Lattice Calculation of Parton Distributions

Fernanda Steffens
DESY – Zeuthen

On behalf of the collaboration: C. Alexandrou, K. Cichy, V. Drach, K. Hadjiyiannakou, K. Jansen, C. Wiese

Outline

- Quark distributions
- Quasi-quark distributions
- Relation between quark and quasi-quark distributions
- Calculation of the matrix elements of the quasi distributions
- Results

Main References

- X. Ji, “Parton Physics on a Euclidean Lattice,” PRL 110 (2013) 262002.

- X. Xiong, X. Ji, J.-H. Zhang and Y. Zhao,
  “One loop matching for parton distributions:Nonsinglet case,”PRD90 (2014) 014051.

- H.-W. Lin, J.-W. Chen, S. D. Cohen and X. Ji,
  “Flavor Structure of the Nucleon Sea from Lattice QCD,” Phys. Rev. D91 (2015) 054510.

- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, k. Hadjiyiannakou, K. Jansen, FS and
  C. Wiese, “A Lattice Calculation of Parton Distributions,” Phys. Rev. D92 (2015) 014502

Light cone quark distributions

QCD

↓

OPE

↓

Moments of structure functions /quark distributions in terms of matrix element of local operators

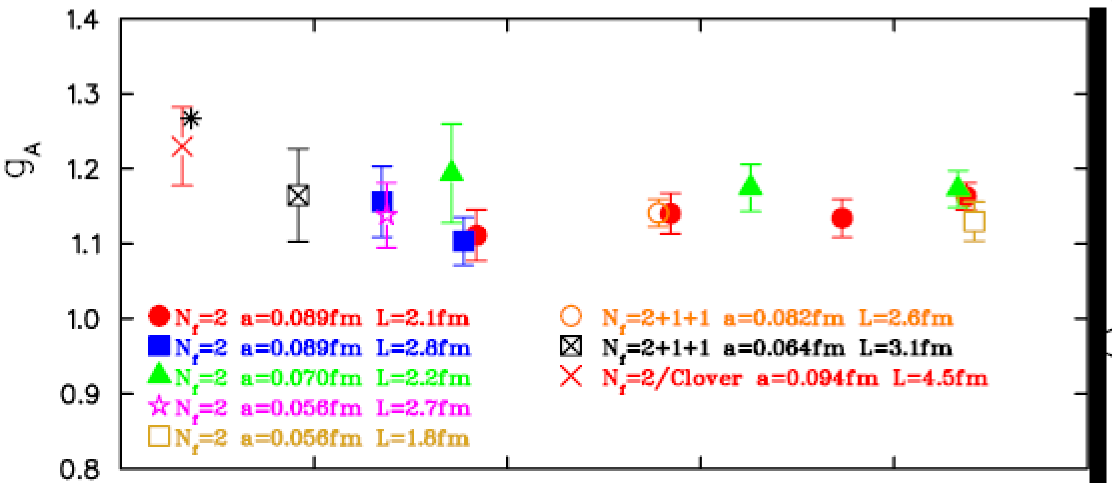
$$\int dx\, x^{n-1} q(x) = \langle N | \mathcal{O}^{\{\mu_1 \ldots \mu_n\}} | N \rangle ,$$

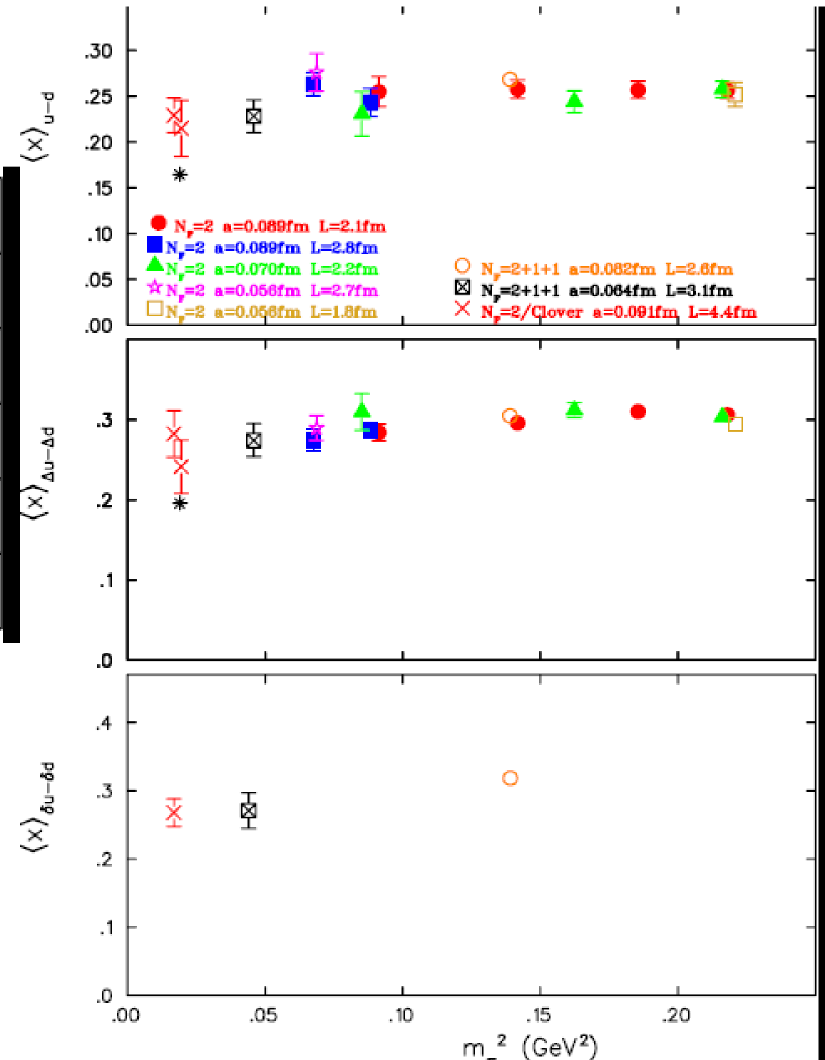
$$\mathcal{O}^{\{\mu_1 \ldots \mu_n\}} = \bar{\psi} \left(\gamma^{\{\mu_1} i \overleftrightarrow{D}^{\mu_2} \ldots i \overleftrightarrow{D}^{\mu_n\}} \right) \frac{\tau^a}{2} \psi .$$

Example – isovector quark momentum fraction ($q(x) = u(x) - d(x)$):

$$\langle x \rangle_{u-d} = \int dx\, x \, \left(q(x) + \bar{q}(x) \right) .$$

- If a sufficient number of moments are calculated, one can reconstruct the x dependence of the distributions
- Hard to simulate high order derivatives on the lattice
- Nevertheless, the first few moments as well as charges can and have been calculated





