Rapidity factorization and evolution of gluon TMD

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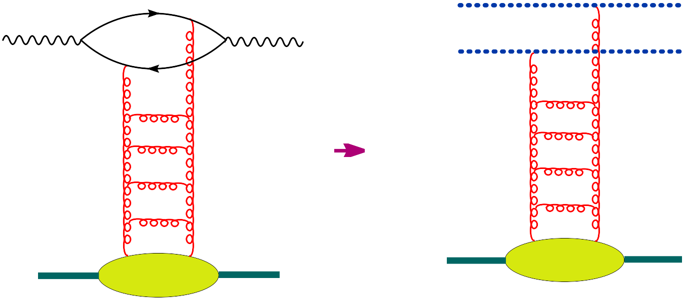
Outline

- Reminder: rapidity factorization and evolution of color dipoles
- Definitions of small-x and "moderate-x" gluon TMDs
- Method of calculation: shock-wave approach + light-cone expansion.
- One loop: real corrections and virtual corrections.
- One-loop evolution of gluon TMD
- DGLAP, Sudakov and BK limits of TMD evolution equation
- Conclusions and outlook

DIS at high energy: Wilson lines and color dipoles

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig \int_{-\infty}^{\infty} du\, n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

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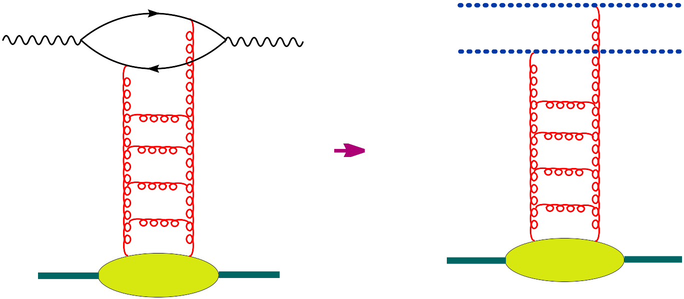

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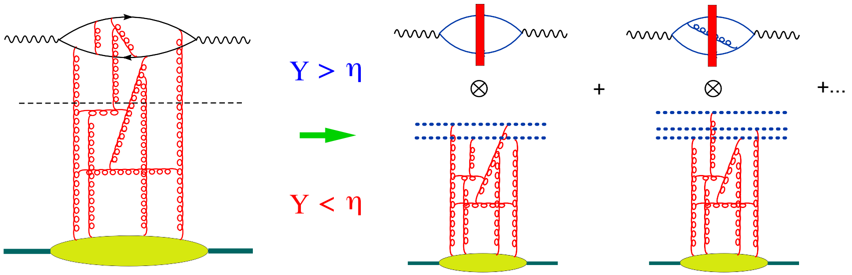
Formally, ➔ means the operator expansion in Wilson lines

Rapidity factorization



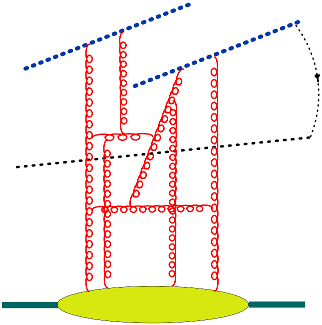
η - rapidity factorization scale

Rapidity Y > η - coefficient function (“impact factor”)
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

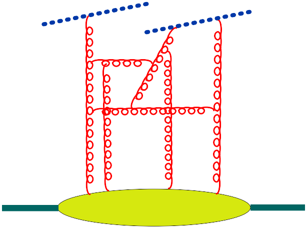
$$U_x^\eta = \text{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big], \quad A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Reminder: evolution of color dipoles at small x

To get the evolution equation for color dipoles, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

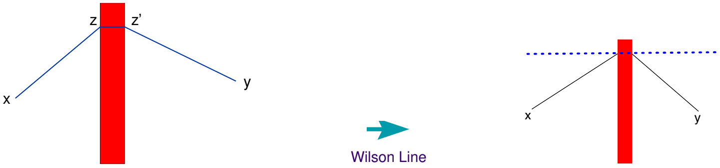


$\alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \otimes$



Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

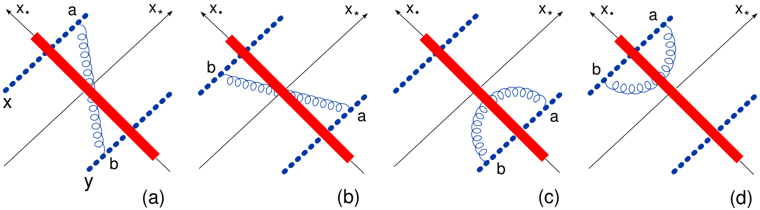


[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

Rapidity evolution of color dipoles in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear (BK equation)

$$\frac{d}{d\eta}\mathrm{Tr}\{U_{z_1}U_{z_2}^\dagger)\} = \frac{\alpha_s}{2\pi}\int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{U_{z_1}U_{z_3}^\dagger)\}\mathrm{Tr}\{U_{z_3}U_{z_2}^\dagger)\} - N_c\mathrm{Tr}\{U_{z_1}U_{z_2}^\dagger)\}]$$

At small x - Weizsacker-Williams unintegrated gluon distribution

$$\sum_X \mathrm{tr}\langle p|U\partial^i U^\dagger(z_\perp)|X\rangle\langle X|U\partial_i U^\dagger(0_\perp)\}|p\rangle$$

Rapidity factorization: each gluon has rapidity $\leq \ln x_B$.
Rewrite (later $n \equiv p_1$)

$$\begin{aligned}
\alpha_s\mathcal{D}(x_B,z_\perp) &= -\frac{\alpha_s}{2\pi(p\cdot n)x_B}\int du \sum_X \langle p|\tilde{\mathcal{F}}^a_\xi(z_\perp+un)|X\rangle\langle X|\mathcal{F}^{a\xi}(0)|p\rangle \\
\mathcal{F}^a_\xi(z_\perp+un) &\equiv [\infty n+z_\perp, un+z_\perp]^{am} n^\mu F^m_{\mu\xi}(un+z_\perp) \\
\tilde{\mathcal{F}}^a_\xi(z_\perp+un) &\equiv n^\mu F^m_{\mu\xi}(un+z_\perp)[un+z_\perp,\infty n+z_\perp]^{ma}
\end{aligned}$$

and define the “WW unintegrated gluon distribution”

$$\mathcal{D}(x_B,k_\perp) = \int d^2z_\perp\, e^{-i(k,z)_\perp}\mathcal{D}(x_B,z_\perp) \qquad x_B s \gg k_\perp^2 \gg \Lambda^2_{\mathrm{QCD}}$$

NB: $\alpha_s\mathcal{D}(x_B,z_\perp)$ is renorm-invariant.

Gluon TMD at moderate x_B

$$\mathcal{D}(x_B, k_\perp, \eta) = \int d^2 z_\perp \; e^{-i(k,z)_\perp} \mathcal{D}(x_B, z_\perp, \eta),$$

$$\alpha_s \mathcal{D}(x_B, z_\perp, \eta)$$

$$= \frac{-x_B^{-1} \alpha_s}{2\pi (p \cdot n)} \int du \; e^{-ix_B u(pn)} \sum_X \langle p | \tilde{\mathcal{F}}^a_\xi (z_\perp + un) | X \rangle \langle X | \mathcal{F}^{a\xi}(0) | p \rangle$$

There are more involved definitions with the above TMD multiplied by some Wilson-line factors but we will discuss the “primordial” TMD.

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The above TMD will have double-logarithmic contributions of the type $(\alpha_s \eta \ln x_B)^n$ while the WW distribution has only single-log terms $(\alpha_s \ln \eta)^n$ described by the BK evolution.

Some definitions

Lightcone variables:

$$k = \alpha p_1 + \beta p_2 + k_\perp \qquad \text{Sudakov variables}$$
$$x_* \equiv x_\mu p_2^\mu = \sqrt{\frac{s}{2}} x_+, \quad x_\bullet \equiv x_\mu p_1^\mu = \sqrt{\frac{s}{2}} x_-$$

Gluon operators ($x_B \equiv x_B$ for DIS and $-x_B \equiv \frac{1}{z}$ for annihilation)

$$\mathcal{F}_i^a(k_\perp, x_B) = \int d^2 z_\perp \, e^{-i(k,z)_\perp} \mathcal{F}_i^a(z_\perp, x_B),$$
$$\mathcal{F}_i^a(z_\perp, x_B) \equiv \frac{2}{s} \int dz_* \, e^{i x_B z_*} [\infty, z_*]_z^{am} F_{\bullet i}^m(z_*, z_\perp)$$

and similarly

$$\tilde{\mathcal{F}}_i^a(k_\perp, x_B) = \int d^2 z_\perp \, e^{i(k,z)_\perp} \tilde{\mathcal{F}}_i^a(z_\perp, x_B),$$
$$\tilde{\mathcal{F}}_i^a(z_\perp, x_B) \equiv \frac{2}{s} \int dz_* \, e^{-i x_B z_*} F_{\bullet i}^m(z_*, z_\perp) [z_*, \infty]_z^{ma}$$

In this talk we study gluon TMDs with Wilson lines stretching to $+\infty$ (like in SIDIS).

Double fun. interval for cross sections

$$\langle p|\tilde{\mathcal{F}}^a_i(k'_\perp, x'_B)\mathcal{F}^{ai}(k_\perp, x_B)|p\rangle \equiv \sum_X \langle p|\tilde{\mathcal{F}}^a_i(k'_\perp, x'_B)|X\rangle\langle X|\mathcal{F}^{ai}(k_\perp, x_B)|p\rangle$$
$$= -2\pi\delta(x_B - x'_B)(2\pi)^2\delta^{(2)}(k_\perp - k'_\perp)2\pi x_B\mathcal{D}(x_B = x_B, k_\perp, \eta)$$

Short-hand notation

$$\langle p|\tilde{\mathcal{O}}_1...\tilde{\mathcal{O}}_m\mathcal{O}_1...\mathcal{O}_n|p\rangle \equiv \sum_X \langle p|\tilde{T}\{\tilde{\mathcal{O}}_1...\tilde{\mathcal{O}}_m\}|X\rangle\langle X|T\{\mathcal{O}_1...\mathcal{O}_n\}|p\rangle$$

This matrix element can be represented by a double functional integral

$$\langle\tilde{\mathcal{O}}_1...\tilde{\mathcal{O}}_m\mathcal{O}_1...\mathcal{O}_n\rangle$$
$$= \int D\tilde{A}D\bar{\tilde{\psi}}D\tilde{\psi}\; e^{-iS_{\rm QCD}(\tilde{A},\tilde{\psi})}\int DAD\bar{\psi}D\psi\; e^{iS_{\rm QCD}(A,\psi)}\tilde{\mathcal{O}}_1...\tilde{\mathcal{O}}_m\mathcal{O}_1...\mathcal{O}_n$$

The boundary condition $\tilde{A}(\vec{x}, t = \infty) = A(\vec{x}, t = \infty)$ (and similarly for quark fields) reflects the sum over all intermediate states X.

