Holographic methods for condensed matter physics

Sean Hartnoll

Harvard University

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Lecture III

Away from equilibrium

1. Expectation values
2. Retarded Green's functions
3. Example: Electrical and thermal conductivity
4. Example: Collisionless to Collision-dominated crossover
5. Example: Cyclotron resonance
6. Example: Impurities in AdS/CFT

- We argued last time that

$$\begin{array}{c}\text{operator } \mathcal{O} \\ \text{(field theory)}\end{array} \quad \leftrightsquigarrow \quad \begin{array}{c}\text{dynamical field } \phi \\ \text{(bulk)}\,,\end{array}$$

such that

$$Z_{\text{bulk}}[\phi \to \delta\phi_{(0)}] = \langle \exp\left(i \int d^d x \sqrt{-g_{(0)}}\, \delta\phi_{(0)} \mathcal{O} \right) \rangle_{\text{F.T.}}\,.$$

where

$$\phi(r) = \left(\frac{r}{L}\right)^{d-\Delta} \phi_{(0)} + \cdots \quad \text{as} \quad r \to 0\,,$$

- I.e. Boundary value of field $\to$ source for dual operator.
- Δ is the scaling dimension of the operator $\mathcal{O}$.
- Can see that if $\mathcal{O}$ is relevant, $\Delta < d$, then $\phi \to 0$ near the boundary.

Expectation values

- From previous formula clear that

$$\langle \mathcal{O} \rangle = -i \frac{\delta Z_{\text{bulk}}[\phi_{(0)}]}{\delta \phi_{(0)}} = \frac{\delta S[\phi_{(0)}]}{\delta \phi_{(0)}} \,.$$

- Useful to make a Hamilton-Jacobi-esque identification

$$\frac{\delta S[\phi_{(0)}]}{\delta \phi_{(0)}} = - \lim_{r \to 0} \frac{\delta S[\phi_{(0)}]}{\delta \partial_r \phi_{(0)}} \equiv \lim_{r \to 0} \Pi[\phi_{(0)}] \,.$$

- Straightforward to check (adding appropriate counterterms) that if

$$\phi(r) = \left(\frac{r}{L}\right)^{d-\Delta} \phi_{(0)} + \left(\frac{r}{L}\right)^{\Delta} \phi_{(1)} + \cdots \quad \text{as} \quad r \to 0 \,.$$

- Then

$$\langle \mathcal{O} \rangle = \frac{2\Delta - d}{L} \phi_{(1)} \,.$$

Retarded Green's functions

- Basic object describing perturbations away from equilibrium

$$\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)\,.$$

- From previous expression

$$G^R_{\mathcal{O}_A\mathcal{O}_B} = \left.\frac{\delta\langle\mathcal{O}_A\rangle}{\delta\phi_{B(0)}}\right|_{\delta\phi=0} = \lim_{r\to 0}\left.\frac{\delta\Pi_A}{\delta\phi_{B(0)}}\right|_{\delta\phi=0} = \frac{2\Delta_A - d}{L}\frac{\delta\phi_{A(1)}}{\delta\phi_{B(0)}}\,.$$

- Near the boundary require: $\delta\phi_A(r) = r^{d-\Delta}\delta\phi_{A(0)} + \cdots$.
- Regularity on the future horizon → ingoing boundary conditions

$$\delta\phi_A(r) = C_A e^{-i4\pi\omega/T\log(r-r_+)} + \cdots \quad \text{as} \quad r\to r_+\,.$$

Example: Electrical and Thermal conductivity

- Want: zero momentum conductivity with a chemical potential.
- Chemical potential mixes thermal and electric conductivities

$$\left(\begin{array}{c} \langle J_x \rangle \\ \langle Q_x \rangle \end{array} \right) = \left(\begin{array}{cc} \sigma(\omega) & \alpha(\omega) T \\ \alpha(\omega) T & \bar{\kappa}(\omega) T \end{array} \right) \left(\begin{array}{c} E_x \\ -(\nabla_x T)/T \end{array} \right) ,$$

where

$$Q_x = T_{tx} - \mu J_x \,.$$

- Why the extra term in Q_x? Chemical potential gives term in action

$$S_\mu = \int d^{d-1}x dt \sqrt{-g_{(0)}} \left(g^{tt} \mu J_t + g^{ti} \mu J_i \right) \,.$$