Source: C. Alexandrou et al. 1405.5213

The x dependence of the distributions

Inverse Mellin transform

$$q(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, x^{-n} a_n \qquad a_n = \int dx\, x^{n-1} q(x)$$

$$q(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-} \langle P|\bar{\psi}(\xi^-)\Gamma\mathcal{L}(\xi^-,0)\psi(0)|P\rangle$$

$$= e^{-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)}$$

- Light cone correlations in the nucleon rest frame
- Equivalent to distributions in the IMF
- Light cone dominated: $\xi^2 = t^2 - z^2 \sim 0$
- Not calculable on Euclidian Lattice as $t^2 + z^2 \sim 0$

Quasi Distributions

Matrix elements $$\langle P|O^{\mu_1\mu_2\ldots\mu_n}|P\rangle = 2a_n^{(0)}\Pi^{\mu_1\mu_2\ldots\mu_n}$$ $$P = (P_0, 0, 0, P_3).$$

where $$\Pi^{\mu_1\mu_2\ldots\mu_n} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\{g\ldots gP\ldots P\}_{k,j}(P^2)^j$$

Setting $$\mu_1 = \mu_2 = \ldots = \mu_{2k} = 3$$

$$\Pi^{3\ldots3} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\frac{(2k)!}{2^j j!(2k-2j)!}(-1)^j(P_3^2)^{k-j}(M^2)^j$$

$$\langle P|O^{3\ldots3}|P\rangle = 2\tilde{a}_{2k}^{(0)}(P_3)^{2k}\sum_{j=0}^{k}\mu^j\begin{pmatrix}2k-j\\ j\end{pmatrix} \equiv 2\tilde{a}_{2k}(P_3)^{2k}$$

With $$\mu = M^2/4(P_3)^2$$

Defining:

$$\tilde{a}_n(\Lambda, P_3) = \int_{-\infty}^{+\infty} x^{n-1}\tilde{q}(x,\Lambda,P_3)dx$$

Mellin transformation implies in

$$\tilde{q}(x,\Lambda,P_3) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{-izk_3} \langle P|\bar{\psi}(0,z)\gamma^3 W(z)\psi(0,0)|P\rangle$$

Parton momentum $k_3 = xP_3$

Wilson line $W(z) = e^{-ig\int_0^z dz' A_3(z')}$

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated on a lattice

What are these quasi-distributions? Do they have a partonic interpretation?

The light cone distributions:

$$x = \frac{k^+}{P^+}$$

$$0 \leq x \leq 1$$

Distributions can be defined in an infinite momentum frame: $p^z, p^+ \to \infty$

Quasi distributions:

P^z large but finite

Some constituents can be moving backward or even with momentum greater than P^z

$x < 0$ or $x > 1$ is possible

Usual partonic interpretation is lost

But they can be related to each other!

Extracting the quark distributions

Infrared region untouched when going from a finite to an infinite momentum

Infinite momentum frame: $P_3 \to \infty, \quad \Lambda$ fixed

$$q(x,\mu) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} Z_F(\mu)\right\} + \frac{\alpha_s}{2\pi}\int_x^1 q^{(1)}(x/y,\mu)\, q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

Finite momentum: $\Lambda \to \infty, \quad P_3$ fixed

$$\tilde{q}(x,\Lambda,P_3) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi}\tilde{Z}_F(\Lambda,P_3)\right\} + \frac{\alpha_s}{2\pi}\int_{x/x_c}^1 \tilde{q}^{(1)}(x/y,\Lambda,P_3)\, q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

$x_c \sim \frac{\Lambda}{P_3}$ Largest value at which the self energy and vertex corrections are meaningful

And we get:

$$\tilde{q}(x,\Lambda,P_3) = q(x,\mu) + \frac{\alpha_s}{2\pi} q(x,\mu)\left\{\tilde{Z}_F(\Lambda,P_3) - Z_F(\mu)\right\} + \frac{\alpha_s}{2\pi}\int_{x/x_c}^{1}\left(\tilde{q}^{(1)}(x/y,\Lambda,P_3) - q^{(1)}(x/y,\mu)\right) q(y,\mu)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

Including antiquarks: $\bar{q}(x) = -q(-x)$

$$q(x,\mu) = \tilde{q}(x,\Lambda,P_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,\Lambda,P_3)\delta Z_F^{(1)}\left(\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right) - \frac{\alpha_s}{2\pi}\int_{-1}^{1} Z^{(1)}\left(\frac{x}{y},\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right)\tilde{q}(y,\Lambda,P_3)\frac{dy}{|y|} + \mathcal{O}(\alpha_s^2)$$

$$\delta Z^{(1)} = \tilde{Z}_F^{(1)} - Z_F^{(1)}$$

$$Z^{(1)} = \tilde{q}^{(1)} - q^{(1)}$$

Nucleon Mass Corrections

$$\langle P|O^{3\ldots3}|P\rangle = 2\tilde{a}^{(0)}_{2k}(P_3)^{2k}\sum_{j=0}^{k}\mu^j \begin{pmatrix} 2k-j \\ j \end{pmatrix} \equiv 2\tilde{a}_{2k}(P_3)^{2k}$$

Defining: $$\tilde{a}^{(0)}_n \doteq \int_{-\infty}^{+\infty} x^{n-1}\tilde{q}^{(0)}(x,P_z)dx$$

$$\tilde{q}(x,P_z) = \frac{1}{1+\mu\xi^2}\tilde{q}^{(0)}(\xi,P_z)$$

$$\xi = \frac{2x}{1+\sqrt{1+4\mu x^2}}$$

Matching Condition

- Relating finite to infinite momentum
- Axial gauge $A_3 = 0$
- UV divergence regulate with $|k_T| \leq \Lambda \sim \frac{1}{a}$
- Renormalization scale μ

$$q(x,\mu) = \tilde{q}(x,\Lambda,P_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,\Lambda,P_3)\delta Z_F^{(1)}\left(\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right) - \frac{\alpha_s}{2\pi}\int_{-1}^{1} Z^{(1)}\left(\frac{x}{y},\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right)\tilde{q}(y,\Lambda,P_3)\frac{dy}{|y|} + \mathcal{O}(\alpha_s^2)$$

Desired quantity

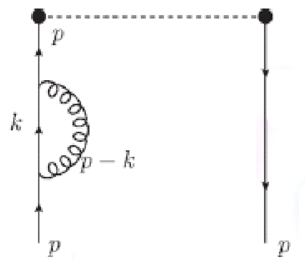
From the lattice

From pQCD. How to calculate them?

Calculating Z

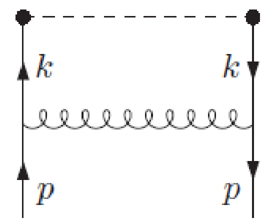
$$Z(y) = Z^0\left(y, \frac{\mu}{Pz}\right) + Z^{(1)}\left(y, \frac{\mu}{Pz}\right) + \cdots$$

$$q(x,\Lambda) = \left(1 + Z_F^{(1)}(\Lambda)\right)\delta(x-1) + q^{(1)}(x,\Lambda) + \cdots$$



Self-energy correction

Vertex correction



$$\tilde{q}(x,\Lambda,P_3) = \left(1 + \tilde{Z}_F^{(1)}(\Lambda,P_3)\right)\delta(x-1) + \tilde{q}^{(1)}(x,\Lambda,P_3)$$

Results

$\Lambda \to \infty, \quad P_3$ fixed

$$\tilde{q}^{(1)}(x,\Lambda,P^z) = \frac{\alpha_S C_F}{2\pi}\begin{cases} \frac{1+x^2}{1-x}\ln\frac{x}{x-1} + 1 + \frac{\Lambda}{(1-x)^2P^z}, & x>1, \\ \frac{1+x^2}{1-x}\ln\frac{(P^z)^2}{m^2} + \frac{1+x^2}{1-x}\ln\frac{4x}{1-x} - \frac{4x}{1-x} + 1 + \frac{\Lambda}{(1-x)^2P^z}, & 0<x<1, \\ \frac{1+x^2}{1-x}\ln\frac{x-1}{x} - 1 + \frac{\Lambda}{(1-x)^2P^z}, & x<0, \end{cases}$$

$$\tilde{Z}_F^{(1)}(\Lambda,P^z) = \frac{\alpha_S C_F}{2\pi}\int dy \begin{cases} -\frac{1+y^2}{1-y}\ln\frac{y}{y-1} - 1 - \frac{\Lambda}{(1-y)^2P^z}, & y>1, \\ -\frac{1+y^2}{1-y}\ln\frac{(P^z)^2}{m^2} - \frac{1+y^2}{1-y}\ln\frac{4y}{1-y} + \frac{4y^2}{1-y} + 1 - \frac{\Lambda}{(1-y)^2P^z}, & 0<y<1, \\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y} + 1 - \frac{\Lambda}{(1-y)^2P^z}, & y<0. \end{cases}$$

$P_3 \to \infty, \quad \Lambda$ fixed

$$q^{(1)}(x,\Lambda) = \frac{\alpha_S C_F}{2\pi}\begin{cases} 0, & x>1 \quad \text{or} \quad x<0, \\ \frac{1+x^2}{1-x}\ln\frac{\Lambda^2}{m^2} - \frac{1+x^2}{1-x}\ln(1-x)^2 - \frac{2x}{1-x}, & 0<x<1, \end{cases}$$

$$Z_F^{(1)}(\Lambda) = \frac{\alpha_S C_F}{2\pi}\int dy \begin{cases} 0, & y>1 \quad \text{or} \quad y<0, \\ -\frac{1+y^2}{1-y}\ln\frac{\Lambda^2}{m^2} + \frac{1+y^2}{1-y}\ln(1-y)^2 + \frac{2y}{1-y}, & 0<y<1, \end{cases}$$

X. Xiong et al., Phys. Ver. D90,014051 (2014)

Final vertex correction: $Z^{(1)} = \tilde{q}^{(1)} - q^{(1)}$

$$\frac{Z^{(1)}(\xi)}{C_F} = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi}{\xi-1} + 1 + \frac{1}{(1-\xi)^2}\frac{\Lambda}{P_3}$$ $\xi > 1$

$$\frac{Z^{(1)}(\xi)}{C_F} = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{(P_3)^2}{\mu^2} + \left(\frac{1+\xi^2}{1-\xi}\right)\ln 4\xi(1-\xi) - \frac{2\xi}{1-\xi} + 1 + \frac{1}{(1-\xi)^2}\frac{\Lambda}{P_3}$$ $0 < \xi < 1$

$$\frac{Z^{(1)}(\xi)}{C_F} = \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi-1}{\xi} - 1 + \frac{1}{(1-\xi)^2}\frac{\Lambda}{P_3}$$ $\xi < 0$

Final wave function correction: $\delta Z^{(1)} = \tilde{Z}_F^{(1)} - Z_F^{(1)}$

$$\delta Z^{(1)} = C_F \int_{-\infty}^{\infty} d\xi \, \delta Z^{(1)}(\xi),$$

$$\delta Z^{(1)}(\xi) = -\left(\frac{1+\xi^2}{1-\xi}\right) \ln \frac{\xi}{\xi-1} - 1 - \frac{1}{(1-\xi)^2} \frac{\Lambda}{P_3}$$ $\xi > 1$

$$\delta Z^{(1)}(\xi) = -\left(\frac{1+\xi^2}{1-\xi}\right) \ln \frac{(P_3)^2}{\mu^2} - \left(\frac{1+\xi^2}{1-\xi}\right) \ln 4\xi(1-\xi) - \frac{2\xi(2\xi-1)}{1-\xi} + 1 - \frac{1}{(1-\xi)^2} \frac{\Lambda}{P_3}$$ $0 < \xi < 1$

$$\delta Z^{(1)}(\xi) = -\left(\frac{1+\xi^2}{1-\xi}\right) \ln \frac{\xi-1}{\xi} + 1 - \frac{1}{(1-\xi)^2} \frac{\Lambda}{P_3}$$ $\xi < 0$

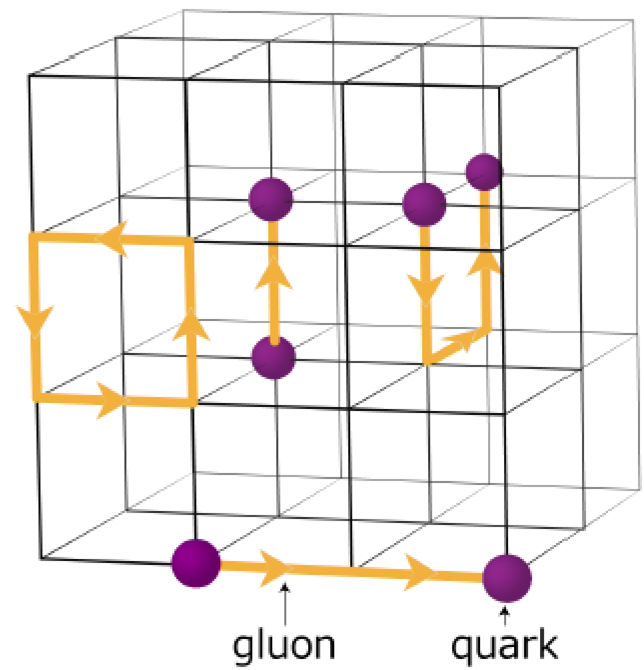
- Contains single and double poles $\xi = 1$
- Single poles cancel between vertex and wave function corrections
- Double pole is reduced to a single pole
- Cauchy principal value regularizes the remaining pole
- Conservation of quark number, requires the integrals to have a cut at $x_c \sim \Lambda/P_3$
- It remains a UV divergence in the wave function correction $\tilde{q}(x)\frac{3}{2}\ln(x_c^2-1)$