Gauge invariance

Due to the boundary condition $\tilde{A}(\vec{x}, t=\infty) = A(\vec{x}, t=\infty)$ the matrix element

$$\begin{aligned}&\langle\tilde{\mathcal{F}}^a_i(z'_\perp, x'_B)[z'_\perp+\infty p_1, z_\perp+\infty p_1]\mathcal{F}^{ai}(z_\perp, x_B)\rangle\\ &= \int D\tilde{A}D\bar{\tilde{\psi}}D\tilde{\psi}\ e^{-iS_{\rm QCD}(\tilde{A},\tilde{\psi})}\int DAD\bar{\psi}D\psi\ e^{iS_{\rm QCD}(A,\psi)}\\ &\tilde{\mathcal{F}}^a_i(z'_\perp, x'_B)[z'_\perp+\infty p_1, z_\perp+\infty p_1]\mathcal{F}^{ai}(z_\perp, x_B)\end{aligned}$$

is gauge invariant

However, the gauge link $[z'_\perp+\infty p_1, z_\perp+\infty p_1]$ does not contribute at least at the one-loop level ($\gamma_{\rm cusp}$ and self-energy diagrams vanish)

We study evolution of $\tilde{\mathcal{F}}_i^{a\eta}(x_\perp, x_B)\mathcal{F}_j^{a\eta}(y_\perp, x_B)$ with respect to rapidity cutoff η

$$A^\eta_\mu(x) \;=\; \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Matrix element of $\tilde{\mathcal{F}}_i^{a}(k'_\perp, x'_B)\mathcal{F}^{ai}(k_\perp, x_B)$ at one-loop accuracy:
diagrams in the "external field" of gluons with rapidity $< \eta$.

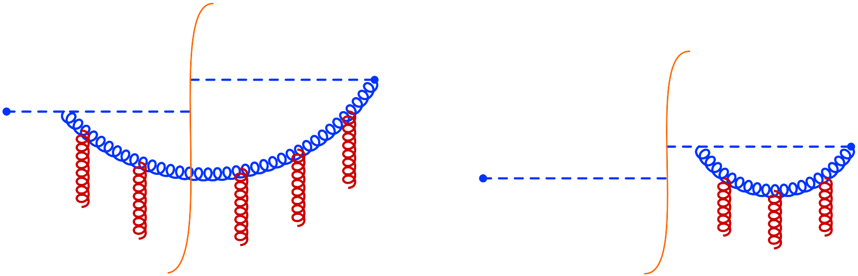
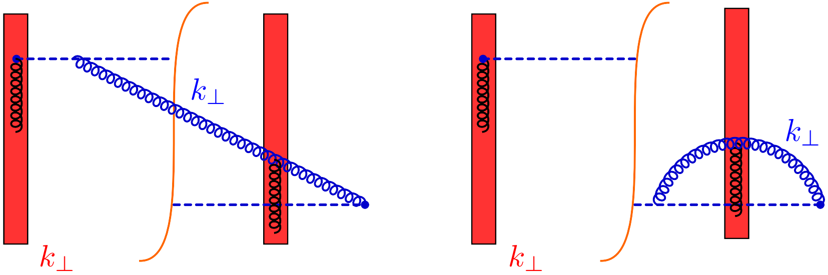


Figure : Typical diagrams for one-loop contributions to the evolution of gluon TMD.

(Fields $\tilde{\mathcal{A}}$ to the left of the cut and $\mathcal{A}$ to the right.)

Shock-wave formalism and transverse momenta

$\alpha \gg \alpha$ and $k_\perp \sim k_\perp \Rightarrow$ shock-wave external field



Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha} \Rightarrow \; x_* \sim \frac{\alpha s}{k_\perp^2}$

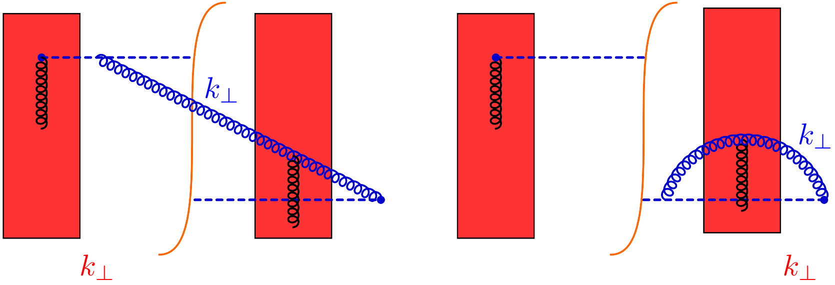
Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha} \Rightarrow \; x_* \sim \frac{\alpha s}{k_\perp^2}$

If $\alpha \gg \alpha$ and $k_\perp^2 \leq k_\perp^2 \Rightarrow x_* \gg x_*$

$\Rightarrow$ Diagrams in the shock-wave background at $k_\perp \sim k_\perp$

Problem: different transverse momenta

$\alpha \gg \alpha$ and $k_\perp \gg k_\perp$ $\Rightarrow$ the external field may be wide



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If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2$ $\Rightarrow$ $x_* \sim x_*$ $\Rightarrow$ shock-wave approximation is invalid

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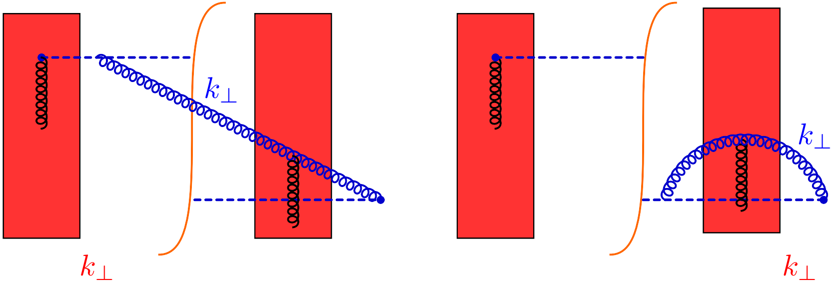

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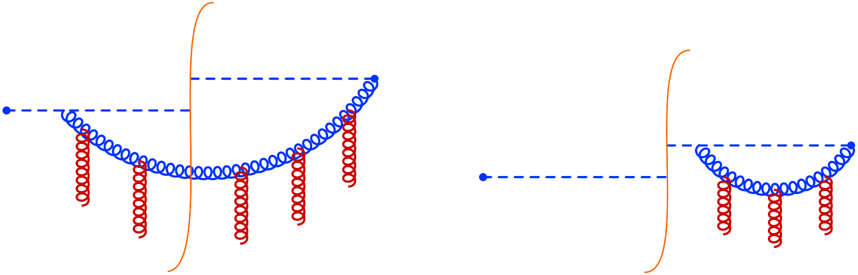
If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2$ $\Rightarrow x_* \sim x_* \Rightarrow$ shock-wave approximation is invalid

Fortunately, at $k_\perp^2 \gg k_\perp^2$ we can use another approximation

$\Rightarrow$ Light-cone expansion of propagators at $k_\perp \gg k_\perp$

Method of calculation

We calculate one-loop diagrams in the fast-field background



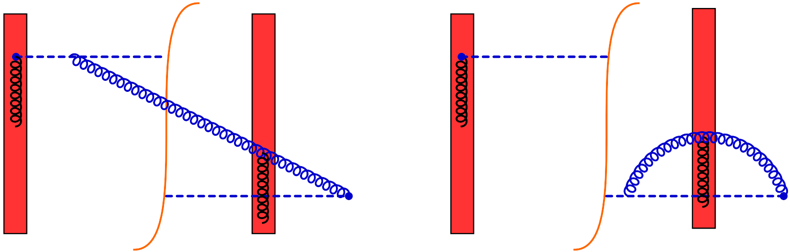
in following way:

if $k_\perp \sim k_\perp \Rightarrow$ propagators in the shock-wave background

if $k_\perp \gg k_\perp \Rightarrow$ light-cone expansion of propagators

We compute one-loop diagrams in these two cases and write down "interpolating" formulas correct both at $k_\perp \sim k_\perp$ and $k_\perp \gg k_\perp$

One-loop corrections in the shock-wave background



(a) *(b)*

Figure : Typical diagrams for one-loop evolution kernel. The shaded area denotes shock wave of background fast fields.

Shock-wave calculation

Reminder:

$$\tilde{\mathcal{F}}_i^a(z_\perp, x_B) \equiv \frac{2}{s}\int dz_* \, e^{-ix_B z_*} F^m_{\bullet i}(z_*, z_\perp)[z_*, \infty]_z^{ma}$$

At $x_B \sim 1$ $e^{-ix_B z_*}$ may be important even if shock wave is narrow.
Indeed, $x_* \sim \frac{\alpha s}{k_\perp^2} \ll x_* \sim \frac{\alpha s}{k_\perp^2} \Rightarrow$ shock-wave approximation is OK,
but $x_B \sigma_* \sim x_B \frac{\alpha s}{k_\perp^2} \sim \frac{\alpha s}{k_\perp^2} \geq 1 \Rightarrow$ we need to “look inside” the shock wave.

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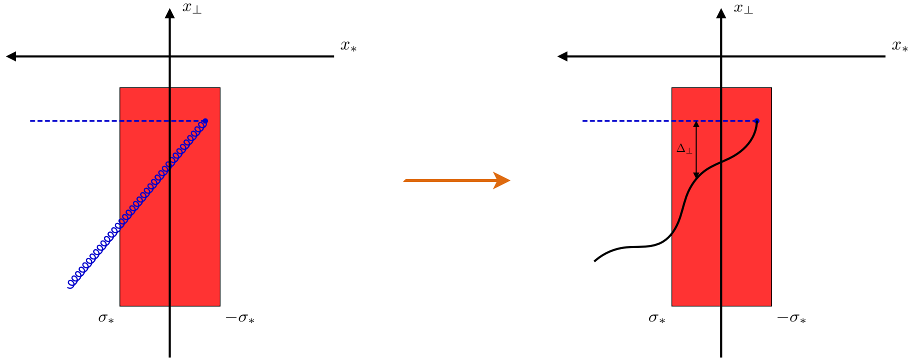
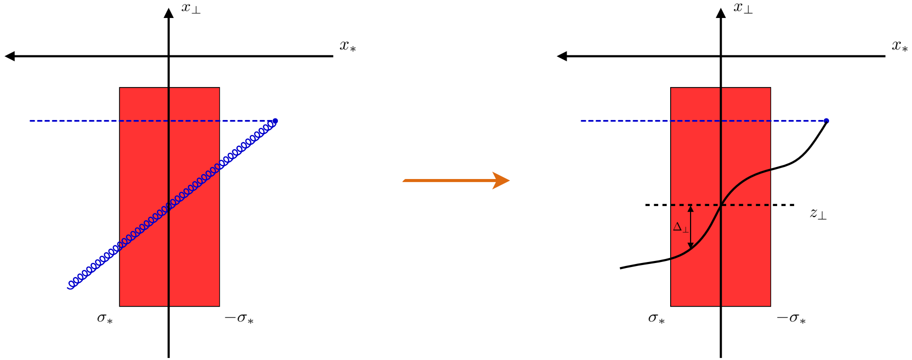
Technically, we consider small but finite shock wave:
take the external field with the support in the interval $[-\sigma_*, \sigma_*]$ (where $\sigma_* \sim \frac{\alpha s}{k_\perp^2}$), calculate diagrams with points in and out of the shock wave, and check that the σ_*-dependence cancels in the sum of “in” and “out” contributions.

Point(s) inside the shock wave: linear terms



$\Delta_\perp$ is small $\Rightarrow$ expansion of $Pe^{ig\int dx_\mu A^m u}$ around $y_\perp$ $\Rightarrow$ same operator $\mathcal{F}(y_\perp, x_B)$ $\Rightarrow$ linear evolution.

Point(s) outside the shock wave: non-linear terms



$\Delta_\perp$ is small $\Rightarrow$ expansion of $Pe^{ig\int dx_\mu A^m u}$ around $z_\perp$
$\Rightarrow$ Wilson line $U_z = [\infty_* p_1 + z_\perp, -\infty_* p_1 + z_\perp]$ in addition to U_y $\Rightarrow$ non-linear terms in the evolution equation

One-loop corrections in the shock-wave background

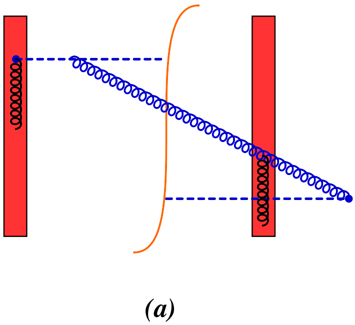


Figure : Typical diagrams for production (a) and virtual (b) contributions to the evolution kernel.

Real corrections: square of “Lipatov vertex”

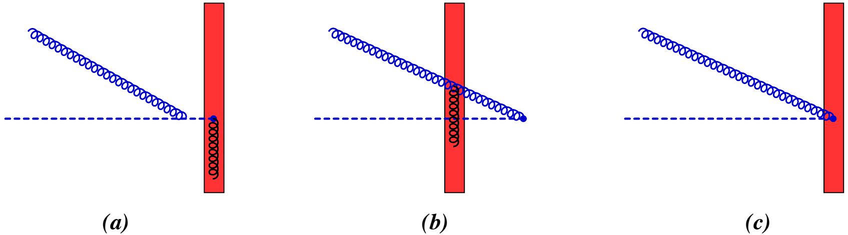


Figure : Lipatov vertex of gluon emission.

Definition

$$L^{ab}_{\mu i}(k, y_\perp, x_B) \; = \; i \lim_{k^2\to 0} k^2 \langle T\{A^a_\mu(k)\mathcal{F}^b_i(y_\perp, x_B)\}\rangle$$

Lipatov vertex in the shock-wave case

Result of calculation (in the background-Feynman gauge)

$$L^{ab}_{\mu i}(k,y_\perp,x_B) \;=\; 2ge^{-i(k,y)_\perp}\Big(\frac{p_{2\mu}}{\alpha s}-\frac{\alpha p_{1\mu}}{k_\perp^2}\Big)[\mathcal{F}_i(x_B,y_\perp)-U_i(y_\perp)]^{ab}$$
$$+\,g(k_\perp|g_{\mu i}\Big(\frac{\alpha x_B s}{\alpha x_B s+p_\perp^2}-U\frac{\alpha x_B s}{\alpha x_B s+p_\perp^2}U^\dagger\Big)+2\alpha p_{1\mu}\Big(\frac{p_i}{\alpha x_B s+p_\perp^2}-U\frac{p_i}{\alpha x_B s+p_\perp^2}U^\dagger\Big)$$
$$+\,\Big[2ix_Bp_{2\mu}\partial_iU-2i\partial^\perp_\mu Up_i+\frac{2p_{2\mu}}{\alpha s}\partial_\perp^2Up_i\Big]\frac{1}{\alpha x_B s+p_\perp^2}U^\dagger-\frac{2\alpha p_{1\mu}}{p_\perp^2}U_i|y_\perp)^{ab}$$

$U_i\equiv\mathcal{F}_i(0)=i(\partial_iU)U^\dagger$.