Therefore under a perturbation of $\delta g_{tx(0)}$

$$\delta S = \int d^{d-1}x dt \sqrt{-g_{(0)}} \left(T^{tx} - \mu J^x \right) \delta g_{tx(0)} \,.$$

- Background electric field:

$$E_x = i\omega\delta A_{x(0)} \,.$$

- Background thermal gradient:

$$-\frac{\nabla_x T}{T} = i\omega\delta g_{tx(0)} \,.$$

[To see this: rescale time so that the period of Euclidean time is fixed, then $g_{tt(0)} = -\frac{1}{T^2}$. A thermal gradient is then

$$\delta g_{tt(0)} = -\frac{2x\nabla_x T}{T^3} \,.$$

Now to a gauge transformation on the background field $\delta g_{ab(0)} = \partial_a\xi_b + \partial_b\xi_a$, with $\xi_t = x\nabla_x T/\omega T^3$.]

- Therefore:

$$\begin{pmatrix} \langle J_x \rangle \\ \langle Q_x \rangle \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \alpha T & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} i\omega\delta A_{x(0)} \\ i\omega\delta g_{tx(0)} \end{pmatrix} .$$

- So conductivities are Green's functions! For instance

$$\sigma(\omega) = \frac{-iG^R_{J_x J_x}(\omega)}{\omega} .$$

- Need to solve bulk equations for A_x and g_{tx} such that

$$A_x \to A_{x(0)} \, .$$
$$g_{tx} \to L^2/r^2 \, g_{tx(0)} .$$

- Get a decoupled equation for A_x:

$$(f\delta A_x')' + \frac{\omega^2}{f}\delta A_x - \frac{4\mu^2 r^2}{\gamma^2 r_+^2}\delta A_x = 0 \ .$$

- Near boundary:

$$\delta A_x = \delta A_{x(0)} + \frac{r}{L}\delta A_{x(1)} + \cdots \quad \text{as} \quad r \to 0 .$$

- Work out the 'momenta' ($\rho = -\partial\Omega/\partial\mu/V$)

$$\Pi_{g_{tx}} = -\frac{\delta S}{\delta \partial_r g_{tx(0)}} = -\rho\,\delta A_{x(0)} + \frac{2L^2}{\kappa^2 r^3}(1 - f^{-1/2})\delta g_{tx(0)}\,,$$
$$\Pi_{A_x} = -\frac{\delta S}{\delta \partial_r A_{x(0)}} = \frac{f\,\delta A'_{x(0)}}{g^2} - \rho\,\delta g_{tx(0)}\,.$$

- Taking the boundary limit $r \to 0$:

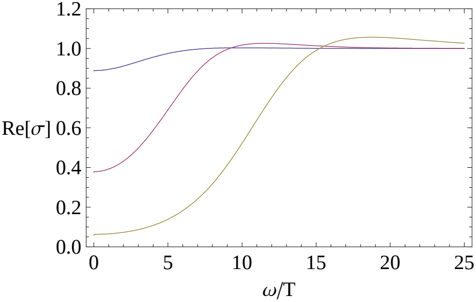
$$\begin{pmatrix} \langle J_x \rangle \\ \langle Q_x \rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{g^2}\frac{\delta A_{x(1)}}{L\delta A_{x(0)}} & -\rho \\ -\rho & -\epsilon \end{pmatrix} \begin{pmatrix} \delta A_{x(0)} \\ \delta g_{tx(0)} \end{pmatrix},$$

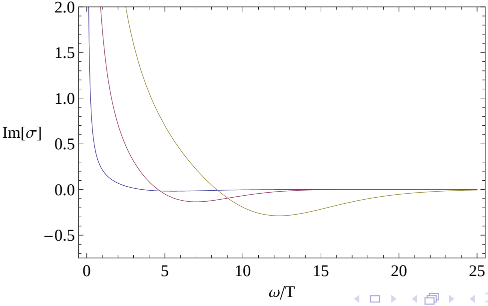
($\epsilon = -2\Omega/V$, energy density).

