1 Topological strings on compact Calabi-Yau, MH, A. Klemm, Quackenbush, hep-th/0612125

- Topological strings: A $N = (2,2)$ supersymmetric non-linear sigma model from world sheet Σ to target space X.

  $$\Phi_i : \Sigma \to X$$

  Topological string theory is the most interesting and free of world sheet anomaly, when the target space X is a Calabi-Yau 3-fold.

- There are two types of topological twistings: A-model and B-model. We are interested in the topological string partition function

  $$Z = \exp(\sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}(t_i))$$

  where t_i are Kahler moduli in the case of A-model, and complex structure moduli in the case of B-model.

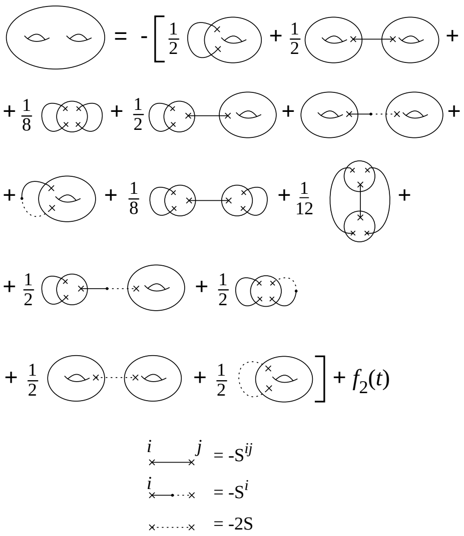
- Topological A-model counts holomorphic curves in target space X, and has a rigorous mathematical formulation known as Gromov-Witten theory. Topological B-model is a complex structure deformation theory known as Kodaira-Spencer theory.

- Topological strings compute physical couplings, world-sheet instanton corrections, R^2 terms in superstring compactifications, geometrically engineer 4-d quantum field theory, etc.

- Mirror symmetry relates topological A-model on manifold X to topological B-model on its mirror manifold. Some very difficult mathematical problems of enumerative geometry can be easily solved by physical methods.

- Many techniques have been developed to study topological string theory. For example, topological strings on a class of non-compact toric Calabi-Yaus are essentially solved to all genera by topological vertex formalism.

- A long standing problem: How to solve topological strings on compact Calabi-Yau spaces? Progress are very limited.



- A famous example: the Quintic manifold, a degree 5 hypersurface in $\mathbb{CP}^4$.
  Candelas et al solve the prepotential, i.e. the counting genus zero curve, using physical idea of mirror symmetry.
  The mirror symmetry results are later proven by mathematicians using Kontsevich's localization methods, Givental; Lian, Liu, Yau.
  At higher genus, the only available approach is the BCOV(Bershadsky, Cecotti, Ooguri, Vafa) method. One use holomorphic anomaly equation to compute $F^{(g)}$ recursively in genus g. This was done by BCOV (in 1993) up to genus 2.

- The BCOV holomorphic anomaly equation

$$\bar{\partial}_{\bar{k}} F^{(g)} = \frac{1}{2} \bar{C}^{ij}_{\bar{k}} \left(D_i D_j F^{(g-1)} + \sum_{r=1}^{g-1} D_i F^{(r)} D_j F^{(g-r)} \right)$$

- An example of BCOV diagrams, at genus 2.

- However, it is difficult to push the BCOV methods to higher genus. Two major difficulties are the followings.

  1. Holomorphic ambiguity problem. The holomorphic anomaly equation only determine $F^{(g)}$ recursively in terms of lower genus results

up to a holomorphic ambiguity, a meromorphic function in the moduli space with a finite number of unknown constants. One need find alternative ways to fix these unknown constants.

2. Computational complexity in BCOV method: the number of diagrams grows exponentially with genus. A normal laptop can handle the diagrams only up to about genus 6, even for the simplest one parameter models such as the quintic.

- The calculation was pushed up to genus 3 for the quintic, using further information from the counting of BPS states known as Gopakumar-Vafa invariants. Katz, Klemm, Vafa, hep-th/9910181.

- In this talk I report major progress in this question.

  1. We solve the holomorphic anomaly equation directly without the BCOV Feynman diagrams, by using the idea of formulating topological strings as polynomials Yamaguchi, Yau, hep-th/0406078. The computational complexity of the method grows only polynomially in genus.
  2. We discover novel boundary conditions at the conifold point of the moduli space, i.e. the "gap" condition c.f. Huang, Klemm, hep-th/0605195, which fix the holomorphic ambiguity to a large extend.

- We are able to solve a class of one-parameter Calabi-Yau models to very high genus, e.g. genus ~ 26 for the quintic.

- Our main example: the quintic. The quintic has one Kahler modulus t and its mirror has one complex structure modulus ψ.

- Picard-Fuchs equation, periods, and mirror map.

$$\{(\psi\partial_\psi)^4 - \psi^{-1}(\psi\partial_\psi - \frac{1}{5})(\psi\partial_\psi - \frac{2}{5})(\psi\partial_\psi - \frac{3}{5})(\psi\partial_\psi - \frac{4}{5})\}\omega = 0$$

The equation can be solved by asymptotic series at $\psi = \infty$,

$$\vec{\Pi} = \begin{pmatrix} \int_{B_1}\Omega \\ \int_{B_2}\Omega \\ \int_{A^1}\Omega \\ \int_{A^2}\Omega \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ X_0 \\ X_1 \end{pmatrix} = \omega_0 \begin{pmatrix} 2F^{(0)} - t\partial_t F^{(0)} \\ \partial_t F^{(0)} \\ 1 \\ t \end{pmatrix}$$

The mirror map has a logarithmic behavior

$$2\pi i t(\psi) = -\log(5^5\psi) + \frac{154}{625\psi} + \frac{28713}{390625\psi^2} + \cdots$$

- The Kahler potential and metric

$$K := -\log i(\bar{X}^i F_i - X^i \bar{F}_i), \quad G_{\psi\bar{\psi}} := \partial_\psi \partial_{\bar{\psi}} K$$

Topological strings as polynomials, Yamaguchi and Yau, hep-th/0406078

- Define the following generators

$$A_p := \frac{(\psi\partial_\psi)^p G_{\psi\bar{\psi}}}{G_{\psi\bar{\psi}}}, \quad B_p := \frac{(\psi\partial_\psi)^p e^{-K}}{e^{-K}}, \quad (p = 1, 2, 3, \cdots)$$
$$C := C_{\psi\psi\psi}\psi^3, \quad X := \frac{1}{1-\psi}$$

These generators satisfy the derivative relations

$$\psi\partial_\psi A_p = A_{p+1} - AA_p, \quad \psi\partial_\psi B_p = B_{p+1} - BB_p, \quad \psi\partial_\psi X = X(X-1)$$

- The independent generators are (A_1, B_1, B_2, B_3, X). One can use the Picard-Fuchs equation and special geometry relation to show B_4 and A_2 are polynomials of (A_1, B_1, B_2, B_3, X).
- Define the topological string amplitudes in "Yukawa coupling frame"

$$P_g := C^{g-1} F^{(g)}, \quad P_g^{(n)} = C^{g-1} \psi^n C_{\psi^n}^{(g)}$$

- We have the initial data and recursion relation in n

$$\begin{aligned} P_{g=0}^{(3)} &= 1 \\ P_{g=1}^{(1)} &= -\frac{31}{3}B + \frac{1}{12}(X-1) - \frac{1}{2}A + \frac{5}{3} \\ P_g^{(n+1)} &= \psi\partial_\psi P_g^{(n)} - [n(A+1) + (2-2g)(B - \frac{1}{2}X)]P_g^{(n)} \end{aligned}$$

- Define a change of variable

$$(A_1, B_1, B_2, B_3, X) \rightarrow (u, v_1, v_2, v_3, X)$$

by the followings

$$\begin{aligned} &B = u, \quad A = v_1 - 1 - 2u, \quad B_2 = v_2 + uv_1, \\ &B_3 = v_3 - uv_2 + uv_1X - \frac{2}{5}uX \end{aligned}$$

- The anti-holomorphic derivative of the generators can be related to each other. Only $\partial_{\bar{\psi}} A_1$ and $\partial_{\bar{\psi}} B_1$ are independent. The BCOV holomorphic anomaly equations are

$$\begin{aligned} \frac{\partial P_g}{\partial u} &= 0 \\ (\frac{\partial}{\partial v_1} - u\frac{\partial}{\partial v_2} - u(u+X)\frac{\partial}{\partial v_3})P_g &= -\frac{1}{2}(P^{(2)}_{g-1} + \sum_{r=1}^{g-1} P^{(1)}_r P^{(1)}_{g-r}) \end{aligned}$$

- The Main Proposition: Each P_g, $(g \geq 2)$ is a degree $3g-3$ inhomogeneous polynomial of v_1, v_2, v_3, X, where one assigns the degree $1, 2, 3, 1$ for v_1, v_2, v_3, X, respectively. Yamaguchi and Yau.

- The number of terms n_g in P_g grows polynomially with genus g.

$$n_g \preceq (3g-3)^4$$

- The generators (A_i, B_i, X) are modular functions of the monodromy group of the quintic, a subgroup of $Sp(4, Z)$.

- We use the holomorphic anomaly equation to compute the P_g recursively, up to a holomorphic ambiguity

$$f^{(g)} = \sum_{i=0}^{3g-3} c_i X^i$$

The degree is fixed by the maximal degree of the poles at the conifold point.

- There are $3g-2$ unknown constants at each genus g.

Boundary conditions

- There are three singular points in the complex structure moduli space: $\psi = 0$, $\psi = 1$, $\psi = \infty$.
- We can expand the topological strings around these singular points. In the holomorphic limit, the Kahler potential and metric go like

$$e^{-K} \sim \omega_0, \qquad G_{\psi\bar{\psi}} \sim \partial_\psi t,$$

  So in the holomorphic limit, the generators A_p and B_p are

$$A_p = \frac{(\psi\partial_\psi)^p(\partial_\psi t)}{\partial_\psi t}, \quad B_p = \frac{(\psi\partial_\psi)^p \omega_0}{\omega_0},$$

- The period ω_0 and mirror map t can be solved asymptotically at each singular point of the moduli space by the Picard-Fuchs equation.
- Boundary condition at the orbifold point $\psi = 0$. The Picard-Fuchs equation has 4 power series solutions that go like $\omega_0 \sim \psi^{\frac{1}{5}}$, $\omega_1 \sim \psi^{\frac{2}{5}}$, $\omega_2 \sim \psi^{\frac{3}{5}}$, $\omega_3 \sim \psi^{\frac{4}{5}}$.
- The topological string amplitudes are

$$F^{(g)}_{\text{orbifold}} = \lim_{\bar{\psi}\to 0} \omega_0^{2(g-1)} (\frac{1-\psi}{\psi})^{g-1} P_g \sim \frac{P_g}{\psi^{\frac{3}{5}(g-1)}}$$

  We expect $F^{(g)}_{\text{orbifold}}$ to be regular at the orbifold point, based on earlier works (e.g. Katz, Klemm , Vafa).
- P_g is a power series of ψ, starting from a constant. This imposes

$$\lceil \frac{3}{5}(g-1) \rceil$$

  number of conditions on the holomorphic ambiguity in P_g.
- Boundary condition at the conifold point $\psi = 1$. Picard-Fuchs equation around $z = \psi - 1$ have four solutions that go like

$$\vec{\Pi} = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 + \mathcal{O}(z) \\ z + \mathcal{O}(z^2) \\ z^2 + \mathcal{O}(z^3) \\ \omega_1 \log(z) + \mathcal{O}(z^4) \end{pmatrix}$$

- We define a dual mirror map $t_D = \frac{\omega_1}{\omega_0}$. We find the topological strings around the conifold point has a "gap" structure in the t_D coordinate

$$\begin{aligned} F^{(g)}_{\text{conifold}} &= \lim_{\bar{z}\to 0} \omega_0^{2(g-1)} (\frac{1-\psi}{\psi})^{g-1} P_g \\ &= \frac{(-1)^{g-1} B_{2g}}{2g(2g-2) t_D^{2g-2}} + \mathcal{O}(t_D^0), \end{aligned}$$

  This fixes $2g-2$ coefficients in the holomorphic ambiguity.

- An arbitrary change of the basis $\omega_0 \to \omega_0 + b_1\omega_1 + b_2\omega_2$ does not affect this gap like structure.

- The leading coefficients of the conifold expansion were actually pointed out long time ago, Ghoshal, Vafa, hep-th/9506122. The gap condition is first observed recently in the context of $SU(2)$ Seiberg-Witten theory, Huang, Klemm, hep-th/0605195.

- Near the conifold point of the moduli space, a D3-brane wrapping a vanishing 3-cycle appears as a charged, BPS, extremal, and nearly massless black hole in space-time, Strominger, hep-th/9504090.

- A physical explanation of the gap condition: Integrating out the massless black hole state in a graviphoton background...

- Gopakumar-Vafa-Schwinger Computation: In $\mathcal{N}=2$ supergravity, we integrate out a charged BPS hypermultiplet of $e = m = \frac{t}{\lambda}$, and Lorentz Group $SO(4) = SU(2)_L \times SU(2)_R$ representation

$$[(\frac{1}{2},0) + 2(0,0)] \bigotimes (j_L, j_R)$$

  in a graviphoton background where the self-dual part of the graviphoton field strength is $F_+ = \lambda$.

- The Gopakumar-Vafa-Schwinger Computation generates the following term in the effective action

$$\begin{aligned} S &= \int d^4x F(t,\lambda) R_+^2, \\ \text{where} \quad F(t,\lambda) &= \int_\epsilon^\infty \frac{ds}{s} \frac{\text{Tr}(-1)^F \exp(-st)\exp(-2s\lambda\sigma_L)}{(2\sin(\frac{s\lambda}{2}))^2} \end{aligned}$$

- In type IIB compactification near the conifold, there is only one light particle: the massless black hole.
- The topological string near the conifold should be, (up to regular terms of the period t),

$$F(\lambda, t) = \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{\exp(-st)}{(2\sin(\frac{s\lambda}{2}))^2} = \sum (\frac{\lambda}{t})^{2g-2} \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)} + \mathcal{O}(t^0)$$

  This is precisely the gap condition.
- Boundary conditions at infinity $\psi = \infty$. The constant map contribution of manifold M, Faber, Pandharipande, math.ag/9810173,

$$\lim_{t\to\infty} F^{(g)}_{\text{A-model}} = \frac{(-1)^{g-1}B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\chi(M)$$

- The world sheet instanton corrections

$$F^{(g)}_{\text{instanton}} = \sum_{\beta\in H_2(M,\mathbb{Z})} r^{(g)}_{\beta} \exp(2\pi i t\beta)$$

  where $r^{(g)}_{\beta}$ are rational numbers, known as the Gromov-Witten invariants of holomorphic maps.
- Re-organize the world sheet instanton contributions

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}_{\text{instanton}} = \sum_{g=0}^{\infty}\sum_{\beta}\sum_{m=1}^{\infty} n^{(g)}_{\beta} (\frac{e^{2\pi i t\beta m}}{m})(2\sin\frac{m\lambda}{2})^{2g-2}$$

- The Gopakumar-Vafa invariants $n^{(g)}_{\beta}$ are integers counting BPS D0-D2 brane bound states.
- The quintic example: one kahler modulus, $\beta = d$ is the degree of the holomorphic map. The GV invariants

g	d=1	d=2	d=3	d=4	d=5
0	2875	609250	317206375	242467530000	229305888887625
1	0	0	609250	3721431625	12129909700200
2	0	0	0	534750	75478987900
3	0	0	0	8625	-15663750
4	0	0	0	0	49250
5	0	0	0	0	1100
6	0	0	0	0	10
7	0	0	0	0	0

- Boundary condition: at each genus, the Gopakumar-Vafa invariants vanish $n_d^{(g)} = 0$ for low degree d holomorphic maps.

Summary of Boundary Conditions at genus g

- Holomorphic ambiguity: $3g-2$ unknown constants.
- The expansion around orbifold point $\psi = 0$ provides $\lceil \frac{3}{5}(g-1) \rceil$ boundary conditions.
  The expansion around conifold point $\psi = 1$ provides $2g-2$ boundary conditions.
  The large complex structure modulus/large volume limit $\psi = \infty$ provides $a_g + 1$ boundary conditions, where a_g is the number of low degree vanishing GV invariants at genus g, sensitive to specific models.
- Count the number of unknown constants

$$3g - 2 - (\lceil \frac{3}{5}(g-1) \rceil + 2g - 2 + 1 + a_g) = [\frac{2}{5}(g-1)] - a_g$$

- We have enough/redundant data to compute topological strings if

$$a_g \geq [\frac{2}{5}(g-1)]$$

- This is true for low genus, (up to $g \sim 51$ for the quintic) . However, asymptotically

$$a_g \sim \sqrt{g}, \quad \text{when} \quad g \to \infty$$

  So far our calculation is limited only by the power of our computational facilities.

- The analysis can be straightforwardly generalized to one-parameter Calabi-Yau models, realized as hypersurfaces or complete intersections in weight projective spaces.

$$X_5(1^5)\quad X_6(1^4,2),\quad X_8(1^4,4),\quad X_{10}(1^3,2,5),\quad X_{3,3}(1^6),$$
$$X_{4,2}(1^6),\quad X_{3,2,2}(1^7),\quad X_{2,2,2,2}(1^8)\quad X_{4,3}(1^5,2),\quad X_{4,4}(1^4,2^2),$$
$$X_{6,2}(1^5,3),\quad X_{6,4}(1^3,2^2,3),\quad X_{6,6}(1^2,2^2,3^2).$$

- We solve all these 13 models to very high genus. The singular behaviors around the conifold point is universal.

- On the other hand, we discover a rich variety of singularity structures around the orbifold point. The 13 models fall into 4 classes.

Four cases

(1). No massless charged state. The F^g are regular at the orbifold point $\psi = 0$, imposing boundary conditions. This includes models $X_5(1^5)$, $X_6(1^4,2)$, $X_8(1^4,4)$, $X_{10}(1^3,2,5)$,$X_{3,3}(1^6)$, $X_{2,2,2,2}(1^8)$, $X_{4,4}(1^4,2^2)$,$X_{6,6}(1^2,2^2,3^2)$.

(2). One massless charged state. The F^g exhibit the "gap structure" similar to the conifold point, imposing boundary conditions. This includes models $X_{4,2}(1^6)$, $X_{6,2}(1^5,3)$.

(3). Two massless charged states. The interactions between massless states destroy the "gap structure", no boundary conditions at the orbifold point. This includes models $X_{3,2,2}(1^7)$.

(4). Multiple massless charged states. The F^g are singular with no obvious structures at the orbifold point. However the scaling of masses of these light states imposes some boundary conditions. This includes the model $X_{4,3}(1^5,2)$, $X_{6,4}(1^3,2^2,3)$.

Applications for black hole physics

- Compactify M-theory on a compact Calabi-Yau 3-fold. The 5-D supergravity has a BPS black hole solution (BMPV black hole) with graviphoton charge Q, angular momentum J of the $SU(2)_L \subset SO(4)$. The classical entropy of the black hole is one quarter of the horizon area

$$S = 2\pi\sqrt{Q^3 - J^2}$$

- There are R^2 correction to the black hole entropy, computable by Wald's formula,

$$\Delta S = 2\pi \int_{\text{Horizon}} \frac{\partial(\Delta \mathcal{L})}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon\rho\sigma \sim Q^{\frac{1}{2}}$$

- An open problem: How to count the black hole microstates? Much more difficult than the Strominger-Vafa black hole.

- Katz, Klemm, Vafa (KKV), 1999: The black hole microstates are counted by topological strings. For a black hole with 2-brane charge d and $SU(2)_L$ angular momentum $J = m$, the number of states are

$$N_d^m = \sum_r n_d^r \begin{pmatrix} 2r+2 \\ r+1+m \end{pmatrix}$$

  The graviphoton charge are related by the supergravity attractor equation $Q = (\frac{2}{9})^{\frac{1}{3}} \frac{d}{\sqrt{\kappa}}$, where κ is the intersection number.

- This is a very natural proposal since the Gopakumar-Vafa invariant n_d^r is a supersymmetric index that remains constant in the moduli space.

- Difficulty: For non-compact Calabi-Yaus, the KKV formula can not be reliably applied to count 5D black hole microstates, since this is not really a compactification to 5D supergravity. There were not much computations of the Gopakumar-Vafa invariants for compact Calabi-Yau available (before our paper).

- We use our new results and the KKV formula the count micro-states. Consider e.g. angular momentum $m = 0$,

$$S = \log(N_d^0) = \frac{4\pi}{3\sqrt{2\kappa}} d^{\frac{3}{2}} + \mathcal{O}(d^{\frac{1}{2}})$$

  Topological string data provide the values

$$f(d) = \frac{\log(N_d^0)}{d^{\frac{3}{2}}} = \frac{4\pi}{3\sqrt{2\kappa}} + \frac{b_1}{d} + \frac{b_2}{d^2} + \cdots$$

  for d up to a finite degree.

- For all 13 models, the KKV formula for counting micro-states confirms the macroscopic black hole prediction of leading coefficient with impressively small error of 1∼ 3 % .

Conclusion and Future Directions

- We have made significant progress in solving topological strings on compact Calabi-Yau spaces.
- It would be interesting to develop algebraic geometric theory to systematically verify or prove our predictions.
- Continue to explore the fascinating implications for the OSV conjecture and black hole physics, higher curvature corrections to black hole entropy.
- Topological string partition functions are modular, almost holomorphic sections of the moduli space, Aganagic et al, Marino et al, Grimm et al. The modular ring for monodromy group of the quintic is still poorly understood.
- Discover more boundary conditions in the modili space of compact Calabi-Yaus. Gromov-Witten theory at the orbifold points? Matrix models around the orbifold or conifold points?