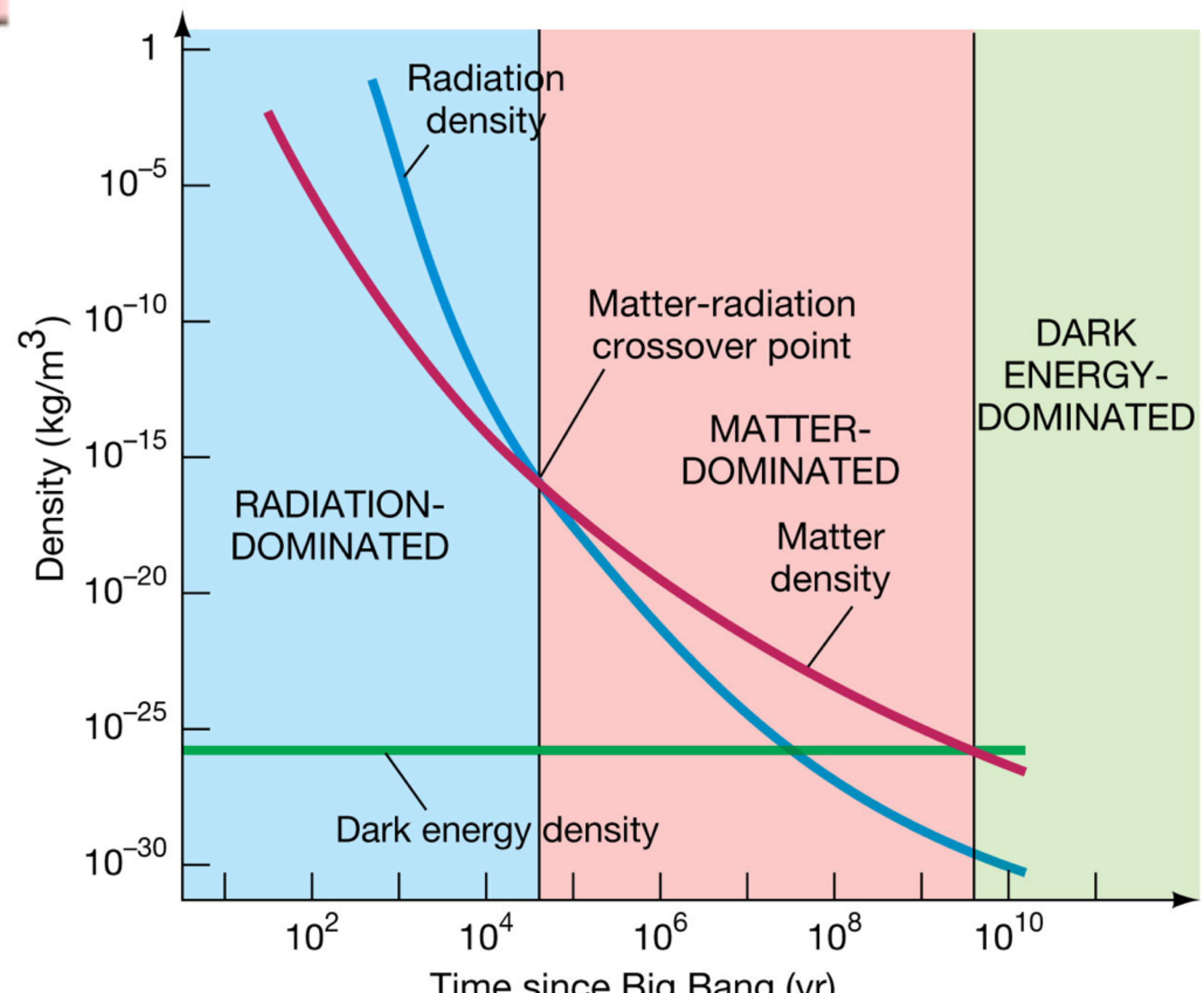
EMT given by $\qquad T_{\mu\nu} = -\Lambda \, g_{\mu\nu} \qquad$ Vacuum energy

Derived from the action $\qquad S = -\int d^4x \sqrt{g} \, \Lambda$



$1 + z = \frac{1}{R}$ → backward in time

NB: a(t) = R



Einstein Equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Symmetry:

0i components are zero

ij components give a single eq.

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3} \rho \qquad \text{00}$$

$$2\frac{\ddot{a}}{a} + H^2 + \frac{\kappa}{a^2} = -8\pi G\, p \qquad \text{ij}$$

Actually from 00 eq. + EMT conservation one gets ij

we can forget about ij eqs.

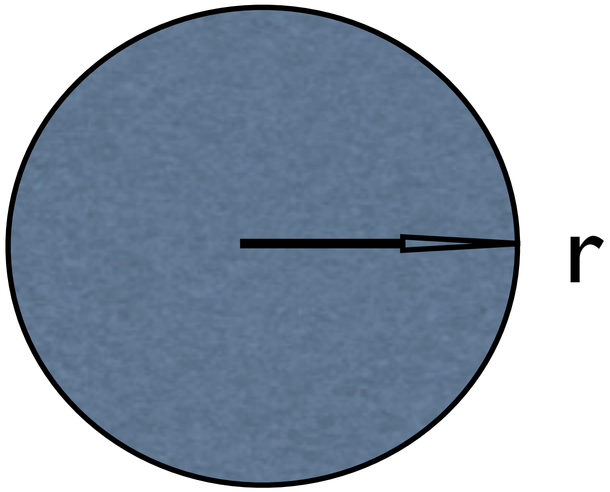
Newtonian "Derivation"

Take a bunch of particles with homogeneous density ρ
the energy of a test particle of mass m at radius r

$$E = \frac{m}{2}\,\dot{r}^2 - \frac{MmG}{r} = \frac{m}{2}\,\dot{r}^2 - \frac{m\,G}{r}\,\rho\,\frac{4\pi\,r^3}{3}$$

$$= \frac{m}{2}\,\dot{a}^2\vec{x}^2 - \frac{4\pi\,G\,m\,\rho\,a^2\,\vec{x}^2}{3}$$

$$\frac{\dot{a}^2}{a^2} - \frac{2\,E}{m\,a^2\,\vec{x}^2} = \frac{8\pi G}{3}\,\rho$$



$$\vec{r} = a(t)\,\vec{x}$$

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\,\rho$$

$$\kappa = -\frac{2\,E}{m\,\vec{x}^2} \qquad H = \frac{\dot{a}}{a}$$

Matter conservation

$$\partial_t\left(\rho\,\frac{4\pi}{3}\,r^3\right) = 0 \qquad \Rightarrow \dot{\rho} + 3H\,\rho = 0$$

EMT conservation
non-rel. matter for p=0

Critical Density

$$\frac{\kappa}{a^2} = \frac{8\pi G}{3}\left(\rho - \frac{3H^2}{8\pi G}\right) = \frac{8\pi G}{3}\left(\rho - \rho_c\right) = \frac{8\pi G\,\rho_c}{3}\left(\Omega - 1\right) = H^2\left(\Omega - 1\right)$$

$$\rho_c = \frac{3\,H^2}{8\pi G} \qquad\qquad \Omega = \frac{\rho}{\rho_c}$$

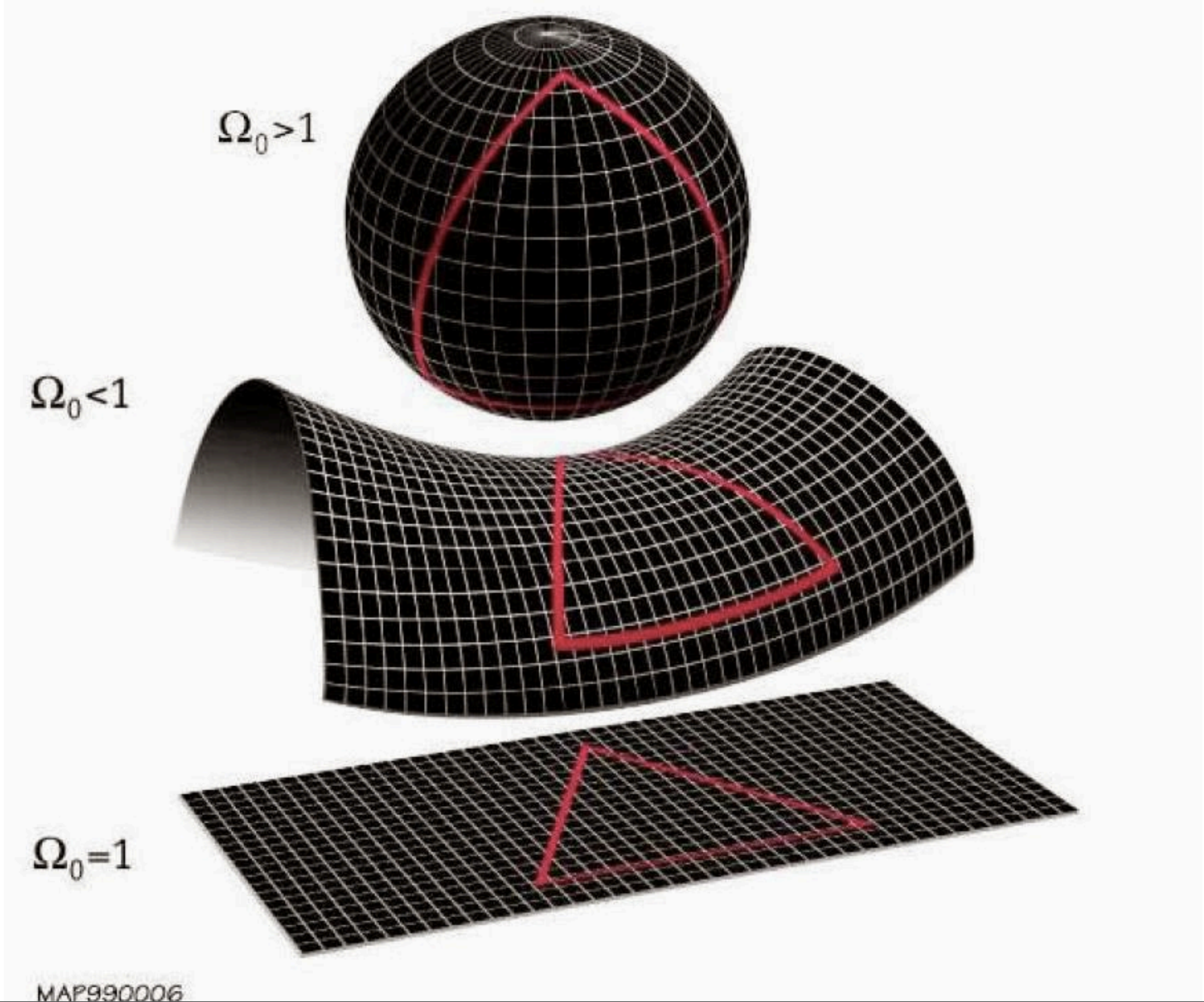
At any time the sign of κ is the same Ω-1
taking $t=t_0$ (now)

$$\kappa > 0 \quad \Rightarrow \rho_0 > \rho_{0c} = \frac{3H_0}{8\pi G} \approx h^2\, 1.8 \times 10^{-29}\text{gr cm}^{-3}$$

$$\kappa < 0 \quad \Rightarrow \rho_0 < \rho_{0c}$$

$$\kappa = 0 \quad \Rightarrow \rho_0 = \rho_{0c}$$

Spatial curvature connected present amount of matter



$\Omega_0 > 1$
$\Omega_0 < 1$
$\Omega_0 = 1$
MAP990006

2d Examples

Take a 2-sphere embedded in a 3d Euclidian space

$$x^2 + y^2 + z^2 = b^2 \qquad\qquad d\ell^2 = dx^2 + dy^2 + dz^2$$

How to find the infinitesimal distance (metric) on the 2-sphere ?

Solve the constraint and plug it back on embedding space metric

$$x = r\ \cos\varphi \qquad x = r\ \sin\varphi \qquad z = \pm(b^2 - r^2)^{1/2}$$

$$\Rightarrow \quad dx = dr\ \cos\varphi - r\, d\varphi\ \sin\varphi \qquad dy = r\, d\varphi\ \cos\varphi + dr\ \sin\varphi$$

$$dz = \mp(b^2 - r^2)^{-1/2}\, r\, dr$$

$$\Rightarrow \quad dl^2 = \frac{dr^2}{1 - \frac{r^2}{b^2}} + r^2\, d\varphi^2$$

That is the case $\kappa=b^{-2}>0$, positive constant curvature

For large very large b, $\kappa=0$, curvature 2d plane

Take an hyperboloid embedded in a 3d Minkowski space

$$x^2 + y^2 - z^2 = -b^2 \qquad dl^2 = -dz^2 + dx^2 + dy^2$$

$$x = r\ \cos\varphi \qquad y = r\ \sin\varphi \qquad z = \pm(b^2 + r^2)^{1/2}$$

$$\Rightarrow\ dx = dr\ \cos\varphi - r\,d\varphi\ \sin\varphi \qquad dy = r\,d\varphi\ \cos\varphi + dr\ \sin\varphi$$

$$dz = \pm(b^2 + r^2)^{-1/2}\, r\, dr$$

$$\Rightarrow \quad dl^2 = \frac{dr^2}{1 - \frac{r^2}{b^2}} + r^2\, d\varphi^2$$

That is the case $\kappa=-b^{-2} <0$, negative constant curvature

Lobachevski space

The fluid can have many component

$$\rho = \sum_{n=1}^{N} \rho_n \, \rho_n \qquad\qquad p = \sum_{n=1}^{N} p_n \qquad p_n = w_n \, \rho_n$$

One can measure ρ_0 by counting matter and $\rho_0 < \rho_{0c}$

CMB observations tell us that κ is very small

The missing component is called

Dark Energy

Today's Content

Ω_{DE}=0.68 Ω_{rad}=10^{-5}

$\Omega_{non\text{-}rel\text{-}matt}$= $\Omega_{dark\ matter}$ + $\Omega_{baryons}$ =0.27+0.048 =0.32

Solutions (k=0)

$$\rho(t) = \frac{\rho_0}{a^{3(w+1)}}$$

generic w

Plug into the equation for a and solve it !

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \frac{\rho_0}{a^{3(w+1)}}$$

$$\Rightarrow \qquad a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(w+1)}}$$

Non-relat. matter dominated Universe, w=0 $\quad a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$

Radiation dominated Universe, w=1/3 $\quad a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}$

Solutions (k=0)

The case w=-1 is special. Universe dominated by a CC

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\Lambda \equiv \lambda^2 \qquad \Rightarrow \qquad a(t) = e^{\lambda\, t}$$

de Sitter Universe

$$ds^2 = -dt^2 + e^{\lambda\, t}(dx^2 + dy^2 + dz^2)$$

In general the 4d scalar curvature is for FLRW

$$R = 6\left(\frac{\ddot{a}}{a} + H^2 + \frac{\kappa}{a^2}\right) \qquad R_{\rm dS} = 12\,\lambda^2 > 0$$

de Sitter Universe has 4d constant positive curvature !