- Compare with above:

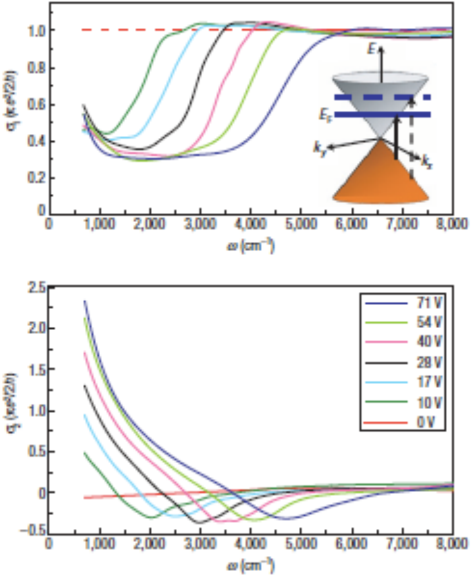
$$\sigma(\omega) = \frac{-i}{g^2\omega}\frac{\delta A_{x(1)}}{L\delta A_{x(0)}}\,; \qquad \alpha(\omega) = \frac{i\rho}{\omega T}\,; \qquad \bar{\kappa}(\omega) = \frac{i\epsilon}{\omega T}\,.$$

- Solve the differential equation for A_x to get electrical conductivity





- For amusement, compare with experimental data on a 2+1 dimensional relativistic theory, graphene!



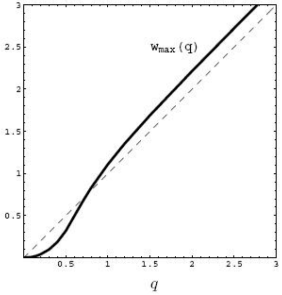
- Note that the real conductivity in the data goes up again at low frequencies, this is the Drude peak due to momentum relaxation from impurities, ions, etc.

- In the AdS/CFT theories (without impurities) there will be a delta function in the conductivity at $\omega = 0$.

- This is because a translation-invariant medium with a net charge cannot relax a DC ($\omega = 0$) current due to momentum conservation.

Collisionless to hydrodynamic crossover

- Say something about finite momentum, but no chemical potential.
- Generic expectations for charge transport in a CFT.
  - Long wavelength $k \ll T$: hydrodynamics $\to$ charge diffusion (collision timescale $\sim 1/T$).
  - Short wavelength $k \gg T$: zero temperature relativistic transport ('phase coherent').
- AdS/CFT the first CFT in which crossover can be exhibited explicitly



Cyclotron resonance

- Simple to add a magnetic field $\rightarrow$ dyonic black hole.
- Can get analytic results at $\omega \ll T$, and $B \sim \omega^{1/2}$:

$$\begin{aligned} \sigma_{xx} &= \sigma_Q \frac{\omega(\omega + i\gamma + i\omega_c^2/\gamma)}{(\omega + i\gamma)^2 - \omega_c^2} \,, \\ \sigma_{xy} &= -\frac{\rho}{B} \frac{-2i\gamma\omega + \gamma^2 + \omega_c^2}{(\omega + i\gamma)^2 - \omega_c^2} \,. \end{aligned}$$

where

$$\omega_c = \frac{B\rho}{\epsilon + P} \,, \quad \gamma = \frac{\sigma_Q B^2}{\epsilon + P} \,,$$

- Exactly the same results can be obtain from hydrodynamics alone: conservation law plus constitutive relations.