$$
\begin{aligned}
q(x,\mu) \;=\; & \tilde{q}(x,\Lambda,P_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,\Lambda,P_3)\delta Z^{(1)}\left(\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right) \\
& -\frac{\alpha_s}{2\pi}\int_{-x_c}^{-|x|/x_c} Z^{(1)}\left(\xi,\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right)\tilde{q}\left(\frac{x}{\xi},\Lambda,P_3\right)\frac{d\xi}{|\xi|} \\
& -\frac{\alpha_s}{2\pi}\int_{+|x|/x_c}^{+x_c} Z^{(1)}\left(\xi,\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right)\tilde{q}\left(\frac{x}{\xi},\Lambda,P_3\right)\frac{d\xi}{|\xi|} + \mathcal{O}(\alpha_s^2)
\end{aligned}
$$

Calculation of the matrix elements in a lattice

C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos,
K. Hadjiyiannakou, K. Jansen, FS, C. Wiese, Lattice 2014 , arXiv:1411.0891

- We introduce a 4D hypercubic lattice:
  - ⋆ quark fields on lattice sites,
  - ⋆ gluon fields on lattice links.
- Gauge invariant objects:
  - ⋆ Wilson loops,
  - ⋆ quarks and antiquarks connected with a gauge link.
- Lattice as a regulator:
  - ⋆ UV cut-off – inverse lat. spac. a^{-1},
  - ⋆ IR cut-off – inverse lat. size L^{-1}.
- Remove the regulator:
  - ⋆ continuum limit $a \to 0$,
  - ⋆ infinite volume limit $L \to \infty$.



Source: JICFuS, Tsukuba

The Wilson twisted mass fermion action for the 2 light (u, d quarks) is given in the so-called twisted basis by: [R. Frezzotti, P. Grassi, G.C. Rossi, S. Sint, P. Weisz, 2000-2004]

$$S_l[\psi, \bar{\psi}, U] = a^4 \sum_x \bar{\chi}_l(x) \big(D_W + m_{0,l} + i\mu_l \gamma_5 \tau_3\big) \chi_l(x),$$

where:

- D_W – Wilson-Dirac operator,
- $m_{0,l}$ and μ_l – bare untwisted and twisted light quark masses,
- the matrix τ^3 acts in flavour space,
- $\chi_l = (\chi_u, \chi_d)$ is a 2-component vector in flavour space, related to the one in the physical basis by a chiral rotation with angle ω:

$$\psi = e^{i\gamma_5 \tau_3 \omega/2} \chi.$$

With maximal twist, $\omega = \pi/2$, automatic O(a)-improvement is achieved.

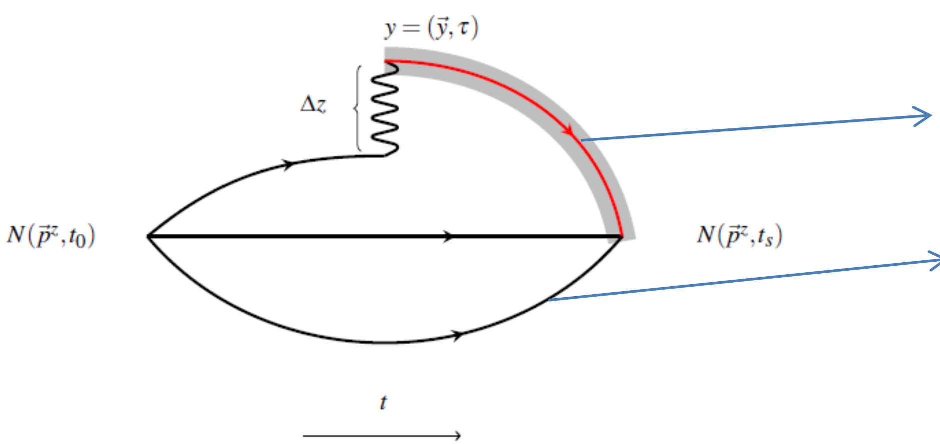
We want: $$h(P_3, z) = \langle P|\overline{\psi}(z)\gamma_3 W_3(z,0)\psi(0)|P\rangle$$

Let : $$C^{3\text{pt}}(t,\tau,0) = \left\langle N_\alpha(\vec{P},t)\mathcal{O}(\tau)\overline{N}_\alpha(\vec{P},0)\right\rangle$$

$$N_\alpha(\vec{P},t) = \Gamma_{\alpha\beta}\sum_{\vec{x}} \mathrm{e}^{i\vec{P}\vec{x}}\epsilon^{abc}u^a_\beta(x)\left(d^{b^T}(x)\mathcal{C}\gamma_5 u^c(x)\right)$$

$$\mathcal{O}(z,\tau,Q^2=0) = \sum_{\vec{y}}\overline{\psi}(y+z)\gamma_3 W_3(y+z,y)\psi(y)$$

All to all propagators needed



Stochastic source method is used

Point source method is used

Flavour structure: u - d

Extraction of the matrix elements

$$\frac{C^{3\mathrm{pt}}(t,\tau,0;\vec{P})}{C^{2\mathrm{pt}}(t,0;\vec{P})} \overset{0\ll\tau\ll t}{=} \frac{-iP_3}{E} h(P_3,\Delta z)$$

$8a, 10a$ Source – sink separation

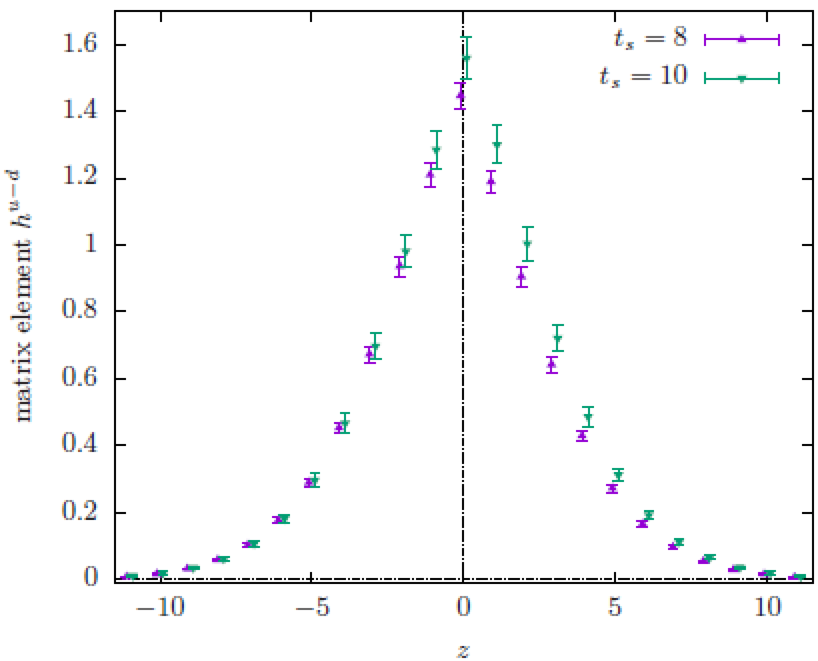
$32^3 \times 64$ Lattice

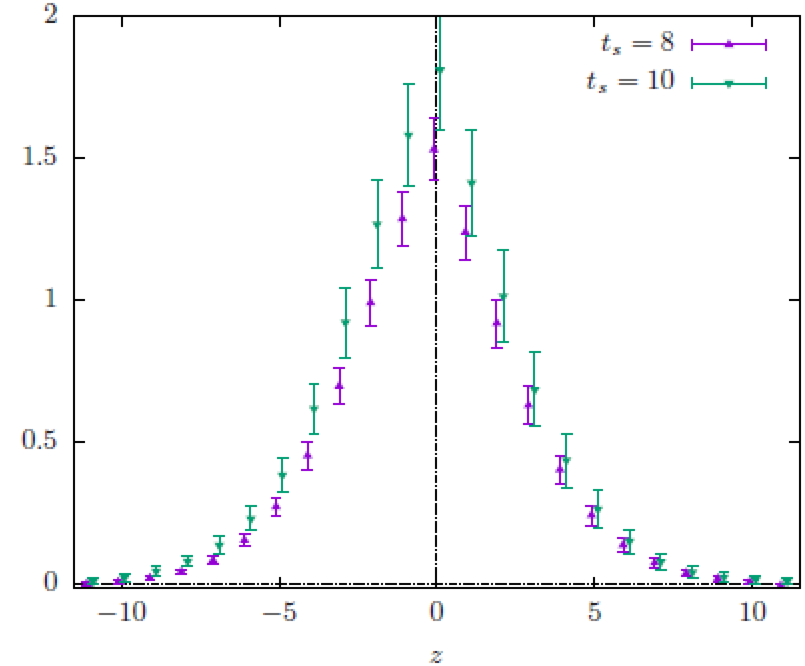
$\beta = \frac{6}{g_0^2} = 1.95$ $a \approx 0.082\, fm$ $N_f = 2+1+1$

Maximally twisted mass ensemble: $a\mu = 0.0055 \Rightarrow m_{PS} \approx 370\ MeV$

$$P_3 = \frac{2\pi}{L}, \frac{4\pi}{L}, \ldots$$

Study of the source –link separation efffect





181 gauge configurations
15 sets of point source forward propagators
02 sets of stochastic propagators

5430 measurements

Compatible within errors, so we choose the smaller source-sink separation

Normalization of the matrix elements

For z = 0, the operator can be identified with local vector current at $Q^2 = 0$

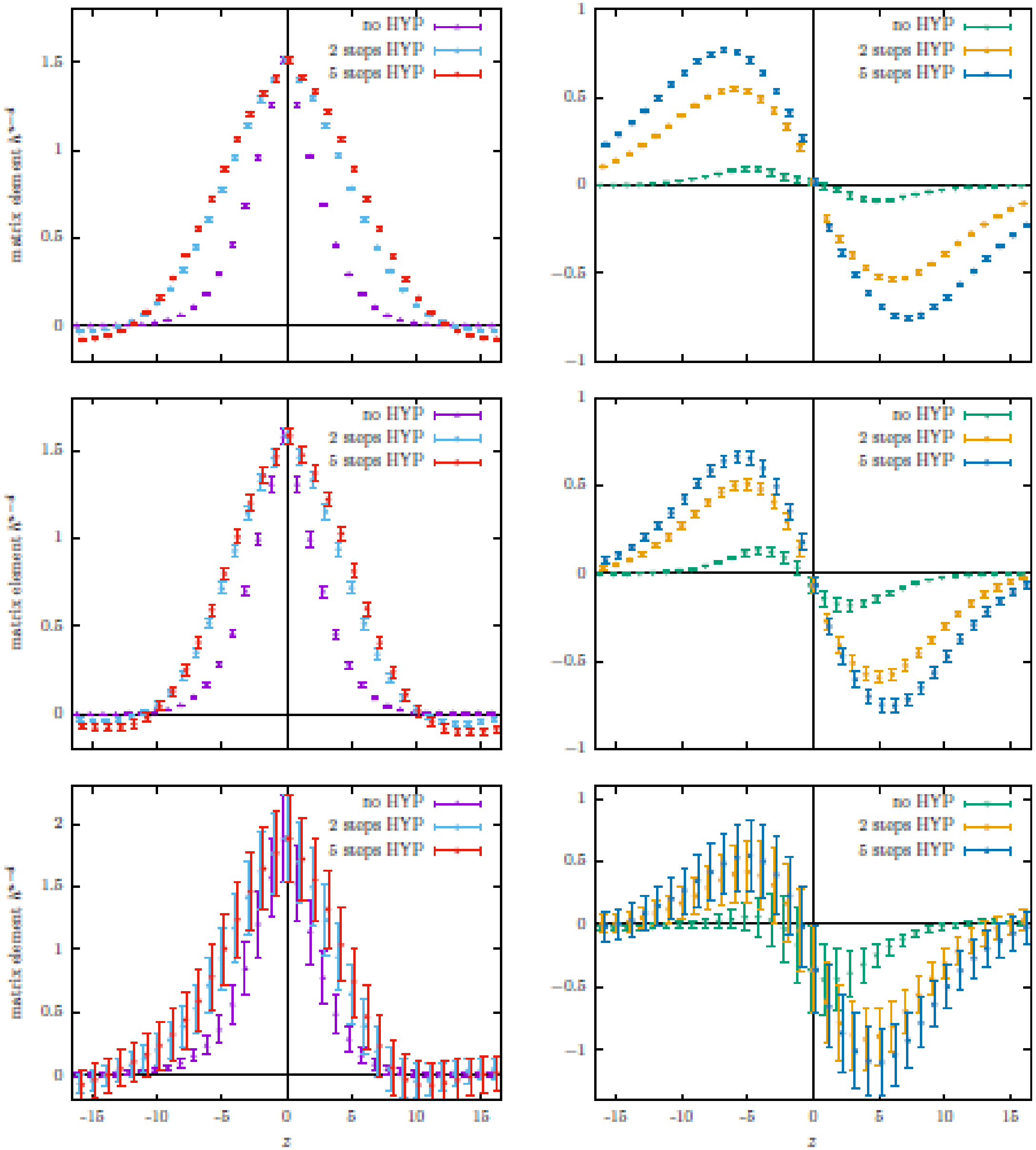
This operator is renormalized with the vector current renormalization constant Z_V

For this ensemble, $Z_V = 0.625(2)$ *C. Alexandrou et al.* Phys.Rev. D88 (2013) 014509

Using this value, we obtain:

$$Z_V h^{u-d}(0) = 0.99(3) \text{ for } P_3 = 4\pi/L$$

$$Z_V h^{u-d}(0) = 1.18(22) \text{ for } P_3 = 6\pi/L$$



$$P_3 = \frac{2\pi}{L}$$

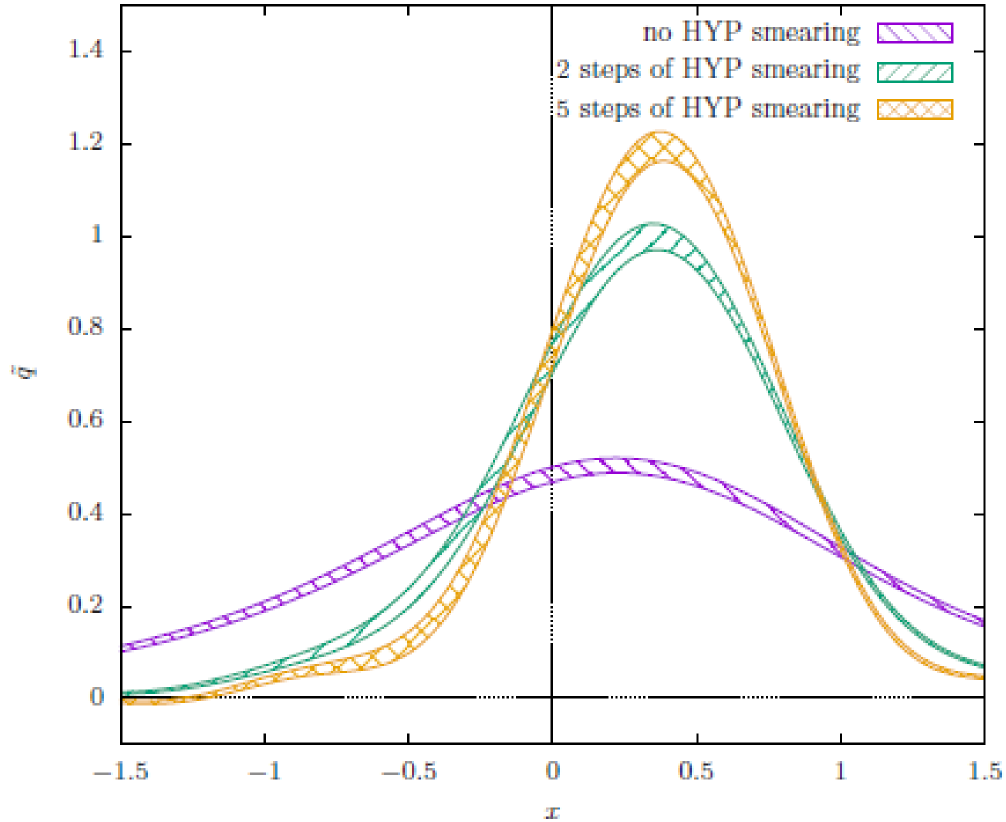
$$P_3 = \frac{4\pi}{L}$$

$$P_3 = \frac{6\pi}{L}$$

HYP Smearing

It replaces a given gauge link with some average over neighbouring links, i.e. ones from hypercubes attached to it