Milne Universe

Take the vacuum: ρ=0

$$H^2 + \frac{\kappa}{a^2} = 0$$

κ has to be <0

$$a(t) = |\kappa|^{1/2}\, t$$

$$ds^2 = -dt^2 + |k|\, t^2 \left[\frac{dr^2}{1+|k|r^2} + r^2 \left(d\theta^2 + \sin\theta^2\, d\varphi^2\right)\right]$$

$$H = t^{-1} \qquad \ddot{a} = 0 \qquad \Rightarrow \qquad R_{\mathrm{Milne}} = 0$$

Setting $\tau = t\left(1+|k|\, r^2\right)^{1/2}$ $\qquad s = |k|^{1/2}\, t\, r$

$$ds^2 = -d\tau^2 + ds^2 + s^2\left(d\theta^2 + \sin\theta^2\, d\varphi^2\right)$$

A portion of Minkowski space in disguise

Acceleration

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho \qquad 2\frac{\ddot{a}}{a} + H^2 + \frac{\kappa}{a^2} = -8\pi G\, p$$

$$-\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(\rho + 3p) \equiv q\, H^2$$

If the Universe is accelerating a certain time t when q(t) <0

or ρ + 3 p <0 ⇒ w <- 1/3 negative pressure

With multiple components

$$q = \frac{4\pi G}{3 H^2}(\rho + 3p) = \frac{1}{2\rho_c}(\rho + 3p) = \frac{1}{2}\left(\Omega + \sum_n 3\, w_n\, \Omega_n\right)$$

acceleration ⇒ $\Omega + \sum_n 3\, w_n\, \Omega_n < 0$

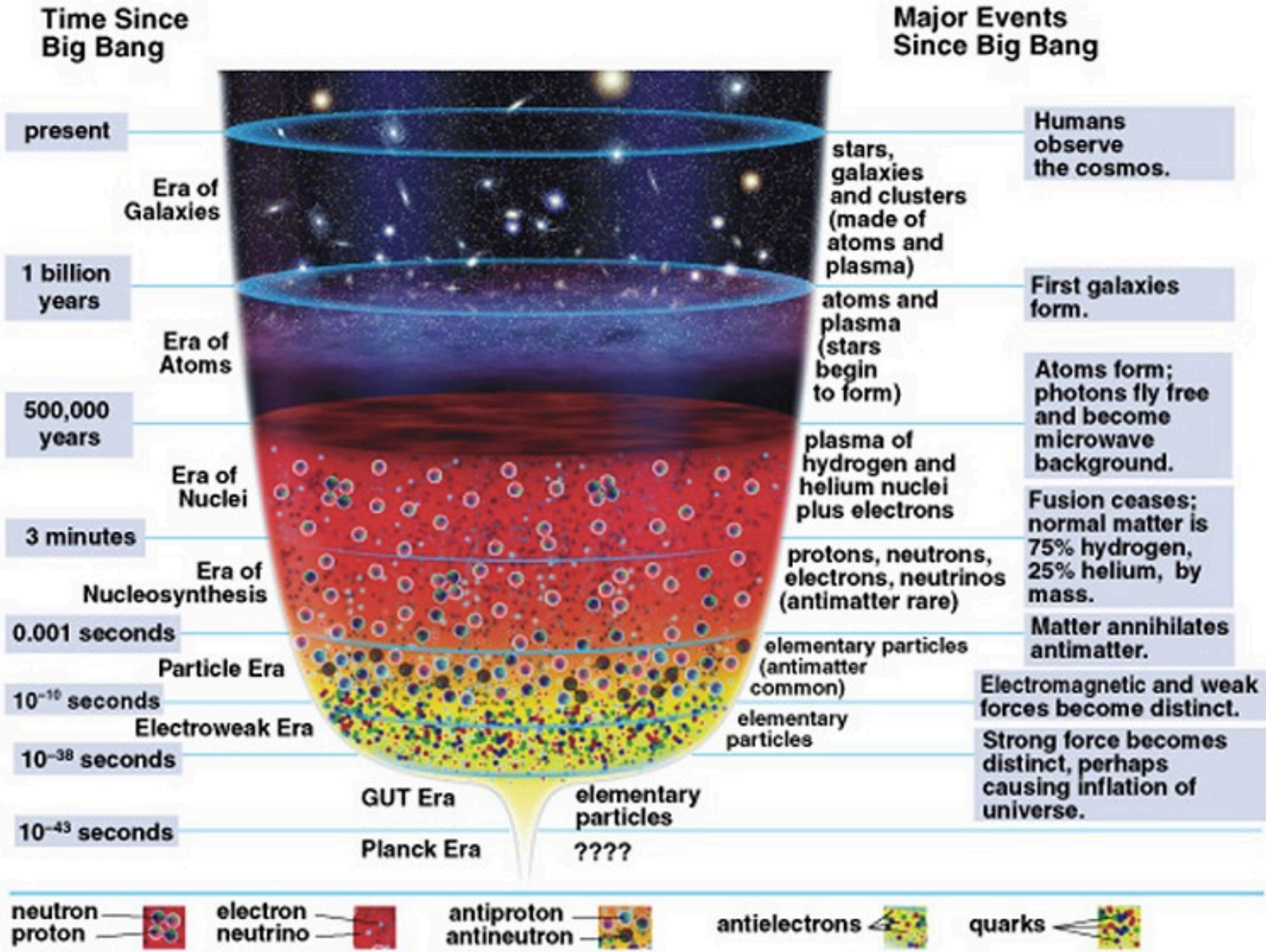
Data (SNe Ia) tell us that presently the Universe (flat) is accelerating

We have NR-matter+ Dark Energy

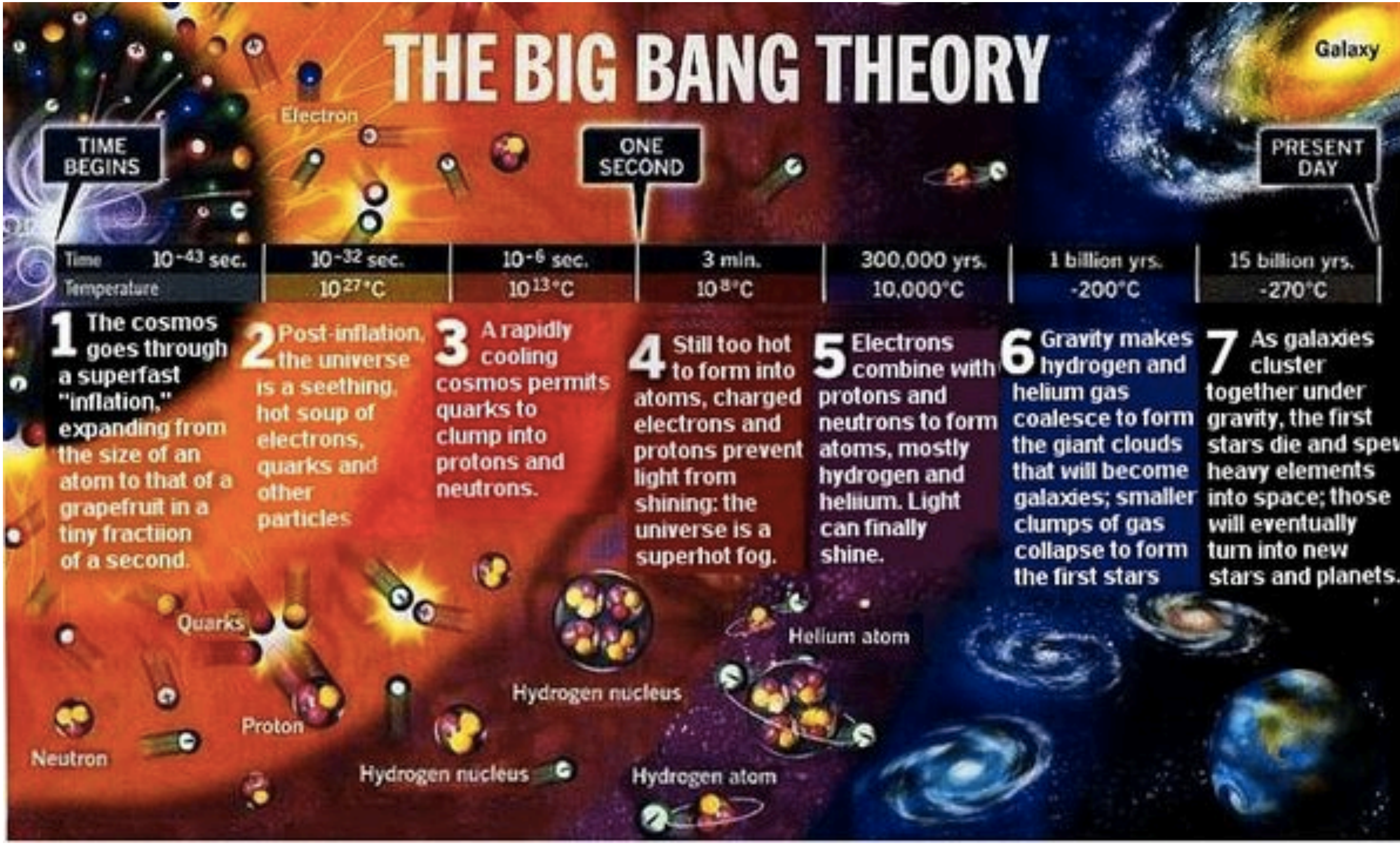
$$q_0 = \frac{1}{2} + \frac{3}{2} w_{DE} \Omega_{DE} < 0$$

If Dark Energy is a Cosmological Constant then $w_{DE} = w_{\Lambda} = -1$

$$\Omega_\Lambda > \frac{1}{3}$$



Time Since Big Bang
Major Events Since Big Bang
present
Humans observe the cosmos.
stars, galaxies and clusters (made of atoms and plasma)
Era of Galaxies
1 billion years
First galaxies form.
atoms and plasma (stars begin to form)
Era of Atoms
500,000 years
Atoms form; photons fly free and become microwave background.
plasma of hydrogen and helium nuclei plus electrons
Era of Nuclei
3 minutes
Fusion ceases; normal matter is 75% hydrogen, 25% helium, by mass.
protons, neutrons, electrons, neutrinos (antimatter rare)
Era of Nucleosynthesis
0.001 seconds
Matter annihilates antimatter.
elementary particles (antimatter common)
Particle Era
10^{-10} seconds
Electromagnetic and weak forces become distinct.
elementary particles
Electroweak Era
10^{-38} seconds
Strong force becomes distinct, perhaps causing inflation of universe.
GUT Era
elementary particles
10^{-43} seconds
Planck Era
????
neutron
proton
electron
neutrino
antiproton
antineutron
antielectrons
quarks



THE BIG BANG THEORY
Galaxy
Electron
TIME BEGINS
ONE SECOND
PRESENT DAY
Time 10-43 sec. | 10-32 sec. | 10-6 sec. | 3 min. | 300,000 yrs. | 1 billion yrs. | 15 billion yrs.
Temperature 1027°C | 1013°C | 108°C | 10,000°C | -200°C | -270°C
1 The cosmos goes through a superfast "inflation," expanding from the size of an atom to that of a grapefruit in a tiny fractiion of a second.
2 Post-inflation, the universe is a seething, hot soup of electrons, quarks and other particles
3 A rapidly cooling cosmos permits quarks to clump into protons and neutrons.
4 Still too hot to form into atoms, charged electrons and protons prevent light from shining: the universe is a superhot fog.
5 Electrons combine with protons and neutrons to form atoms, mostly hydrogen and helium. Light can finally shine.
6 Gravity makes hydrogen and helium gas coalesce to form the giant clouds that will become galaxies; smaller clumps of gas collapse to form the first stars
7 As galaxies cluster together under gravity, the first stars die and spew heavy elements into space; those will eventually turn into new stars and planets.
Quarks
Helium atom
Hydrogen nucleus
Proton
Neutron
Hydrogen nucleus
Hydrogen atom