Schwinger's notations $(x_\perp|\mathcal{O}(\hat p_\perp,\hat{X_\perp})|y_\perp)\equiv\int d^2p\,\mathcal{O}(p_\perp,x_\perp)e^{-i(p,x-y)_\perp}$

Lipatov vertex in the light-cone case

Result of calculation (in the background-Feynman gauge)

$$L^{ab}_{\mu i}(k,y_\perp,x_B)\rangle \;=\; \frac{2ge^{-i(k,y)_\perp}}{\alpha x_B s+k_\perp^2}\mathcal{F}^{ab}_l(x_B+\frac{k_\perp^2}{\alpha s},y_\perp)$$
$$\times\Big[\frac{\alpha x_B s}{k_\perp^2}\big(\frac{k_\perp^2}{\alpha s}p_{2\mu}-\alpha p_{1\mu}\big)\delta^l_i-\delta^l_\mu k_i+\frac{\alpha x_B s g_{\mu i}k^l}{k_\perp^2+\alpha x_B s}+\frac{2\alpha k_i k^l}{k_\perp^2+\alpha x_B s}p_{1\mu}\Big]$$

NB:

$$k^\mu L^{ab}_{\mu i}(k,y_\perp,x_B) \;=\; 0$$

for both shock-wave and light-cone Lipatov vertices.

It is convenient to write Lipatov vertex in the light-like gauge $p_2^\mu A_\mu=0$ by replacement $\alpha p_1^\mu\rightarrow\alpha p_1^\mu-k^\mu \;=\; -k_\perp^\mu-\frac{k_\perp^2}{\alpha s}$

$$L^{ab}_{\mu i}(k,y_\perp,x_B)^{\rm light-like} \;=\; 2ge^{-i(k,y)_\perp}$$
$$\times\Big[\frac{k^\perp_\mu\delta^l_i}{k_\perp^2}-\frac{\delta^l_\mu k_i+\delta^l_i k^\perp_\mu-g_{\mu i}k^l}{\alpha x_B s+k_\perp^2}-\frac{k_\perp^2 g_{\mu i}k^l+2k^\perp_\mu k_i k^l}{(\alpha x_B s+k_\perp^2)^2}\Big]\mathcal{F}^{ab}_l(x_B+\frac{k_\perp^2}{\alpha s},y_\perp) \;+\; O(p_{2\mu})$$

Lipatov vertex at arbitrary momenta

"Interpolating formula" between the shock-wave and light-cone Lipatov vertices

$$L^{ab}_{\mu i}(k,y_\perp,x_B)^{\rm light-like}$$
$$= g(k_\perp|\mathcal{F}^j\big(x_B+\frac{k_\perp^2}{\alpha s}\big)\Big\{\frac{\alpha x_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha x_B s+k_\perp^2}(k_jU+Up_j)\frac{1}{\alpha x_B s+p_\perp^2}U^\dagger$$
$$-2k^\perp_\mu U\frac{g_{ij}}{\alpha x_B s+p_\perp^2}U^\dagger-2g_{\mu j}U\frac{p_i}{\alpha x_B s+p_\perp^2}U^\dagger+\frac{2k^\perp_\mu}{k_\perp^2}g_{ij}\Big\}|y_\perp)^{ab}+O(p_{2\mu})$$

This formula is actually correct (within our accuracy $\alpha_{\rm fast}\ll\alpha_{\rm slow}$) in the whole range of x_B and transverse momenta

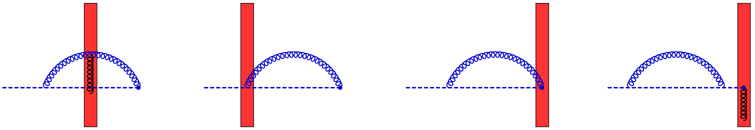


Figure : Virtual gluon corrections.

Result of the calculation (in light-like and background-Feynman gauges)

$$\langle\mathcal{F}_i^n(y_\perp,x_B)\rangle^{\text{Fig. 5}} = -ig^2f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\!{}^-\alpha}{\alpha}(y_\perp|-\frac{p^j}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)$$
$$\times(2\delta_j^k\delta_i^l-g_{ij}g^{kl})U\frac{1}{\alpha x_Bs+p_\perp^2}U^\dagger+\mathcal{F}_i(x_B)\frac{\alpha x_Bs}{p_\perp^2(\alpha x_Bs+p_\perp^2)}|y_\perp)^{kl}$$

NB: with $\alpha<\sigma$ cutoff there is no UV divergence.

Rapidity *vs* UV cutoff

Regularizing the IR divergence with a small gluon mass m^2 we obtain

$$\int_0^\sigma \frac{d\alpha}{\alpha}\int d^2p_\perp \frac{\alpha x_B s}{(p_\perp^2+m^2)(\alpha x_B s+p_\perp^2+m^2)} \simeq \frac{\pi}{2}\ln^2\frac{\sigma x_B s+m^2}{m^2} \qquad (1)$$

Simultaneous regularization of UV and rapidity divergence is a consequence of our specific choice of cutoff in rapidity.

For a different rapidity cutoff we may have the UV divergence in the remaining integrals which has to be regulated with suitable UV cutoff.

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Simultaneous regularization of UV and rapidity divergence is a consequence of our specific choice of cutoff in rapidity.
For a different rapidity cutoff we may have the UV divergence in the remaining integrals which has to be regulated with suitable UV cutoff.
We calculated

$$\int \frac{\not{d}\alpha \not{d}\beta \not{d}\beta' \not{d}^2 p_\perp}{(\beta-i\epsilon)(\beta'+x_B-i\epsilon)(\alpha\beta s-p_\perp^2-m^2+i\epsilon)(\alpha\beta' s-p_\perp^2-m^2+i\epsilon)}$$

by taking residues in the integrals over Sudakov variables β and β' and cutting the obtained integral over α from above by the cutofl by $\alpha<\sigma$

Rapidity *vs* UV cutoff

Instead, let us take the residue over α:

$$ix_B\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int d\!\!{}^-\beta d\!\!{}^-\beta'\frac{\theta(\beta)\theta(-\beta')-\theta(-\beta)\theta(\beta')}{(\beta'+x_B-i\epsilon)(\beta-i\epsilon)(\beta'-\beta)}$$

$$=\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int\frac{d\!\!{}^-\beta d\!\!{}^-\beta'}{\beta'+x_B-i\epsilon}\frac{ix_B\theta(\beta)}{(\beta-i\epsilon)(\beta'-\beta+i\epsilon)}\;=\;x_B\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int_0^\infty\frac{d\!\!{}^-\beta}{\beta(\beta+x_B)}$$

which is integral (1) with change of variable $\beta=\frac{p_\perp^2}{\alpha s}$.

Rapidity *vs* UV cutoff

Instead, let us take the residue over α:

$$ix_B \int \frac{d\!\!{}^-{}^2 p_\perp}{m^2+p_\perp^2} \int d\!\!{}^-\beta d\!\!{}^-\beta' \frac{\theta(\beta)\theta(-\beta') - \theta(-\beta)\theta(\beta')}{(\beta'+x_B-i\epsilon)(\beta-i\epsilon)(\beta'-\beta)}$$

$$= \int \frac{d\!\!{}^-{}^2 p_\perp}{m^2+p_\perp^2} \int \frac{d\!\!{}^-\beta d\!\!{}^-\beta'}{\beta'+x_B-i\epsilon} \frac{ix_B\theta(\beta)}{(\beta-i\epsilon)(\beta'-\beta+i\epsilon)} = x_B \int \frac{d\!\!{}^-{}^2 p_\perp}{m^2+p_\perp^2} \int_0^\infty \frac{d\!\!{}^-\beta}{\beta(\beta+x_B)}$$

which is integral (1) with change of variable $\beta = \frac{p_\perp^2}{\alpha s}$.

A conventional way of rewriting this integral in the framework of collinear factorization approach is

$$x_B \int \frac{d\!\!{}^-{}^2 p_\perp}{m^2+p_\perp^2} \int_0^\infty \frac{d\beta}{\beta(\beta+x_B)} = \int \frac{d\!\!{}^-{}^2 p_\perp}{m^2+p_\perp^2} \int_0^1 \frac{dz}{1-z}$$

where $z = \frac{x_B}{x_B+\beta}$ is a fraction of momentum $(x_B+\beta)p_2$ of “incoming gluon” (described by $\mathcal{F}_i$ in our formalism) carried by the emitted “particle” with fraction $x_B p_2$.

If we cut the rapidity of the emitted gluon by cutoff in fraction of momentum z, we would still have the UV divergent expression which must be regulated by a suitable UV cutoff.

Evoltuion equation for the gluon TMD operator

A. Tarasov and I.B.

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\big(\tilde{\mathcal{F}}_i^a(x_\perp,x_B)\mathcal{F}_j^a(y_\perp,x_B)\big)^{\ln\sigma}\\
&= -\alpha_s\int d\!\!\!{}^-{}^2k_\perp\ \mathrm{Tr}\{\tilde{L}_i^{\ \mu}(k,x_\perp,x_B)^{\mathrm{light-like}}L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}\\
&-\alpha_s\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(x_\perp,x_B)(y_\perp| -\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta_m^k\delta_j^l-g_{jm}g^{kl})U\frac{1}{\sigma x_Bs+p_\perp^2}U^\dagger\\
&\qquad\qquad +\mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|y_\perp)\\
&+(x_\perp|\tilde{U}\frac{1}{\sigma x_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta_i^k\delta_m^l-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(x_B)\frac{p^m}{p_\perp^2}\\
&\qquad\qquad +\tilde{\mathcal{F}}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}\ +\ O(\alpha_s^2)
\end{aligned}
$$

This expression is UV and IR convergent.
It describes the rapidity evolution of gluon TMD operator in for any x_B and transverse momenta!

Evoltuion equation for the gluon TMD

$$\frac{d}{d\ln\sigma}\langle p|\big(\tilde{\mathcal{F}}_i^a(x_\perp,x_B)\mathcal{F}_j^a(y_\perp,x_B)\big)^{\ln\sigma}|p\rangle$$

$$= -\alpha_s\int d\!\!\!{}^-{}^2k_\perp\ \langle p|\mathrm{Tr}\{\tilde{L}_i^{\ \mu}(k,x_\perp,x_B)^{\mathrm{light-like}}\theta\Big(1-x_B-\frac{k_\perp^2}{\alpha s}\Big)L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}|p\rangle$$

$$-\ \alpha_s\langle p|\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(x_\perp,x_B)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta_m^k\delta_j^l-g_{jm}g^{kl})U\frac{1}{\sigma x_Bs+p_\perp^2}U^\dagger$$

$$+\ \mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|y_\perp)$$

$$+\ (x_\perp|\tilde{U}\frac{1}{\sigma x_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta_i^k\delta_m^l-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(x_B)\frac{p^m}{p_\perp^2}$$

$$+\ \tilde{\mathcal{F}}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}|p\rangle\ +\ O(\alpha_s^2)$$

The factor $\theta\big(1-x_B-\frac{k_\perp^2}{\alpha s}\big)$ reflects kinematical restriction that the fraction of initial proton's momentum carried by produced gluon should be smaller than $1-x_B$

Light-cone limit

$$\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma} = \frac{\alpha_s}{\pi}N_c\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\int_0^\infty d\beta\,\Big\{\theta(1-x_B-\beta)$$
$$\times\Big[\frac{1}{\beta}-\frac{2x_B}{(x_B+\beta)^2}+\frac{x_B^2}{(x_B+\beta)^3}-\frac{x_B^3}{(x_B+\beta)^4}\Big]\langle p|\tilde{\mathcal{F}}_i^n(x_B+\beta,x_\perp)$$
$$\times\,\mathcal{F}^{ni}(x_B+\beta,x_\perp)|p\rangle^{\ln\sigma'}-\frac{x_B}{\beta(x_B+\beta)}\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma'}\Big\}$$

In the LLA the cutoff in σ $\Leftrightarrow$ cutoff in transverse momenta

$$\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{k_\perp^2<\mu^2} = \frac{\alpha_s}{\pi}N_c\int_0^\infty d\beta\int_{\frac{\mu'^2}{\beta s}}^{\frac{\mu^2}{\beta s}}\frac{d\alpha}{\alpha}\,\{\text{same}\}$$

$\Rightarrow$ DGLAP equation $\Rightarrow$ $(z'\equiv\frac{x_B}{x_B+\beta})$ DGLAP kernel

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(x_B,0_\perp,\eta) = \frac{\alpha_s}{\pi}N_c\int_{x_B}^1\frac{dz'}{z'}\Big[\Big(\frac{1}{1-z'}\Big)_+ +\frac{1}{z'}-2+z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{x_B}{z'},0_\perp,\eta\Big)$$

Low-x case: BK evolution of the WW distribution

Low-x regime: $x_B = 0$ + characteristic transverse momenta $p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$
$\Rightarrow$ in the whole range of evolution ($1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$) we have $\frac{p_\perp^2}{\sigma s} \ll 1 \Rightarrow$ the kinematical constraint $\theta\big(1 - \frac{k_\perp^2}{\alpha s}\big)$ can be omitted

$\Rightarrow$ non-linear evolution equation

$$\frac{d}{d\eta}\tilde{U}_i^a(z_1)U_j^a(z_2)$$
$$= -\frac{g^2}{8\pi^3}\mathrm{Tr}\{(-i\partial_i^{z_1} + \tilde{U}_i^{z_1})\Big[\int d^2z_3(\tilde{U}_{z_1}\tilde{U}_{z_3}^\dagger - 1)\frac{z_{12}^2}{z_{13}^2 z_{23}^2}(U_{z_3}U_{z_2}^\dagger - 1)\Big](i\overleftarrow{\partial}_j^{z_2} + U_j^{z_2})\}$$

where $\eta \equiv \ln\sigma$ and $\frac{z_{12}^2}{z_{13}^2 z_{23}^2}$ is the BK kernel

This eqn holds true also at small x_B up to $x_B \sim \frac{(x-y)_\perp^{-2}}{s}$ since in the whole range of evolution $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ one can neglect $\sigma x_B s$ in comparison to $p_\perp^2$ in the denominators $(p_\perp^2 + \sigma x_B s) \Leftrightarrow$ effectively $x_B = 0$.

Sudakov double logs

Sudakov limit: $x_B \equiv x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim$ few GeV.

One can show that the non-linear terms are power suppressed $\Rightarrow$

$$\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle$$
$$= 4\alpha_s N_c \int \frac{d^2 p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(x_B+\frac{p_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^a\Big(x_B+\frac{p_\perp^2}{\sigma s},y_\perp\Big)|p\rangle$$
$$- \frac{\sigma x_B s}{\sigma x_B s + p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

Double-log region: $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s \gg p_\perp^2 \gg (x-y)_\perp^{-2}$

$$\Rightarrow \frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) = -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2 p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big]$$

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$$- \frac{\sigma x_B s}{\sigma x_B s + p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

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$\Rightarrow$ Sudakov double logs

$$\mathcal{D}(x_B,k_\perp,\ln\sigma) \sim \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\ln^2\frac{\sigma s}{k_\perp^2}\Big\}\mathcal{D}(x_B,k_\perp,\ln\frac{k_\perp^2}{s})$$

Conclusions and outlook

1. Conclusions
   - The evolution equation for gluon TMD at any x_B and transverse momenta.
   - Interpolates between linear DGLAP and Sudakov limits and the non-linear low-x BK regime
2. Outlook
   - The evolution equation for gluon TMD with Wilson lines to $-\infty$ (work in progress)
   - Transition between collinear factorization and k_T factorization.