Holographic methods for condensed matter physics

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Lecture I

Part I: Why AdS/CMT?

1. Motivations
2. Quantum criticality
3. Nonconventional superconductors

Part II: Geometric duals for scale invariant theories

1. The AdS/CFT logic
2. Scale invariance and z
3. Scale invariant geometries
4. What is z in the real world?

Future lectures II, III, IV

Lecture II

1. Away from scale invariance
2. Away from equilibrium – transport

Lecture III

1. The physics of spectral functions
2. Examples from experiments and from AdS/CFT

Lecture IV

1. Holographic superconductivity
2. Landscape of superconducting membranes

Part I: Why AdS/CMT?

1. Motivations
2. Quantum criticality
3. Nonconventional superconductors

The logic of these lectures

- Traditional condensed matter physics considers weakly interacting ‘quasiparticles’. Extremely successful:
  - e.g. Landau liquid theory + BCS theory of superconductivity.
- Doesn’t work for materials with strongly correlated electrons.
  - e.g. High temperature superconductors.
  - e.g. Near quantum critical points.
- AdS/CFT allows computations at strong coupling, including at finite temperature, and a novel conceptual framework.
- Quantum critical points are scale invariant
  $\rightarrow$ ideal kinematics for AdS/CFT.

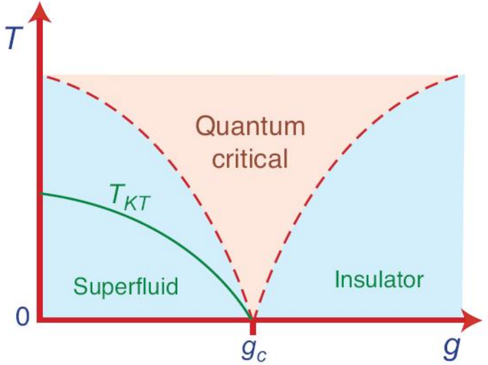
Why condensed matter?

Some 'big picture' comments

- There is a unique Lagrangian for particle physics in our universe.
- There are many Lagrangians in condensed matter systems, and an increasing range can be engineered.
  - e.g. Atoms in optical lattices.
- One day: SUSY in a lab? Experimental AdS/CFT?
- AdS/CFT suggests that there is not a unique 'fundamental' formulation of physics, but different duality frames.
  - e.g. Superconductivity $\leftrightarrow$ new types of hairy black hole.
  - A unified approach to physics.

Quantum criticality

- Quantum critical points: continuous phase transitions at $T = 0$.
- Lead to a strongly coupled quantum critical region of phase diagram:



Comments on phase transitions and dimensionality

- Coleman-Mermin-Wagner-Hohenberg theorem: no second order phase transitions in 2 dimensions. I.e.
  - 2+1 at finite temperature.
  - 1+1 at zero temperature.
- Massless goldstone bosons have an IR divergence below 2 dimensions: 'Fluctuations destroy order'.
- Exception I: At infinite N, no fluctuations. Pin system to extremum.
- 'Exception' II: Berezinsky-Kosterlitz-Thouless transition in 2+1 dimensions at finite temperature (infinite order transition, condensation of vortices, algebraically quasi-ordered phase).
- We will often be working in 2+1 dimensions.

Example I: Wilson-Fisher fixed point

- Let Φ be an N dimensional vector:

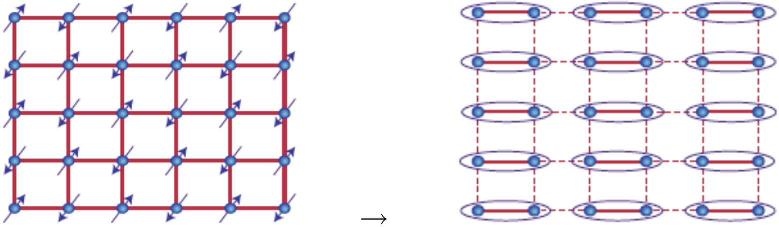
$$S[\Phi] = \int d^3x \left((\partial\Phi)^2 + r\Phi^2 + u\left(\Phi^2\right)^2 \right) .$$

- Flows to strongly coupled fixed point when $r \to r_c$ (quantum critical).
- This, relativistic, theory emerges at a critical point in an insulating quantum magnet

$$H_{\text{AF}} = \sum_{\langle ij \rangle} J_{ij} S_i \cdot S_j .$$

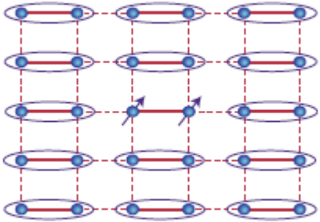
- S_i: spin on a 2D square lattice.
- Antiferromagnetic: $J_{ij} > 0$.

- Take couplings J or J/g as shown below right (dashed $= J/g$).
- Tune g from 1 to ∞. Ground state:



- $g = 1$ ground state is Néel ordered: $\langle \sum_i (-1)^i S_i \rangle = \Phi \neq 0$.
- $g = \infty$ ground state is decoupled, spin singlet, dimers. Spin rotation preserved.
- Therefore: $\exists$ Quantum phase transition! Numerically: $g_c \approx 1.91$.

- Excitations about Néel state: spin density waves. Φ^4 theory with $N = 3$ in symmetry broken phase (Φ is the order parameter).
- Excitations about dimer state: triplons



- Three polarisations of triplon: Φ^4 theory with $N = 3$ in symmetric phase.
- Low energy dynamics of entire phase diagram described by Φ^4 theory, including the Wilson-Fisher fixed point.

Example II: spinons and emergent photons

- Let A be an (emergent 2+1 dimensional) photon, z a complex spinor

$$S[z, A] = \int d^3x \left(|(\partial - iA)\, z|^2 + r|z|^2 + u \left(|z|^2\right)^2 + \frac{1}{2e_0^2} F^2 \right) .$$

- Flows to fixed point at $r \to r_c$. For $r < r_c$ can map to previous example via

$$\Phi = \bar{z}_\alpha \sigma_{\alpha\beta} z_\beta .$$

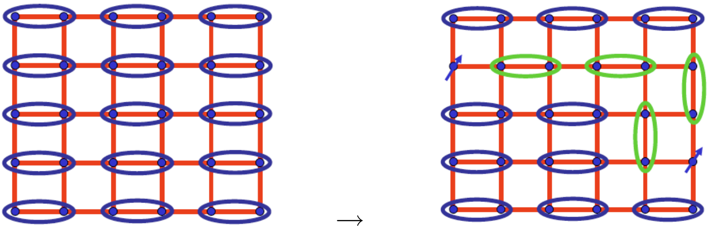
- Taking again an insulating quantum magnet ($J, Q > 0$)

$$H_{\text{VBS}} = J \sum_{\langle ij \rangle} S_i \cdot S_j - Q \sum_{\langle ijkl \rangle} \left(S_i \cdot S_j - \frac{1}{4} \right) \left(S_k \cdot S_l - \frac{1}{4} \right) ,$$

- Preserves lattice rotations ($\mathbb{Z}_4$), unlike previous example.
- Second term favours 4 spins on a plaquette forming 2 singlet pairs.

- For $J/Q \gg 1$, back to Néel state. Already discussed.
  - Order parameter: $\Phi = \langle \sum_i (-1)^i S_i \rangle \neq 0$.
  - Breaks spin rotation but preserves lattice rotation.

- For $J/Q \ll 1$ get valence bond solid (VBS).
  - Preserves spin rotation but breaks lattice rotation.
  - Order parameter: $\Psi = (-1)^{j_x} S_j \cdot S_{j+\hat{x}} + i(-1)^{j_y} S_j \cdot S_{j+\hat{y}} \neq 0$.
  - Has a spin half excitation: spinon.



- Emergent photon has a beautiful description in the VBS phase.
- In 2+1 dimensions we can dualise the photon to a scalar

$$\star_3 F = d\zeta \,.$$

Revealing a 'dual' symmetry $\zeta \to \zeta + \delta\zeta$.
- The VBS order parameter turns out to be a condensate of monopoles

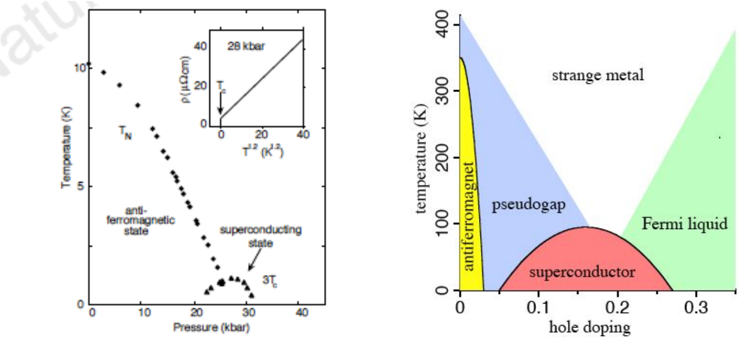
$$\Psi \sim e^{i2\pi\zeta/e_0^2} \,.$$

- The spontaneously broken symmetry: $\zeta \to \zeta + \delta\zeta$ is in fact the rotational symmetry of the lattice.
- The dual photon ζ is precisely the goldstone boson for spatial rotations.
- The dual photon is massive for $r > r_c$ because the lattice breaks $U(1) \to \mathbb{Z}_4$, but becomes massless at the quantum critical point.

Nonconventional superconductors

- Conventional superconductivity is described by BCS theory (1957).
- BCS theory: Lattice vibrations (phonons) mediate an attractive force between (dressed) electrons. At low temperatures an electron bilinear condenses: $\langle \Psi\Psi \rangle \neq 0$.
- Many superconductors not describable by BCS theory (eg. high-T_c).
- Non-BCS superconductivity might mean:
  - weak: 'pairing mechanism' does not involve phonons but a different 'glue' such as paramagnons (mediate spin-spin forces).
  - strong: inherently strongly coupled. No charged quasiparticles to 'pair' in the first place.
- Candidates for the stronger case if superconductivity close to a quantum critical point.

Two phase diagrams:



- Left: 'heavy fermion' compound. Clear connection to quantum critical point.
- Right: high-T_c cuprate. Experimental evidence for nearby VBS order and possible quantum critical point.
- Quantum critical points similar to those we discussed (antiferromagnetism).

Part II: Geometric duals for scale invariant theories

1. The AdS/CFT logic
2. Scale invariance and z
3. Scale invariant geometries
4. What is z in the real world?

The AdS/CFT logic

- The large N limit is a classical limit.
- For gauge theories: what are the classical saddles?
- AdS/CFT: for some theories, at least, the classical saddles are solutions to a theory of gravity in one (or more) higher dimensions!
- AdS/CFT geometrises the energy scale. (cf. renormalsation group equations suggest 'locality' in energy).
- Minimal AdS/CFT structure:

Large N gauge theory d spacetime dimensions	$\leftrightsquigarrow$	Classical gravitational theory $d+1$ spacetime dimensions.

Scale invariance and z

- Field theories can be defined with a cutoff or at a UV fixed point.
- At a fixed point, theory is invariant under space and time scaling

$$t \to \lambda^z t \,, \qquad \vec{x} \to \lambda \vec{x} \,.$$

- z is the dynamical critical exponent. There is no reason for $z = 1$.
- Minimal algebra has $\{M_{ij}, P_k, H, D\}$. Dilatations act

$$[D, M_{ij}] = 0 \,, \qquad [D, P_i] = i P_i \,, \qquad [D, H] = i z H \,.$$

- Sometimes called the Lifshitz algebra.

Scale invariant geometries

- AdS/CFT asks: can we realise the Lifshitz algebra geometrically?
- Kachru et al. reply (2008):

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dx^i dx^i}{r^2} + \frac{dr^2}{r^2} \right) .$$

- Killing vectors generating algebra

$$M_{ij} = -i(x^i \partial_j - x^j \partial_i) , \quad P_i = -i\partial_i , \quad H = -i\partial_t ,$$
$$D = -i(z\, t\, \partial_t + x^i \partial_i + r\, \partial_r) .$$

- $z = 1$ is AdS_{d+1}, ehancement to Lorentzian conformal algebra.

Comments on the Lifshitz geometry

- If $z > 1$ then near the boundary g_{tt} diverges faster than $g_{x_i x_i}$. Lightcones flatten. Effective speed of light diverges near the boundary
  $\rightarrow$ nonrelativistic causality of field theory dual.

- If $z \neq 1$ the spacetimes are singular at the 'horizon' $r = \infty$. (null 'pp' singularity – implies no global extension. Physics?).

- The case $z = 1$ is a solution to Einstein gravity

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{L^2} \right) .$$

Other cases need additional matter.

More symmetry! – adding Galilean boosts

- In classical mechanics Galilean boosts: $\{x_i \to x_i + v_i t, t \to t\}$ satisfy

$$[M_{ij}, K_k] = i(\delta_{ik} K_j - \delta_{jk} K_i) , \quad [P_j, K_i] = 0 , \quad [H, K_i] = -iP_i .$$

- Quantum mechanical representations require a central extension

$$[P_j, K_i] = 0 \quad \rightsquigarrow \quad [P_j, K_i] = -i\delta_{ij} N .$$

  N can be interpreted as the particle number/total mass.

- Galilean algebra is consistent with scale invariance if

$$[D, K_i] = i(1 - z)K_i , \quad [D, N] = i(2 - z)N .$$

- Algebra $\{M_{ij}, P_i, H, D, K_i, N\}$ sometimes called Schrödinger algebra.

Comments on the Schrödinger algebra

- When $z = 2$, symmetry can be further extended by a 'special' conformal generator.
- Dilatation only commutes with the particle number (also: the total mass) if $z = 2$. This is because mass is dimensionless if $z = 2$.
- Implication: cannot have a discrete spectrum for the particle number in a scale invariant theory if $z \neq 2$.

Galilean scale-invariant geometry

- The Schrödinger geometry (Son, McGreevy + Balasubramanian 2008)

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} - \frac{2 dt d\xi}{r^2} + \frac{dx^i dx^i}{r^2} + \frac{dr^2}{r^2} \right) .$$

- Killing vectors generating the new symmetries

$$K_i = -i(-t\partial_i + x^i \partial_\xi) \,, \quad N = -i\partial_\xi \,,$$
$$D = -i(zt\partial_t + x^i\partial_i + (2-z)\xi\partial_\xi + r\partial_r) \,.$$

- Extra dimension ξ for particle number symmetry. To get discrete particle spectrum

$$\xi \sim \xi + 2\pi L_\xi \,.$$

Comments on Schrödinger geometries

- When $z \neq 2$, dilatation is not a symmetry if ξ identified.
- ξ is a null direction. Identification is dangerous (e.g. wound strings).
- The sign of g_{tt} can be changed leaving the solution Lorentzian. However, with the opposite sign lightcones no longer collapse near the boundary. Furthermore, the spacetime appears to be unstable in this case.
- For $z \neq 1$, need matter fields to solve equations of motion.

A physical consequence of z

$$S = \int \frac{d^{d-1}k d\omega}{(2\pi)^d} \left(r + k^2 + \omega^{2/z} \right) |\Phi(\omega, k)|^2 + \int d^{d-1}x dt \, u \, (\Phi^2)^2 \, .$$

- The scaling dimension of the coupling u can be worked out to be

$$[u] = (5 - d - z) \, .$$

- Therefore the coupling u is irrelevant if

$$d > d_c = 5 - z \, .$$

- For $z > 1$ critical dimension is lowered!
- E.g. in $d = 3 = 2 + 1$:
    - $z = 1$: u relevant.
    - $z = 2$: u marginal.
    - $z = 3$: u irrelevant.
- Higher z more amenable to perturbative treatment in $d = 2 + 1$.

Examples of z in the real world

- $\mathbf{z = 1}$:
  - Insulating quantum antiferromagnets (relevant for high-T_c).
  - Bose Hubbard-like models at p/q filling (e.g. optical lattices).
- $\mathbf{z = 2}$: Itinerant anitferromagnetism (heavy fermions).

$$S_{iA.F.} = \int dt d^{d-1}x \left[-\gamma \Phi \partial_t \Phi + (\partial_x \Phi)^2 + r\Phi^2 + u \left(\Phi^2\right)^2 \right] .$$

- $\mathbf{z = 3}$: Itinerant ferromagnetism (heavy fermions).

$$S_{iF.} = - \int \frac{dt\, d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{v|k|} \Phi \partial_t \Phi + \int dt d^{d-1}x \left[(\partial_x \Phi)^2 + r\Phi^2 + u \left(\Phi^2\right)^2 \right]$$

- **z nonuniversal**: 'local quantum criticality' (heavy fermions).
  E.g. $z \approx 2.6$ in $CeCu_{6-x}Au_x$ at critical doping.

A choice of z

For the remainder of these lectures I will focus on $z = 1$ for the following reasons:

- Plenty of interesting quantum critical points in nature with $z = 1$.
- In 2+1 dimensions and within certain classes of models, $z = 1$ theories are the most likely to strongly coupled.
- The AdS/CFT dictionary is not yet fully developed for $z \neq 1$.
- The $z = 1$ background solves the Einstein equations without extra matter. This may make predictions more robust.

Summary so far

- Quantum critical phases have an underlying scale invariant and often strongly coupled field theory.
- May be important for nonconventional superconductivity.
- Aim to study these theories using AdS/CFT.
- Discussed geometric realisations of scale invariance for different values of dynamical critical exponent z.
- Options for $z \neq 1$: With and without Galilean symmetry
- Galilean symmetry most natural if $z = 2$.