Crude substitute for renormalization



$$P_3 = \frac{4\pi}{L}$$

Parameters

$$\alpha_s = 6/(4\pi\beta) \approx 0.245$$

$$\Lambda = 1/a \cong 2.5\,\mathrm{GeV}$$

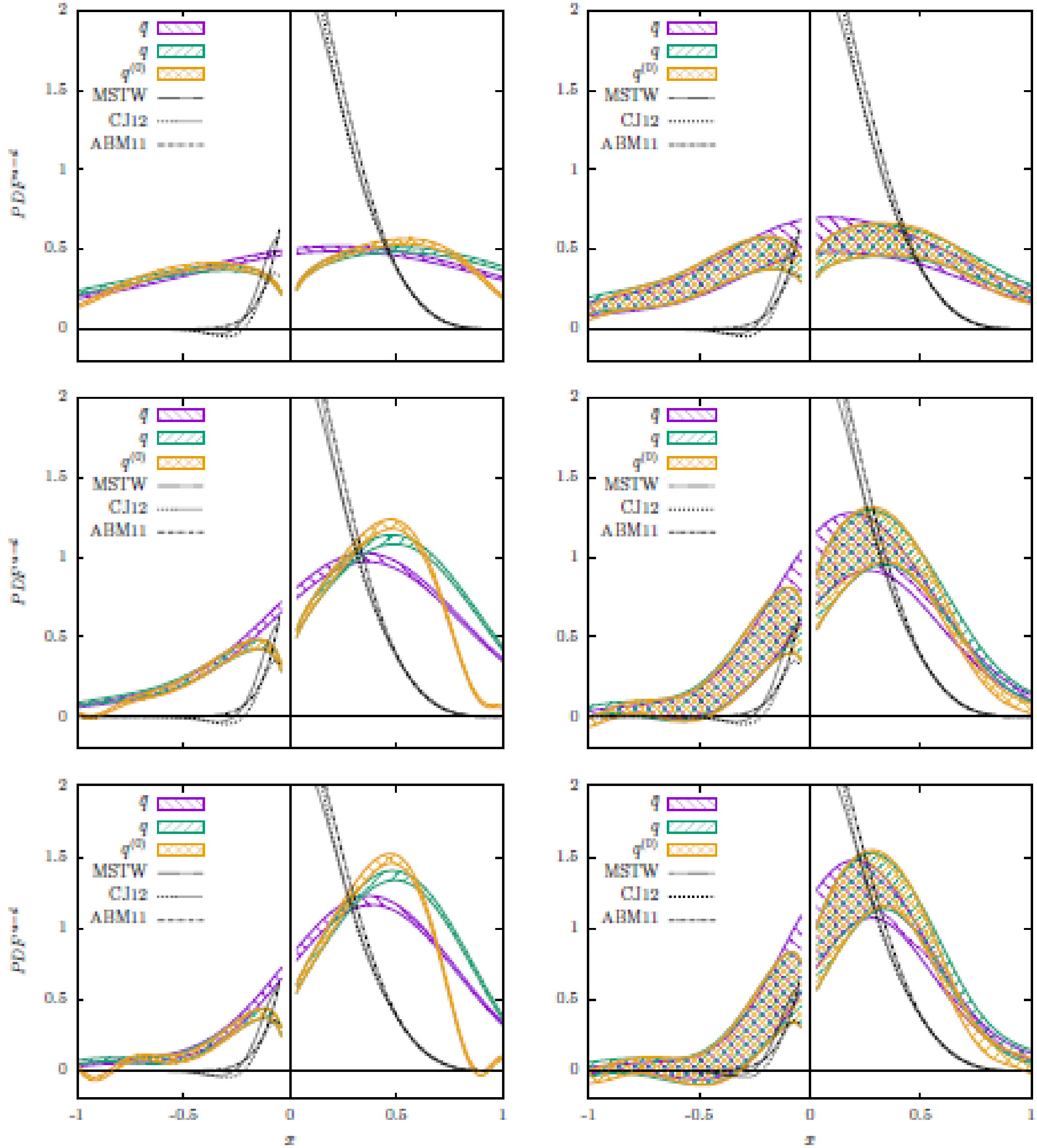
Results for

$\mu = \Lambda$

No HYP

2 steps of HYP

5 steps of HYP



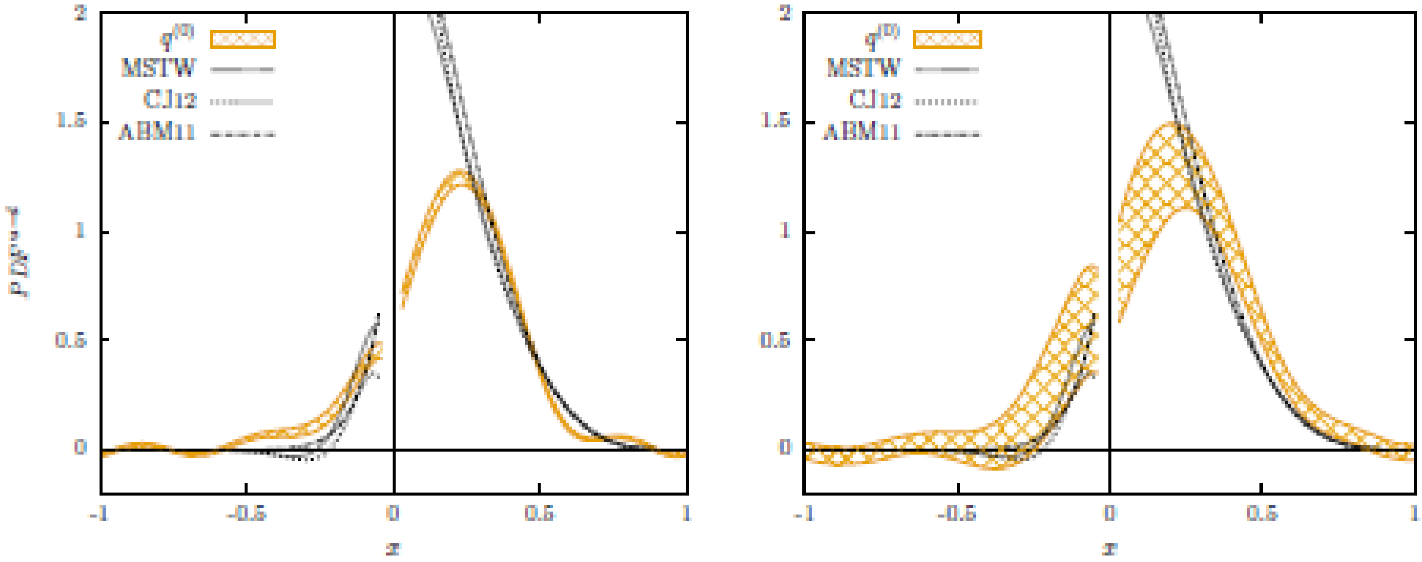
$P_3 = \frac{4\pi}{L}$

$P_3 = \frac{6\pi}{L}$

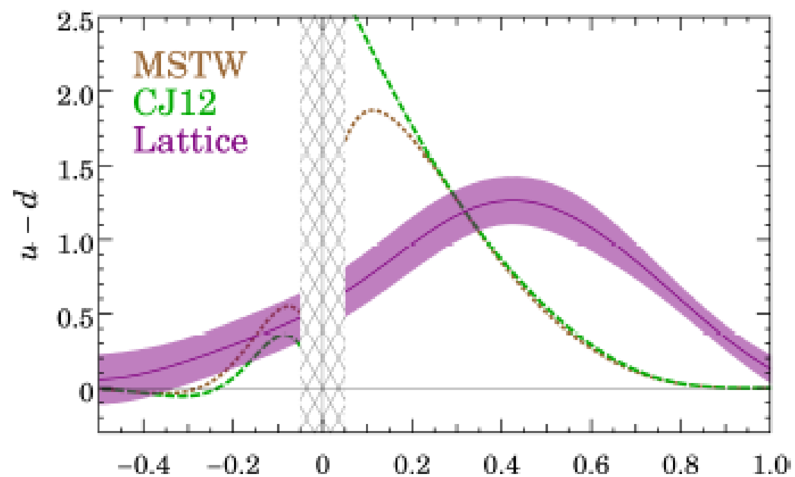
Mixed setup scheme

Matrix elements from $P_3 = \frac{4\pi}{L}, \frac{6\pi}{L}$

Fourier transformation and matching using $P_3 = \frac{8\pi}{L}$



Only other result



Huey-Wen Lin et al. Phys. Ver. D91 (2015) 054510

$$24^3 \times 48$$

$$a \approx 0.12 fm \qquad N_f = 2+1+1$$

$$m_{PS} \approx 310\ MeV$$

Uses highly improved staggered quarks and HYP smearing

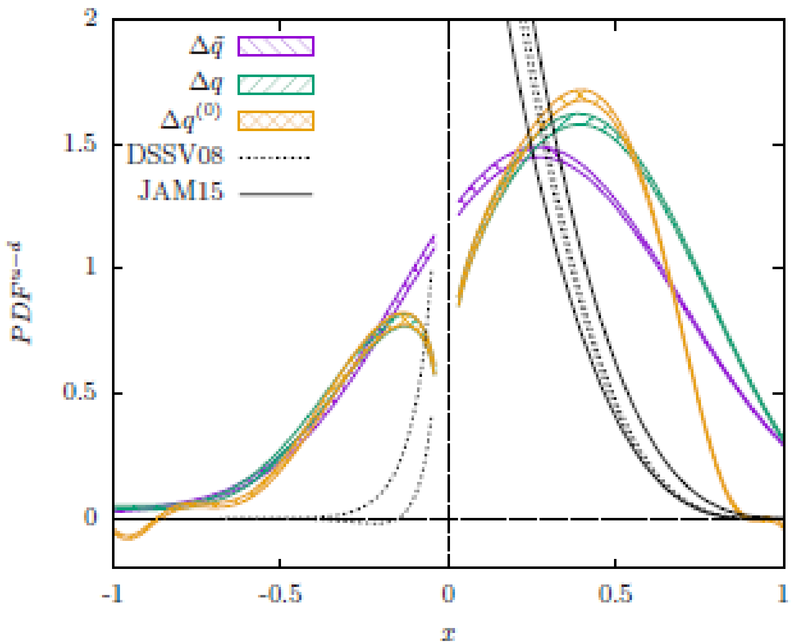
Polarized Sector

γ_5 matrix inserted in the former matrix elements

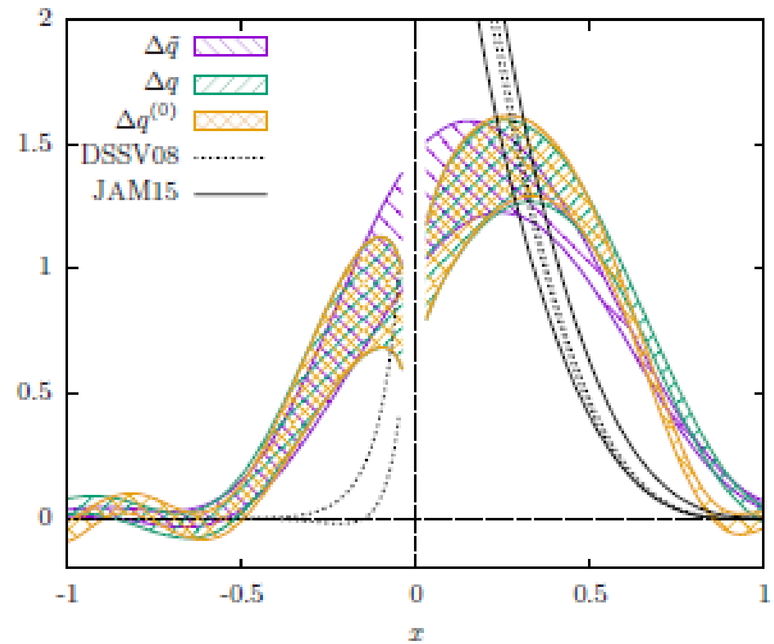
694 gauge configurations

15 point source forward propagators

01 Stochastic propagator



$$P_3 = \frac{4\pi}{L}$$



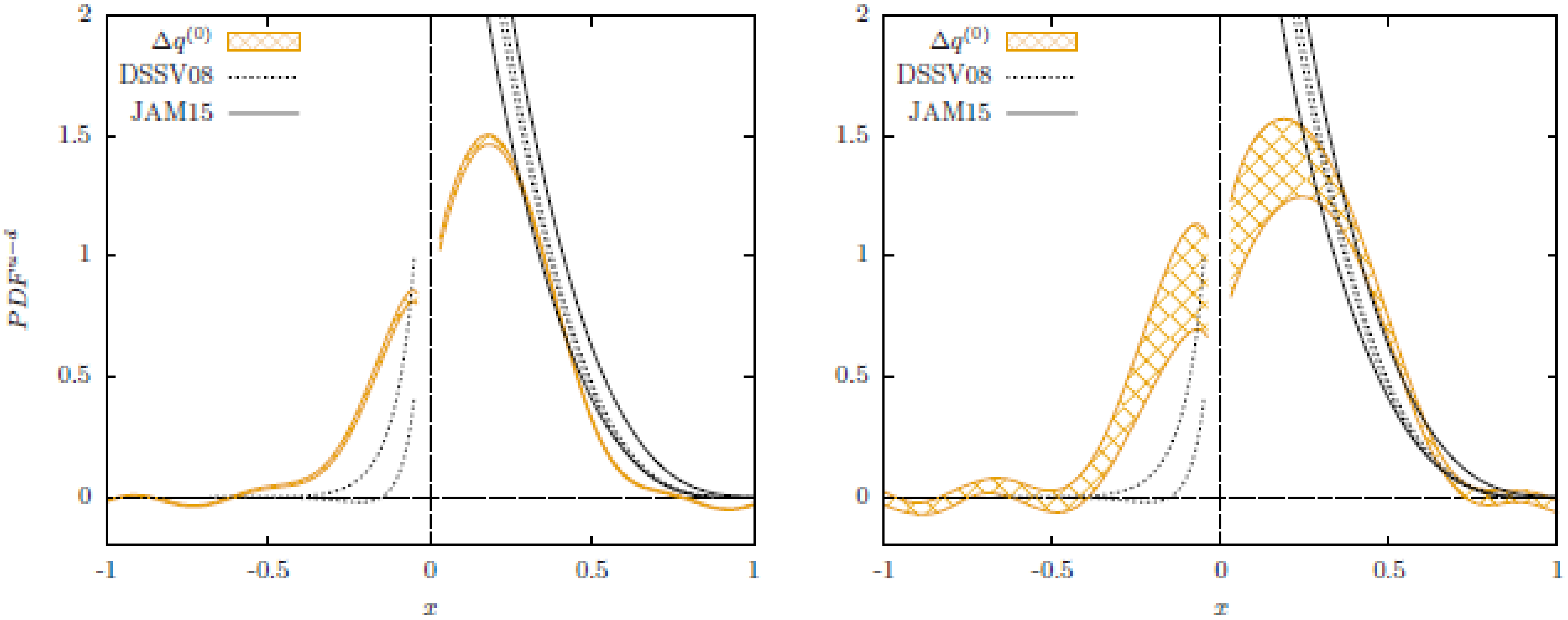
$$P_3 = \frac{6\pi}{L}$$

Mixed setup scheme

Nonsinglet combination $\Delta u(x) - \Delta d(x)$

Matrix elements from $P_3 = \frac{4\pi}{L}, \frac{6\pi}{L}$

Fourier transformation and matching using $P_3 = \frac{8\pi}{L}$



Summary & Outline

- First attempts of a direct QCD calculation of quark distributions;
- Valuable information from intermediate to large x region;
- Asymmetric sea appears naturally. Imaginary part plays a fundamental role;
- Renormalization;
- Higher order correction;
- Go to up 30000 measurements;
- Compute at the physical mass – smaller number of configurations available at the moment;
- Go to the continuum;
- Transverse distributions, singlet combinations, etc
- Much to be done!