Holographic methods for condensed matter physics

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Lecture II

Part I: Away from scale invariance

1. Finite temperature
2. Finite chemical potential
3. Relevant operators
4. Expectation values

Part II: Away from equilibrium

1. Retarded Green's functions
2. Example: Electrical conductivity
3. Example: Thermal conductivity

Part I: Away from scale invariance

1. Finite temperature
2. Finite chemical potential
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4. Expectation values

Finite temperature

- The zero temperature background is AdS_{d+1}

$$ds^2 = L^2 \left(-\frac{dt^2}{r^2} + \frac{dr^2}{r^2} + \frac{dx^i dx^i}{r^2} \right) .$$

- Which is a solution to the theory

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{L^2} \right) .$$

- Want relevant deformations, break scale invariance in the IR.
- Therefore expect geometry of the form

$$ds^2 = L^2 \left(-\frac{f(r)dt^2}{r^2} + \frac{g(r)dr^2}{r^2} + \frac{h(r)dx^i dx^i}{r^2} \right) .$$

- Most universal deformation: temperature.

Finite temperature

- Only one nontrivial solution to Einstein equations of this form:

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^i dx^i\right) \,,$$

where

$$f(r) = 1 - \left(\frac{r}{r_+}\right)^d \,.$$

- Asymptotically AdS as $r \to 0$. (UV)
- Horizon at $r = r_+$. (IR)
- Corresponds to a temperature (from e.g. Euclidean solution)

$$T = \frac{d}{4\pi r_+} \,.$$

- All $T \neq 0$ equivalent: $(r,t,x^i) \to r_+(r,t,x^i)$ eliminates r_+.

Finite temperature

- By computing the action of the Euclidean solution

$$F = -T \log Z = T S_E[g_\star] = -\frac{(4\pi)^d L^{d-1}}{2\kappa^2 d^d} V_{d-1} T^d \,.$$

- Characterised by one number ('central charge'):

$$\frac{(4\pi)^d L^{d-1}}{2\kappa^2 d^d} \sim N^{\#} \,.$$

- Can then compute: Energy, Entropy, etc.

Finite chemical potential

- Want physics of a $U(1)$ symmetry. E.g. electricity!
- In nature $U(1)$ is gauged. In many condensed matter setups, sufficient to work with global symmetry.
  - Photons are screened in a charged medium.
  - Sufficient to consider external sources (no virtual photons).
- What is the dual to a global $U(1)$ in field theory?
- Take cue from global Lorentz invariance. Dual to part of the diffeomorphism invariance of the bulk. Suggests:

Global symmetry (field theory) d spacetime dimensions	$\leftrightsquigarrow$	Gauged symmetry (gravity) $d+1$ spacetime dimensions.

- Natural: QFT global symmetry are 'large' gauge symmetries in bulk.

- Therefore: Need bulk Maxwell field. Minimal action

$$S = \int d^{d+1}x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{4g^2}F^2\right].$$

- Symmetries allow (magnetic term in $d = 2+1$ only)

$$A = A_t(r)dt + B(r)x\,dy\,.$$

- Put $B = 0$ for the moment. Metric solution

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^i dx^i\right),$$

where

$$f(r) = 1 - \left(1 + \frac{r_+^2\mu^2}{\gamma^2}\right)\left(\frac{r}{r_+}\right)^d + \frac{r_+^2\mu^2}{\gamma^2}\left(\frac{r}{r_+}\right)^{2(d-1)}.$$

- Scalar potential is ($A_t(r_+) = 0$ for regularity)

$$A_t = \mu\left[1 - \left(\frac{r}{r_+}\right)^{d-2}\right].$$

- Dimensionless constant

$$\gamma^2 = \frac{(d-1)g^2 L^2}{(d-2)\kappa^2}\,,.$$

- Temperature

$$T = \frac{1}{4\pi r_+}\left(d - \frac{(d-2)r_+^2\mu^2}{\gamma^2}\right).$$

- Near the boundary

$$A_\mu(r) = A_{(0)\mu} + \cdots \quad \text{as} \quad r \to 0\,.$$

$$\text{(cf. } g_{\mu\nu}(r) = \tfrac{L^2}{r^2} g_{(0)\mu\nu} + \cdots \quad \text{as} \quad r \to 0\,.\text{)}$$

- $A_{(0)\mu}$ is background gauge field. $A_{(0)t} = \mu$ is the chemical potential.

- There is now a physical dimensionless temperature T/μ.
- Limit $T/\mu \to 0$ can be taken continuously. Extremal black hole.
- Free energy

$$\Omega = -T \log Z = \mathcal{F}\left(\frac{T}{\mu}\right) V_{d-1} T^d \,.$$

- $\mathcal{F}\left(\frac{T}{\mu}\right)$ is a nontrivial function that is an output of AdS/CFT.
- Entropy: $S = -\frac{\partial \Omega}{\partial T}$, charge density: $\rho = -\frac{1}{V_2}\frac{\partial \Omega}{\partial \mu}$.
- At low temperature: $\Omega \sim a\mu^d + b\mu^{d-1} T + c\mu^{d-2} T^2 + \cdots$. Implies entropy $\to$ constant as $T \to 0$. Discomforting:
  - Large N effect?
  - Weak gravity conjecture $\to$ should be unstable?

Relevant operators

- What is dual to adding a relevant operator to the theory?
- Take inspiration from metric. If $g \to g_{(0)} + \delta g_{(0)}$:

$$\delta S = \int d^d x \sqrt{-g_{(0)}} \delta g_{(0)\mu\nu} T^{\mu\nu} \,.$$

- Equality of bulk and boundary partition functions should remains:

$$Z_{\text{bulk}}[g \to g_{(0)} + \delta g_{(0)}] = \langle \exp \left(i \int d^d x \sqrt{-g_{(0)}} \delta g_{(0)\mu\nu} T^{\mu\nu} \right) \rangle_{\text{F.T.}} \,.$$

- Similarly for the gauge field. If $A \to \delta A_{(0)}$:

$$\delta S = \int d^d x \sqrt{-g_{(0)}} \delta A_{(0)\mu} J^{\mu} \,.$$

- Thus

$$Z_{\text{bulk}}[A \to \delta A_{(0)}] = \langle \exp \left(i \int d^d x \sqrt{-g_{(0)}} \delta A_{(0)\mu} J^{\mu} \right) \rangle_{\text{F.T.}} \,.$$

- Suggests a general correspondence

$$\begin{array}{ccc} \text{operator } \mathcal{O} & & \text{dynamical field } \phi \\ \text{(field theory)} & \leftrightsquigarrow & \text{(bulk)}\,, \end{array}$$

such that

$$Z_{\text{bulk}}[\phi \to \delta\phi_{(0)}] = \langle \exp\left(i \int d^d x \sqrt{-g_{(0)}} \delta\phi_{(0)} \mathcal{O} \right) \rangle_{\text{F.T.}} \, .$$

where

$$\phi(r) = \left(\frac{r}{L}\right)^{d-\Delta} \phi_{(0)} + \cdots \quad \text{as} \quad r \to 0 \, ,$$

- I.e. Boundary value of field $\to$ source for dual operator.
- Δ is the scaling dimension of the operator $\mathcal{O}$.
- Can see that if $\mathcal{O}$ is relevant, $\Delta < d$, then $\phi \to 0$ near the boundary.

Expectation values

- From previous formula clear that

$$\langle \mathcal{O} \rangle = -i \frac{\delta Z_{\text{bulk}}[\phi_{(0)}]}{\delta \phi_{(0)}} = \frac{\delta S[\phi_{(0)}]}{\delta \phi_{(0)}} \,.$$

- Useful to make a Hamilton-Jacobi-esque identification

$$\frac{\delta S[\phi_{(0)}]}{\delta \phi_{(0)}} = - \lim_{r \to 0} \frac{\delta S[\phi_{(0)}]}{\delta \partial_r \phi_{(0)}} \equiv \lim_{r \to 0} \Pi[\phi_{(0)}] \,.$$

- Straightforward to check (adding appropriate counterterms) that if

$$\phi(r) = \left(\frac{r}{L}\right)^{d-\Delta} \phi_{(0)} + \left(\frac{r}{L}\right)^{\Delta} \phi_{(1)} + \cdots \quad \text{as} \quad r \to 0 \,.$$

- Then

$$\langle \mathcal{O} \rangle = \frac{2\Delta - d}{L} \phi_{(1)} \,.$$

Part II: Away from equilibrium

1. Retarded Green's functions
2. Example: Electrical conductivity
3. Example: Thermal conductivity

Retarded Green's functions

- Basic object describing perturbations away from equilibrium

$$\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)\,.$$

- From previous expression

$$G^R_{\mathcal{O}_A\mathcal{O}_B} = \left.\frac{\delta\langle\mathcal{O}_A\rangle}{\delta\phi_{B(0)}}\right|_{\delta\phi=0} = \lim_{r\to 0}\left.\frac{\delta\Pi_A}{\delta\phi_{B(0)}}\right|_{\delta\phi=0} = \frac{2\Delta_A - d}{L}\frac{\delta\phi_{A(1)}}{\delta\phi_{B(0)}}\,.$$

- Near the boundary require: $\delta\phi_A(r) = r^{d-\Delta}\delta\phi_{A(0)} + \cdots$.
- Regularity on the future horizon $\rightarrow$ ingoing boundary conditions

$$\delta\phi_A(r) = C_A e^{-i4\pi\omega/T\log(r-r_+)} + \cdots \quad \text{as} \quad r\to r_+\,.$$

Example: Electrical and Thermal conductivity

- Leave this for next time!

Summary for today

- Break scale invariance by introducting temperature, chemical potential, relevant operators.
- This 3 cases treated on an equal footing in AdS/CFT.
- Boundary values of bulk fields $\leftrightarrow$ sources for dual operators.
- Global symmetry of field theory $\leftrightarrow$ gauge symmetry of boundary.
- Compute expectation values á la Hamilton-Jacobi.
- Green's functions are a ratio of normalisable by non-normalisable mode.