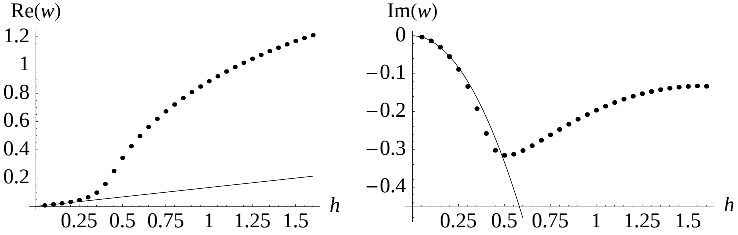
- Cyclotron resonance at $\omega = \omega_c - i\gamma$.
- The fact that cyclotron motion is damped is an inherently relativistic effect (colliding electrons and holes).
- Zero frequency limit gives Hall conductivity

$$\sigma_{xx}(\omega = 0) = 0 \,, \qquad \sigma_{xy}(\omega = 0) = \frac{\rho}{B} \,.$$

- Need a net charge for a Hall effect.

Beyond hydrodynamics

- Hydrodynamics requires small ω and small B. AdS/CFT does not.
- The location of the cyclotron pole is of interest in, for instance, graphene.



- Some qualitatively similar results obtained at weak coupling (=Boltzmann equation) by Müller, Fritz and Sachdev 0805.1413.

Nernst coefficient

- A quantity of recent experimental interest has been the Nernst coefficient:

$$N \equiv \frac{E}{B|\nabla T|} \,.$$

- From the hydrodynamic formulae above one obtains:

$$N = \frac{1}{T}\left(\frac{-i\omega}{(\omega_c^2/\gamma - i\omega)^2 + \omega_c^2}\right) \,.$$

- Vanishes when $\omega \to 0$! $\Rightarrow$ Need impurities.

Impurities

- We need to add the effect of scattering from impurities. Break translational invariance

  $\Rightarrow$ late time non-conservation of momentum

$$\dot{P} \sim -\frac{P}{\tau_{\rm imp}} \,.$$

- Then, at $\omega = 0$,

$$N = \frac{1}{T} \left(\frac{1/\tau_{\rm imp}}{(\omega_c^2/\gamma + 1/\tau_{\rm imp})^2 + \omega_c^2} \right) \,.$$

- We want to compute N(T,B). Need dependence on T and B through $\tau_{\rm imp}$.

- $\tau_{\text{imp}}(T, B)$ needs to be computed from a microscopic theory. No computations known (to me). Use the M2 brane theory!
- What is an impurity?

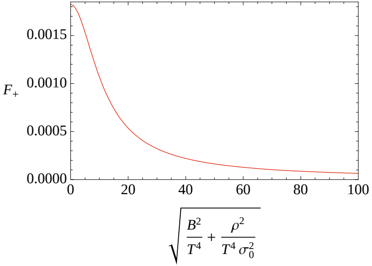
$$\delta H = \int d^2 y V(y) \mathcal{O}(t, y) \,.$$

- V is a random potential

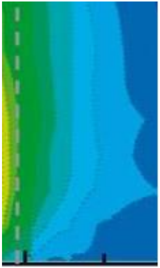
$$\langle V(x) \rangle_{\text{imp}} = 0 \,, \qquad \langle V(x) V(y) \rangle_{\text{imp}} = \bar{V}^2 \delta^{(2)}(x - y) \,.$$

- C. Herzog and I showed (using the 'memory function method') that:

$$\frac{1}{\tau_{\text{imp}}} = \frac{\bar{V}^2}{2(\epsilon + P)} \lim_{\omega \to 0} \int \frac{d^2 k}{(2\pi)^2} \, k^2 \frac{\text{Im}\, G^R_{\mathcal{O}\mathcal{O}}(\omega, k)}{\omega} \,.$$



Experiment:



M2: