TMDs in the Saturation Picture: Quasi-Classical Approximation and Quantum Evolution

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based on work with Matt Sievert

Outline

- Introduction and goals:
  - Classical fields, Wilson lines
  - Non-linear small-x evolution
- TMDs in the quasi-classical approximation at large-x:
  - Unpolarized quark distribution, Sivers function, Boer-Mulders distribution
  - Mixing between the nuclear and nucleon TMDs
- TMD evolution
  - Large-x case
  - Small-x: quark TMDs of an unpolarized target
  - Small-x TMDs of a polarized target: an outlook

Introduction:
calculations in saturation physics

Two-step prescription

To calculate observables in the saturation picture one has to follow the two-step procedure:

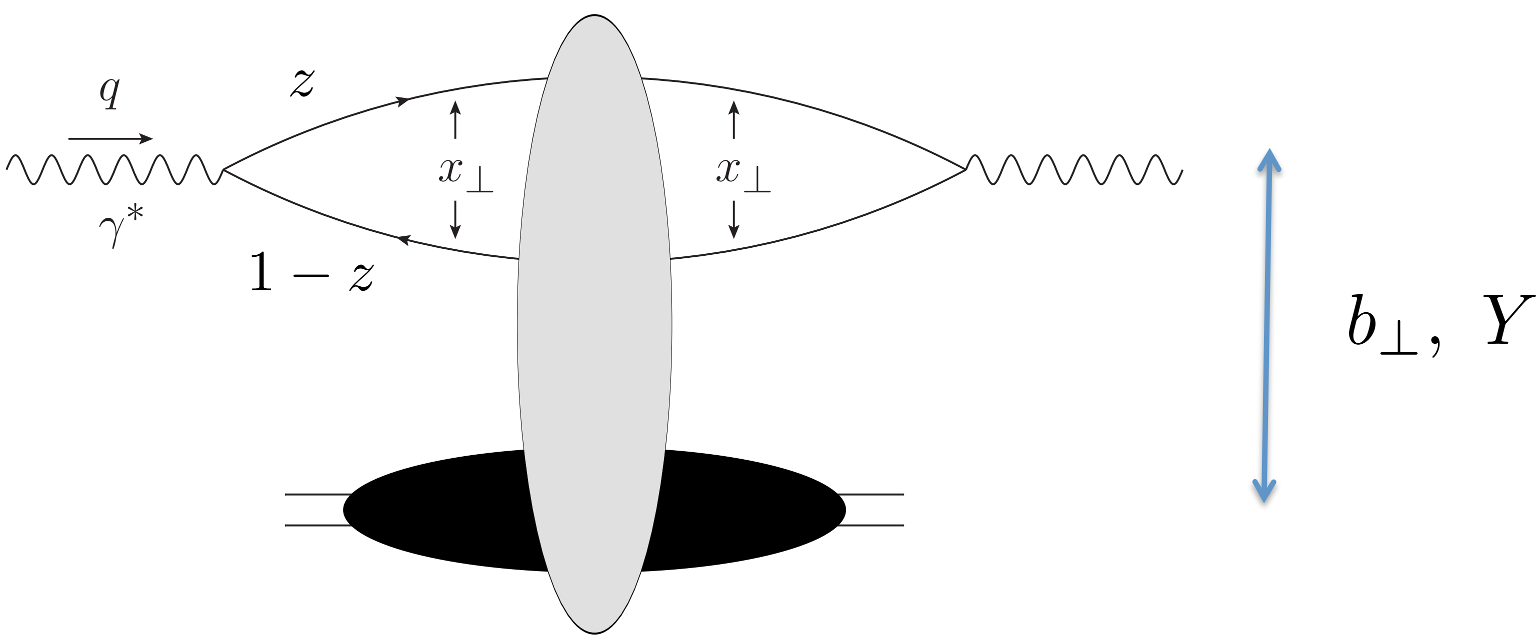
- Calculate the observable in the classical approximation.
- Include nonlinear small-x evolution corrections (BK/JIMWLK), introducing energy-dependence.
- (To compare with experiment, need to at least fix the scale of the running coupling, NLO corrections, etc.)

DIS: Quasi-Classics

Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude N:

$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\pi} \, d^2 b_\perp \int_0^1 \frac{dz}{z\,(1-z)} \, |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2 \; N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



Dipole Amplitude

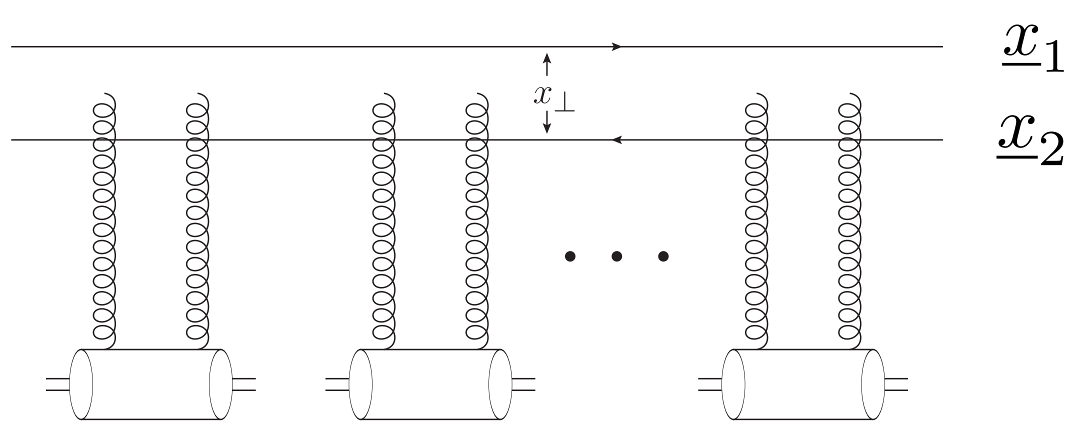
- The quark dipole amplitude is defined by

$$N(\underline{x}_1, \underline{x}_2) = 1 - \frac{1}{N_c} \left\langle \mathrm{tr} \left[V(\underline{x}_1)\, V^\dagger(\underline{x}_2) \right] \right\rangle$$

- Here we use the Wilson lines along the light-cone direction

$$V(\underline{x}) = \mathrm{P} \exp \left[i\, g \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, x^- = 0, \underline{x}) \right]$$

- In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:

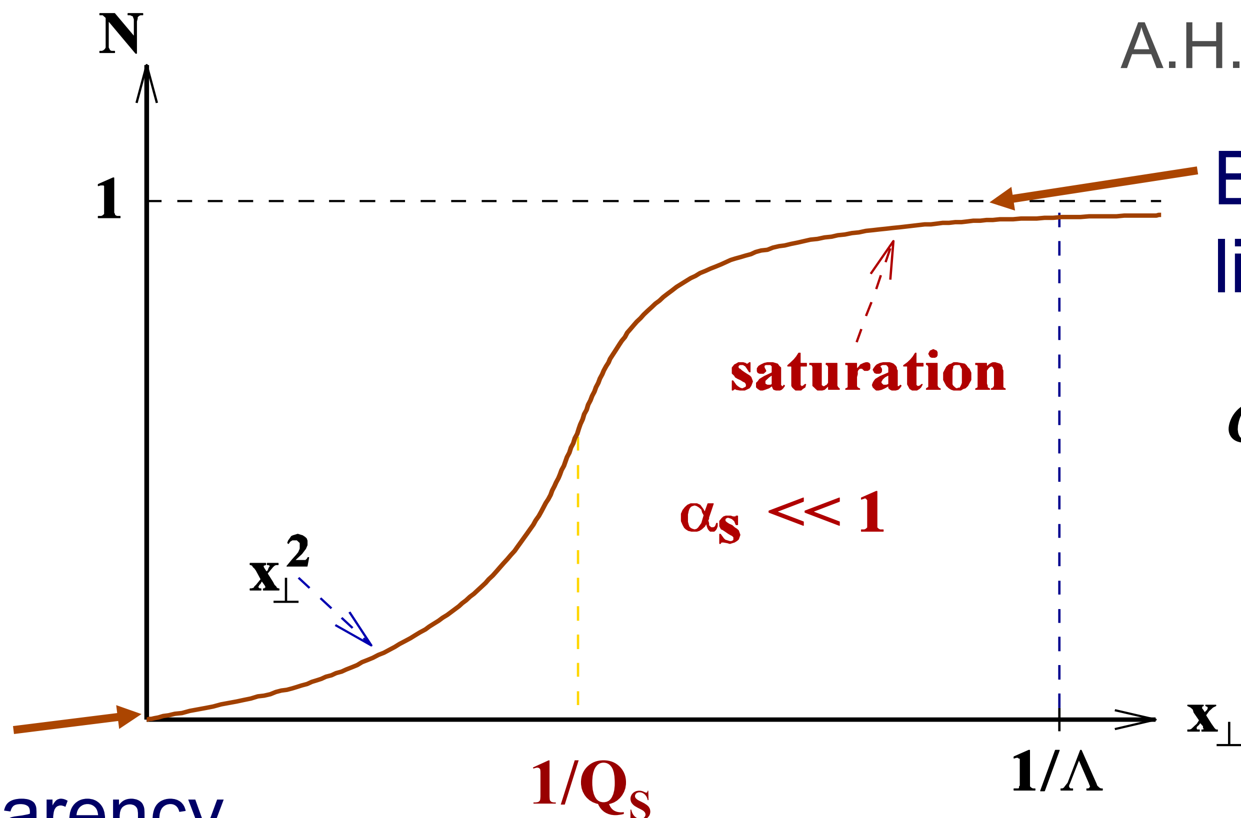


DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

$$N(x_\perp, Y) = 1 - \exp\left[- \frac{x_\perp^2 \, Q_s^2}{4} \, \ln \frac{1}{x_\perp \, \Lambda} \right]$$

A.H. Mueller, '90



Black disk limit,

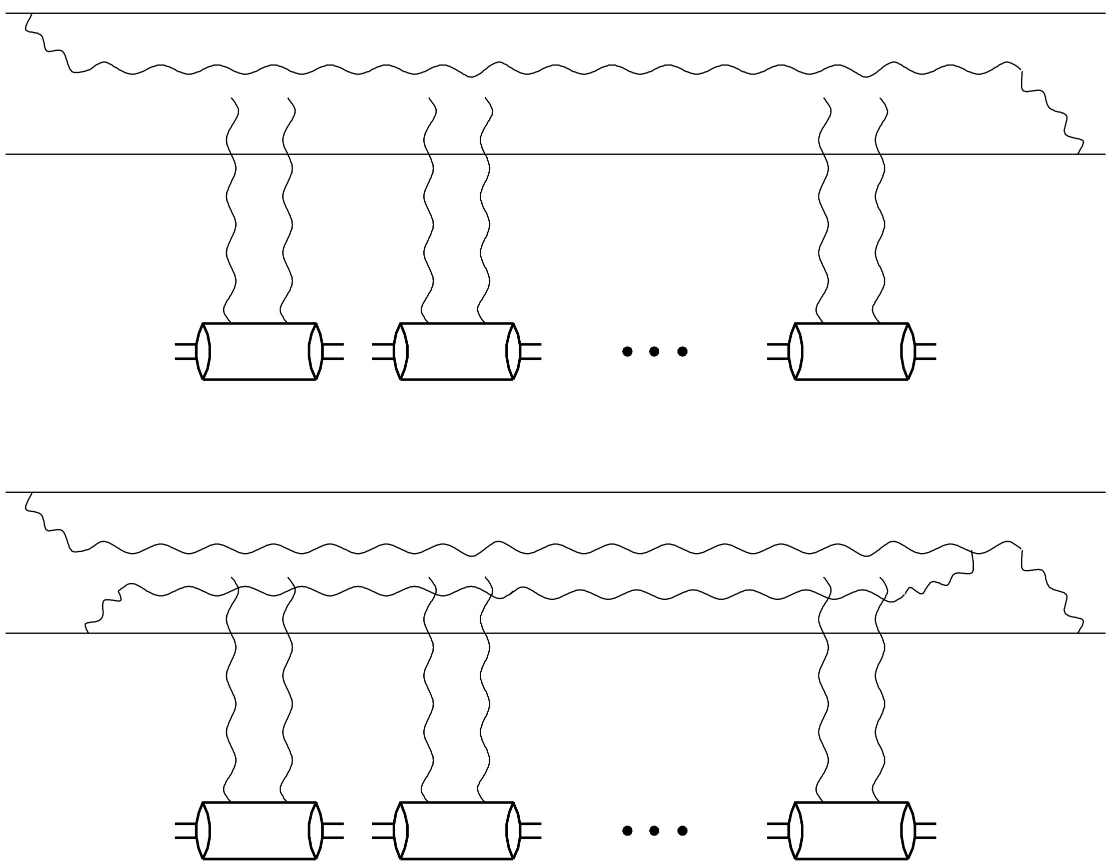
$$\sigma_{tot} < 2\pi R^2$$

Color transparency

DIS: Small-x Evolution

Dipole Amplitude

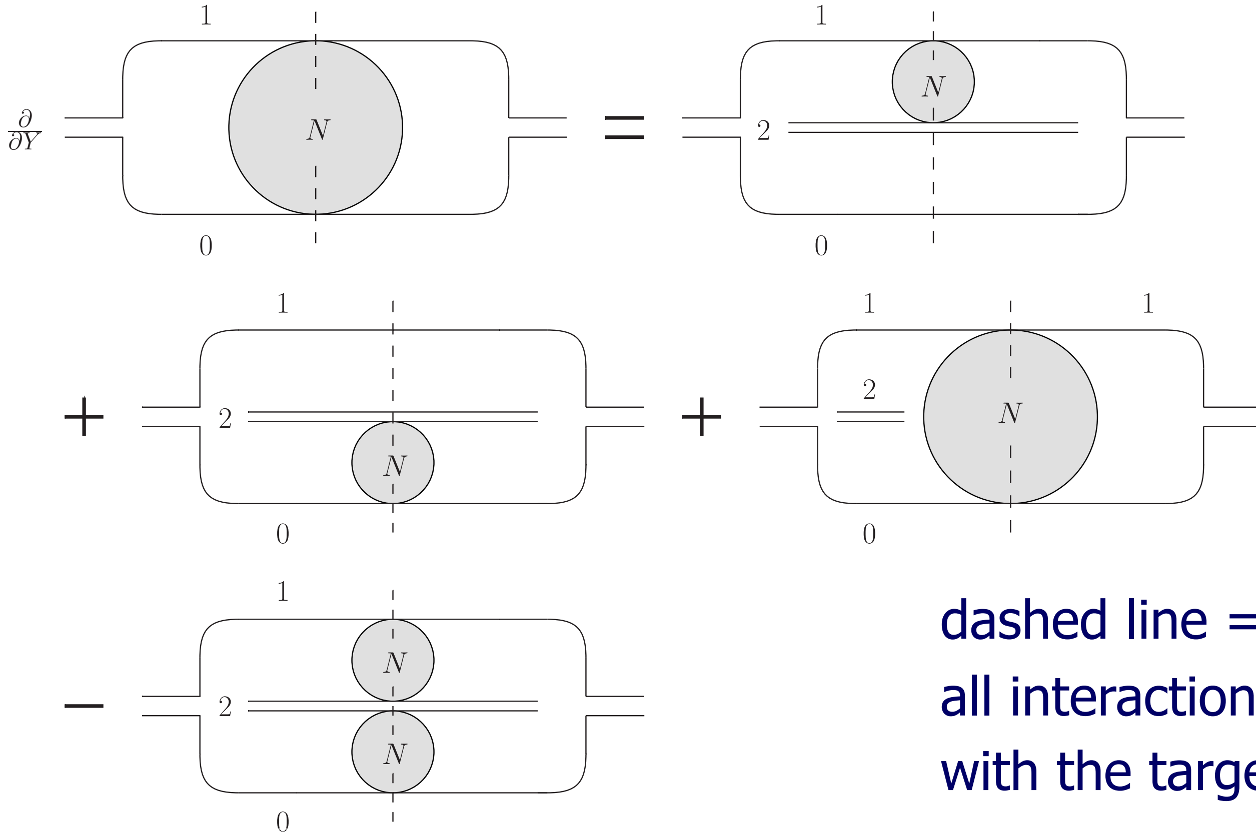
- The energy dependence comes in through nonlinear small-x BK/JIMWLK evolution, which resums the long-lived s-channel gluon corrections:



$$\alpha_s \ln\frac{1}{x} \sim \alpha_s Y \sim 1$$

Nonlinear evolution at large N_c

Here N=1-S (the Im part of the forward scattering amplitude)



dashed line =
all interactions
with the target

$$\partial_Y N_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{21}^2} \left[N_{\mathbf{x}_0,\mathbf{x}_2}(Y) + N_{\mathbf{x}_2,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, N_{\mathbf{x}_2,\mathbf{x}_1}(Y) \right]$$

Balitsky '96, Yu.K. '99

TMDs in the Quasi-Classical Approximation at Large-x

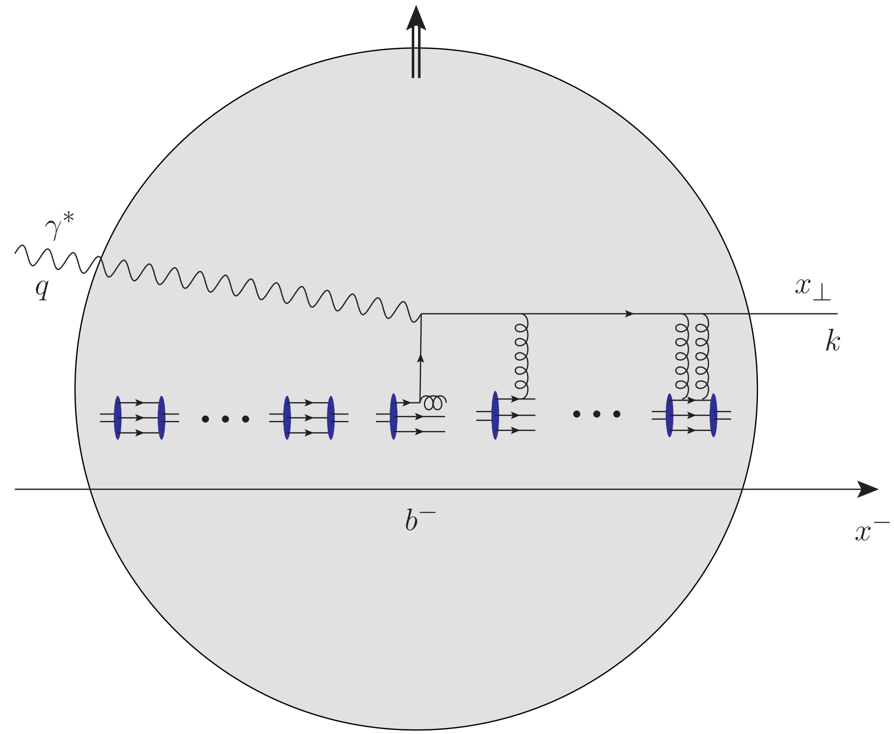
Yu.K., M. Sievert, arXiv:1310.5028 [hep-ph]
arXiv:1505.01176 [hep-ph]

Quark Production in SIDIS

- Start with inclusive classical quark production cross section in SIDIS.

- The kinematics is standard:

$$s \sim Q^2 \gg \perp^2$$



- The result is

Wigner distribution

$$\frac{d\sigma^{\gamma^*+A\to q+X}}{d^2k\,dy} = A \int \frac{dp^+\,d^2p\,db^-}{2(2\pi)^3} \int d^2x\,d^2y\; W\left(p, b^-, \frac{\underline{x}+\underline{y}}{2}\right) \times \int \frac{d^2k'}{(2\pi)^2}\, e^{-i\,(\underline{k}-\underline{k}')\cdot(\underline{x}-\underline{y})}\, \frac{d\hat{\sigma}^{\gamma^*+N\to q+X}}{d^2k'\,dy}(p,q)\, D_{\underline{x}\,\underline{y}}[+\infty, b^-]$$

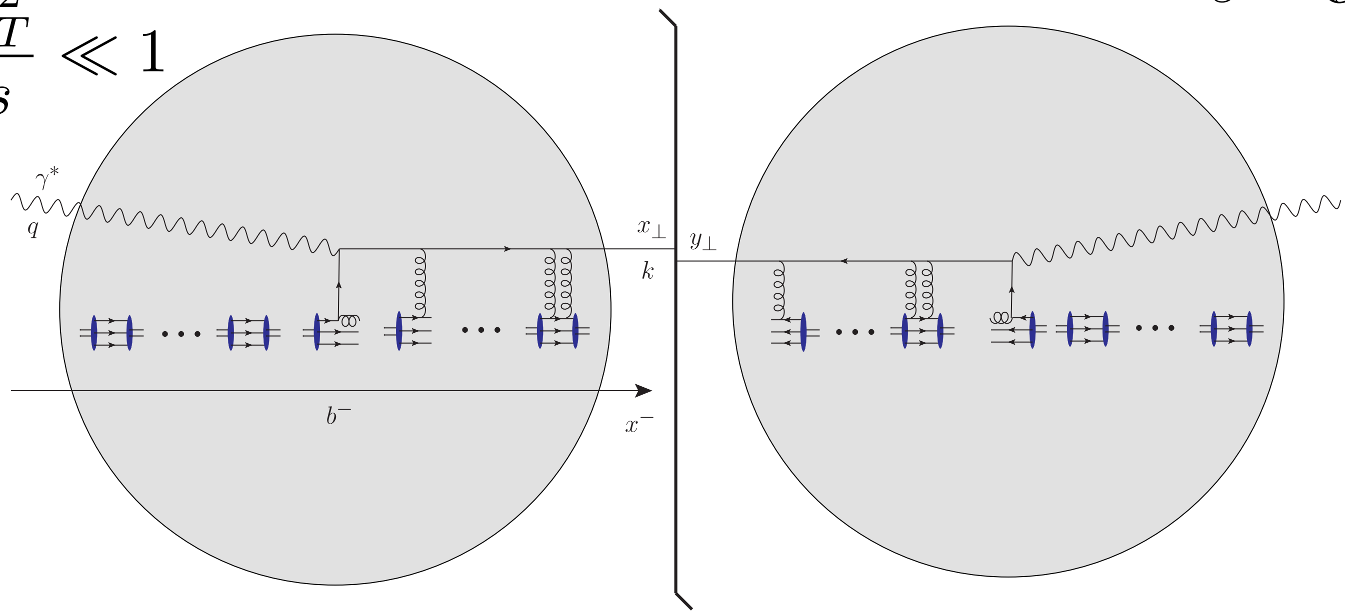
LO cross sect **Wilson lines**

Quark Production

Note: effective x

$$x_{eff} \approx \frac{k_T^2}{s} \ll 1$$

$$s \sim Q^2 \gg \perp^2$$



Wigner distribution

$$\frac{d\sigma^{\gamma^*+A\to q+X}}{d^2k\,dy} = A\int \frac{dp^+\,d^2p\,db^-}{2(2\pi)^3}\int d^2x\,d^2y\;W\!\left(p,b^-,\frac{\underline{x}+\underline{y}}{2}\right) \times \int \frac{d^2k'}{(2\pi)^2}\,e^{-i\,(\underline{k}-\underline{k}')\cdot(\underline{x}-\underline{y})}\,\frac{d\hat{\sigma}^{\gamma^*+N\to q+X}}{d^2k'\,dy}(p,q)\;D_{\underline{x}\,\underline{y}}[+\infty,b^-]$$

LO cross sect **Wilson lines**

Wilson lines

- Here

$$D_{\underline{x}\,\underline{y}}[+\infty, b^-] = \left\langle \frac{1}{N_c} \,\mathrm{Tr}\left[V_{\underline{x}}[+\infty, b^-]\, V_{\underline{y}}^{\dagger}[+\infty, b^-] \right] \right\rangle$$

is the quark dipole scattering S-matrix with

$$V_{\underline{x}}[b^- , a^-] \equiv \mathcal{P} \,\exp\left[\frac{ig}{2} \int\limits_{a^-}^{b^-} dx^- A^+(x^+ = 0, x^-, \underline{x}) \right]$$

denoting Wilson lines. This is the standard ‘staple’ (in a gauge where the link at infinity does not contribute).

- The leading contribution to D_{xy} is x<->y symmetric and will be denoted S_{xy}.

Definitions

- Quark TMDs are defined through the correlator

$$\Phi_{ij}(x,\underline{k};P,S) \equiv \int \frac{dx^- \, d^2x_\perp}{2(2\,\pi)^3} \, e^{i\left(\frac{1}{2}\,x\,P^+\,x^- - \underline{x}\cdot\underline{k}\right)} \, \langle P,S|\bar{\psi}_j(0)\,\mathcal{U}\,\psi_i(x^+ = 0, x^-, \underline{x})|P,S\rangle$$

with the gauge links

$$\mathcal{U}^{SIDIS} = V_{\underline{0}}^\dagger[+\infty\,,\,0]\,V_{\underline{x}}[+\infty\,,\,x^-]$$
$$\mathcal{U}^{DY} = V_{\underline{0}}[0\,,\,-\infty]\,V_{\underline{x}}^\dagger[x^-\,,\,-\infty]$$

- The correlator is decomposed in terms of quark TMDs as

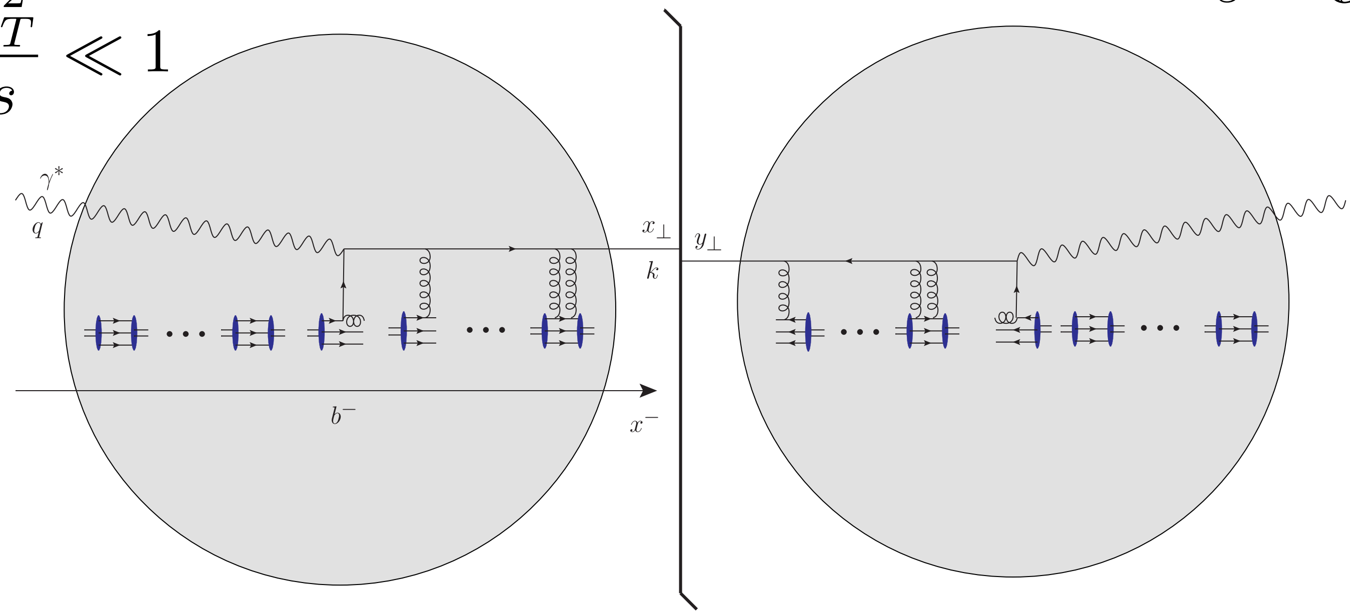
$$\Phi_{ij}(x,\underline{k};P,S) = \frac{M}{2\,P^+}\left[f_1(x,k_T)\,\frac{P\cdot\gamma}{M} + \frac{1}{M^2}\,f_{1\,T}^\perp(x,k_T)\,\epsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\,P^\nu\,k_\perp^\rho\,S_\perp^\sigma - \frac{1}{M}\,g_{1s}(x,\underline{k})\,P\cdot\gamma\,\gamma^5\right.$$
$$\left. -\frac{1}{M}\,h_{1T}(x,k_T)\,i\,\sigma_{\mu\nu}\,\gamma^5\,S_\perp^\mu\,P^\nu - \frac{1}{M^2}\,h_{1s}^\perp(x,\underline{k})\,i\,\sigma_{\mu\nu}\,\gamma^5\,k_\perp^\mu\,P^\nu + h_1^\perp(x,k_T)\,\sigma_{\mu\nu}\,\frac{k_\perp^\mu\,P^\nu}{M^2}\right]_{ij}$$

Quark TMDs of a Nucleus

Note: effective x

$$x_{eff} \approx \frac{k_T^2}{s} \ll 1$$

$$s \sim Q^2 \gg \perp^2$$



Nuclear correlator

$$\Phi^A_{\alpha\beta}(x, \underline{k}; P, S) = A \frac{g_{+-}}{(2\pi)^5} \sum_\sigma \int d^{2+}p\, d^{2-}b\, d^2r\, d^2k'\, e^{-i(\underline{k}-\underline{k}'-\hat{x}\underline{p})\cdot\underline{r}}$$

$$\times W_\sigma(p, b; P, S)\, \phi^N_{\alpha\beta}(\hat{x}, \underline{k}'; p, \sigma)\, S^{[\infty^-, b^-]}_{(r_T, b_T)}$$

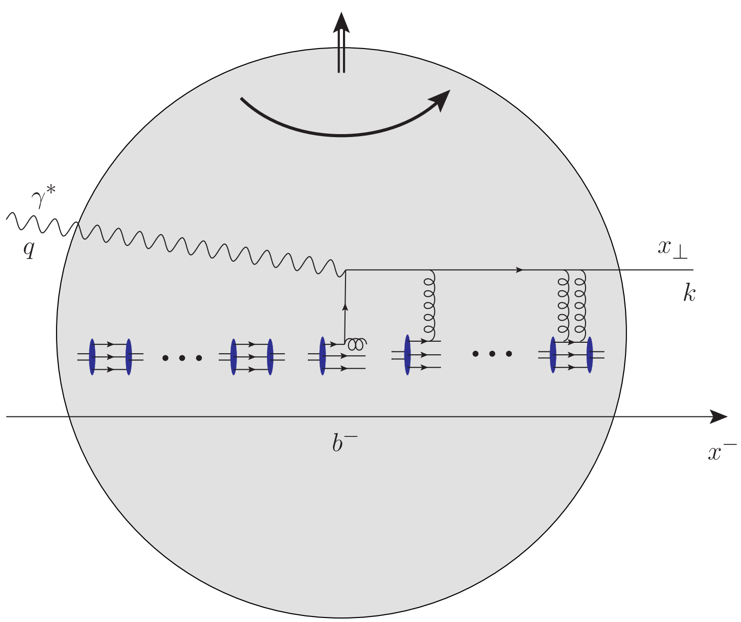
Wigner distribution   Nucleon correlator   Wilson lines

Quasi-Classical Sivers Function

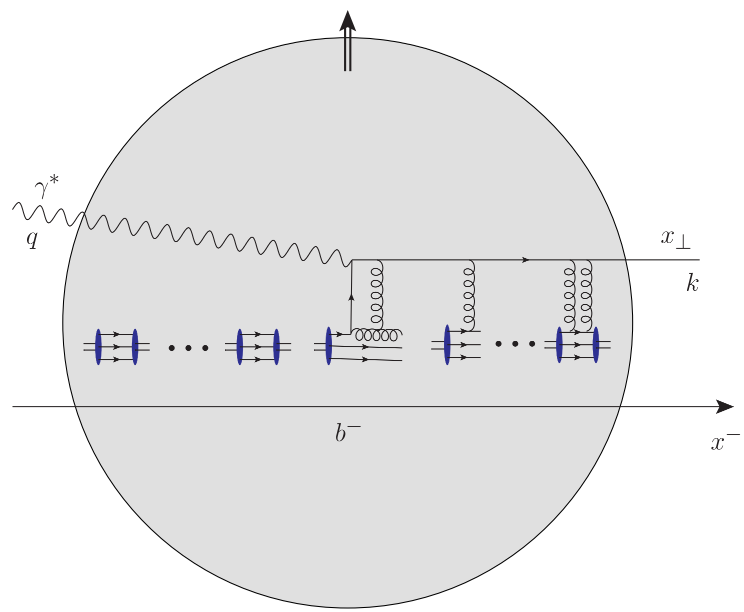
Yu.K., M. Sievert, arXiv:1310.5028 [hep-ph]

Quasi-Classical STSA in SIDIS

- To generate STSA we need spin-dependence and a complex phase. They may come from three sources:
  - Sivers functions of the (polarized) nucleons: transversity channel
  - Orbital rotation due to OAM (just gives real OAM-dependence)
  - Going beyond classical approximation in D_{xy} by including extra rescatterings per nucleon (the odderon): this gives a phase, but is A-suppressed. Hence we will drop it.



OAM Channel



Transversity Channel

Quasi-Classical STSA in SIDIS

- When the dust settles one gets

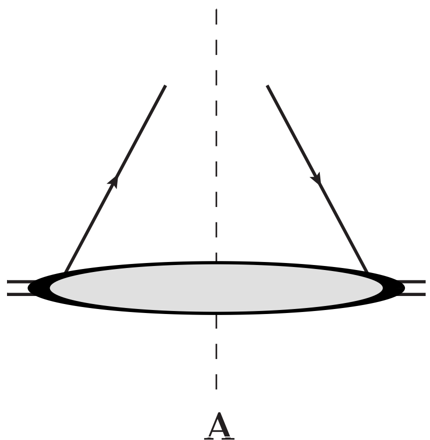
$$\hat{z}\cdot(\underline{J}\times\underline{k})\, f_{1T}^{\perp A}(\bar{x},k_T) = M_A \int \frac{dp^+\, d^2p\, db^-}{2(2\pi)^3}\, d^2x\, d^2y\, \frac{d^2k'}{(2\pi)^2}\, e^{-i\,(\underline{k}-\underline{k}')\cdot(\underline{x}-\underline{y})}$$
$$\times \left\{ i\, x\, \underline{p}\cdot(\underline{x}-\underline{y})\, A\, W_{unp}^{OAM}\left(p^+, \underline{p}, b^-, \frac{\underline{x}+\underline{y}}{2}\right) f_1^N(x, k'_T) \right.$$
$$\left. + \frac{1}{m_N}\, \hat{z}\cdot\left(\underline{S}\times\underline{k}'\right)\, W_{trans}^{symm}\left(p^+, \underline{p}, b^-, \frac{\underline{x}+\underline{y}}{2}\right) f_{1T}^{\perp N}(x, k'_T) \right\} S_{\underline{xy}}[+\infty, b^-]$$

- J = L+S = total spin of the nucleus (analogue of proton spin)
  L = net OAM of the nucleons (analogue of quark and gluon OAM)
  S = net spin of all nucleons (analogue of net spin of quarks and gluons)
- S_{xy} = dipole rescattering S-matrix
- W = Wigner distributions of nucleons, unp = unpolarized (may have <S>=0), trans = transversity, with

$$W^{\binom{symm}{OAM}}(p,b) \equiv \tfrac{1}{2}\left[W(p,b) \pm (\underline{p}\to-\underline{p})\right]$$

LO TMDs

- f_1^N and $f_{1T}^{\perp N}$ are the LO unpolarized quark TMD (in a nucleon) and the "nucleon" Sivers function:



- They can be arbitrary non-perturbative objects: we can always "run" them through the quasi-classical "grinder". ;)
- Resulting quasi-classical Sivers function can be used as initial condition for evolution equations (e.g. CSS for s ~Q^2 or small-x evolution if s>>Q^2).

OAM and Transversity channels

- The final formula is a sum of the contributions of the OAM and Transversity (Sivers function density) channels:

$$\hat{z}\cdot(\underline{J}\times\underline{k})\, f_{1T}^{\perp A}(\bar{x}, k_T) = M_A \int \frac{dp^+\, d^2p\, db^-}{2(2\pi)^3}\, d^2x\, d^2y\, \frac{d^2k'}{(2\pi)^2}\, e^{-i\,(\underline{k}-\underline{k}')\cdot(\underline{x}-\underline{y})}$$

$$\times \left\{ i\, x\, \underline{p}\cdot(\underline{x}-\underline{y})\, A\, W_{unp}^{OAM}\left(p^+, \underline{p}, b^-, \frac{\underline{x}+\underline{y}}{2}\right) f_1^N(x, k'_T) \right.$$

OAM channel

$$\left. + \frac{1}{m_N}\, \hat{z}\cdot\left(\underline{S}\times\underline{k}'\right)\, W_{trans}^{symm}\left(p^+, \underline{p}, b^-, \frac{\underline{x}+\underline{y}}{2}\right) f_{1T}^{\perp N}(x, k'_T) \right\} S_{\underline{x}\underline{y}}[+\infty, b^-]$$

Transversity channel

- The non-perurbative input comes though the Wigner functions and LO TMDs: the dipole scattering is perturbative due to $Q_s >> \Lambda_{QCD}$.

Rigid Rotator Model

- To get the feel for what the result is like, our main formula can be evaluated using a rigid-rotator nucleus with simple (powers of k_T) models of LO TMDs.

- The result is (for k_T not much larger than Q_s)

$$f_{1T}^{\perp A}(\bar{x}, k_T) = \frac{m_N\, N_c}{2\pi\, \alpha_s\, C_F}\, \frac{1}{\beta + \frac{8}{5}\, p_{max}\, R}\, \frac{1}{k_T^2}$$

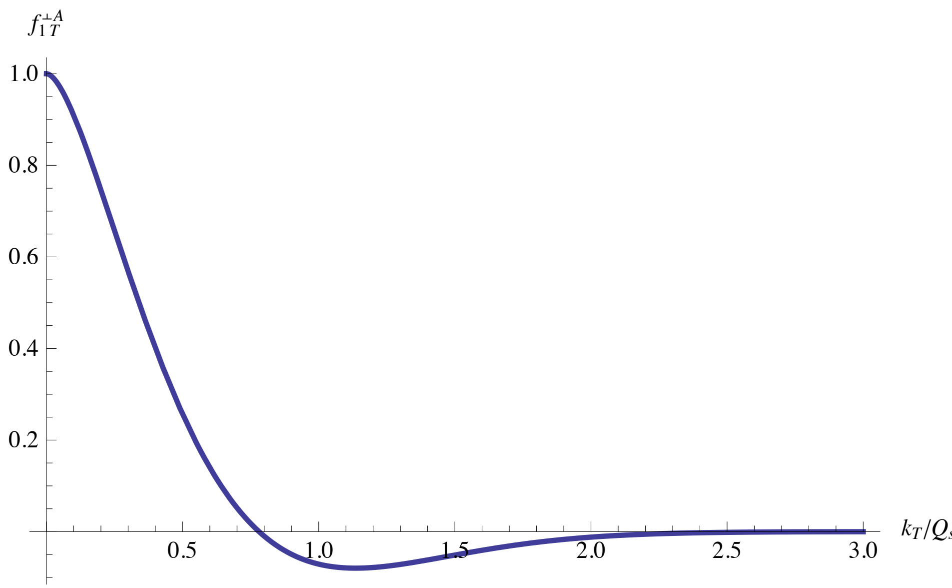
$$\times \int d^2 b \left\{ 4\, \bar{x}\, p_{max}(\underline{b})\, C_1 \left[e^{-k_T^2/Q_s^2(\underline{b})} + 2\, \frac{k_T^2}{Q_s^2(\underline{b})}\, Ei \left(- \frac{k_T^2}{Q_s^2(\underline{b})} \right) \right] + \alpha_s \beta\, m_N\, C_2\, e^{-k_T^2/Q_s^2(\underline{b})} \right\}$$

OAM channel **Transversity channel**

- Note a “new” functional form for Sivers function due to the OAM channel (not a Gaussian).

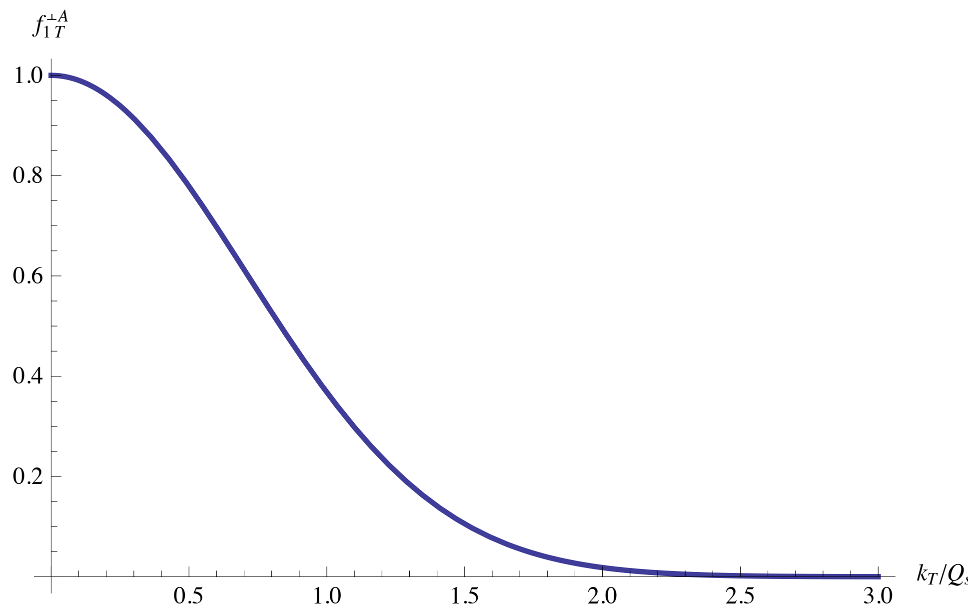
Rigid Rotator Model

- The OAM contribution in this model (!) calculation is shown below (arbitrary units). It changes sign as a function of k_T.



Rigid Rotator Model

- The transversity (Sivers density) contribution comes out to be a simple Gaussian (in this calculation); units are again arbitrary:



STSA at Large-k_T

- For $k_T >> Q_s$ we get

$$f_{1T}^{\perp A}(\bar{x}, k_T)\Big|_{k_T \gg Q_s} = \frac{S}{J}\left[-\frac{4\,\alpha_s\, m_N\, \bar{x}\, C_1}{3\beta\, k_T^6}\, \ln\frac{k_T^2}{\Lambda^2}\int d^2b\, T(\underline{b})\, p_{max}(\underline{b})\, Q_s^2(\underline{b}) + A\, f_{1T}^{\perp N}(\bar{x}, k_T)\right]$$

$$= \frac{\beta}{\beta + \frac{8}{5}\, p_{max}\, R}\left[-\frac{4\,\alpha_s\, m_N\, \bar{x}\, C_1}{3\beta\, k_T^6}\ln\frac{k_T^2}{\Lambda^2}\int d^2b\; T(\underline{b})\, p_{max}(\underline{b})\, Q_s^2(\underline{b}) + \frac{A\,\alpha_s^2\, m_N^2\, C_2}{k_T^4}\,\ln\frac{k_T^2}{\Lambda^2}\right]$$

OAM channel **Transversity channel**

- At high-k_T the transversity (Sivers density) channel dominates over the OAM one.

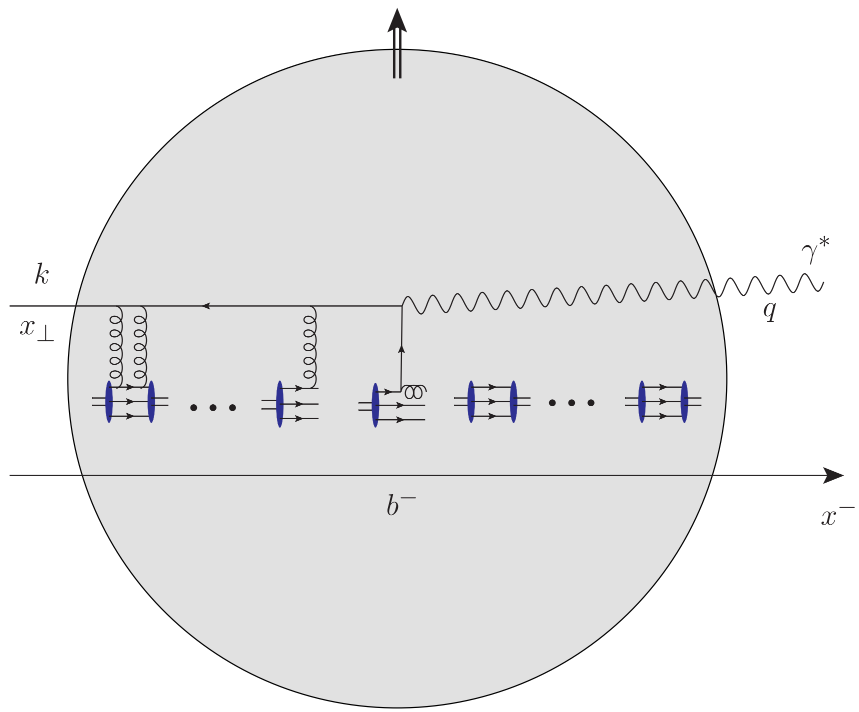
- However, the OAM channel dominates over a fairly broad range

$$k_T < \frac{Q_s}{\sqrt{\alpha_s}}$$

(if $p_T \sim m_N$).

Quasi-Classical STSA in DY

- The DY process in the quasi-classical approximation looks as follows:



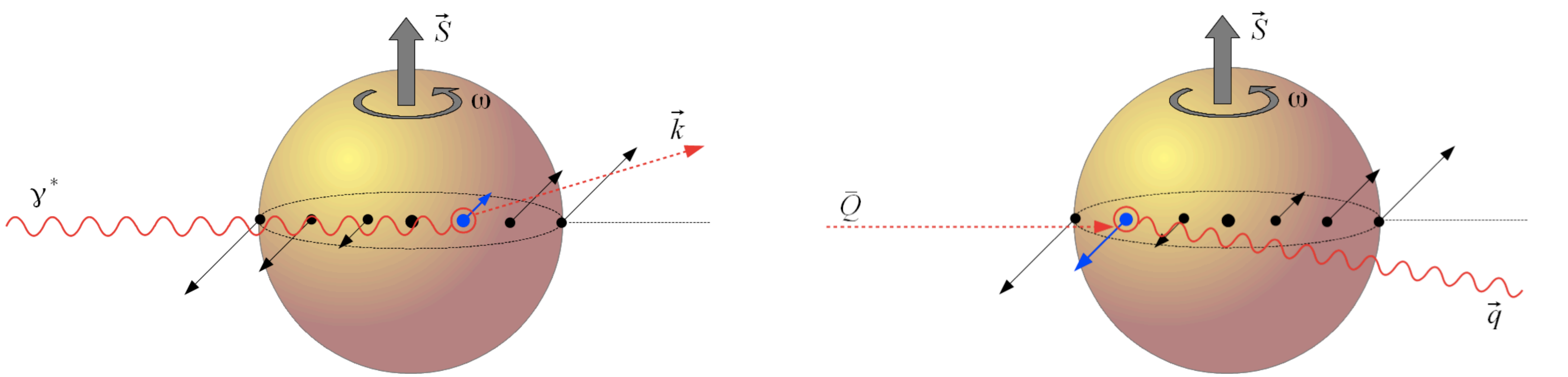
- Note that ordering interactions along the x^- direction makes the reversal of the Wilson line direction between SIDIS and DY explicit.

Origin of Sign Reversal

$$f_{1T}^{\perp A}(x, k_T)\Big|_{SIDIS} = -f_{1T}^{\perp A}(x, k_T)\Big|_{DY}$$

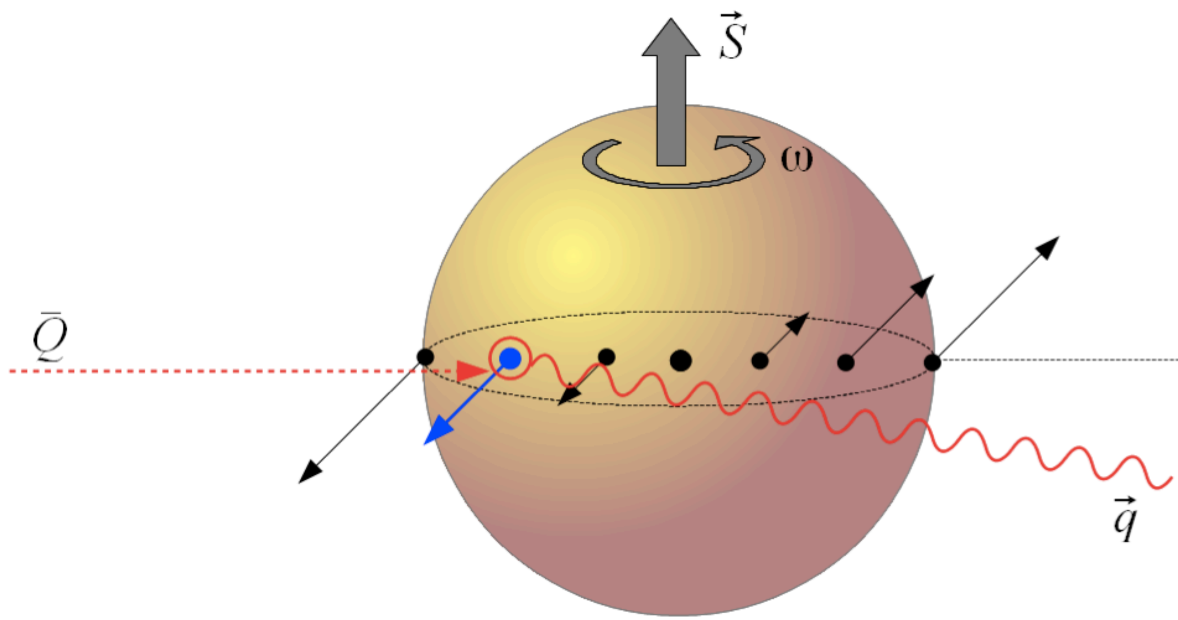
- In the transversity (Sivers density) channel the origin of the sign reversal is simple: the (LO) Sivers function of the nucleon changes sign, and multiple rescatterings do not affect this.

- In the OAM channel the reversal ultimately happens for the simple reason pictured here:



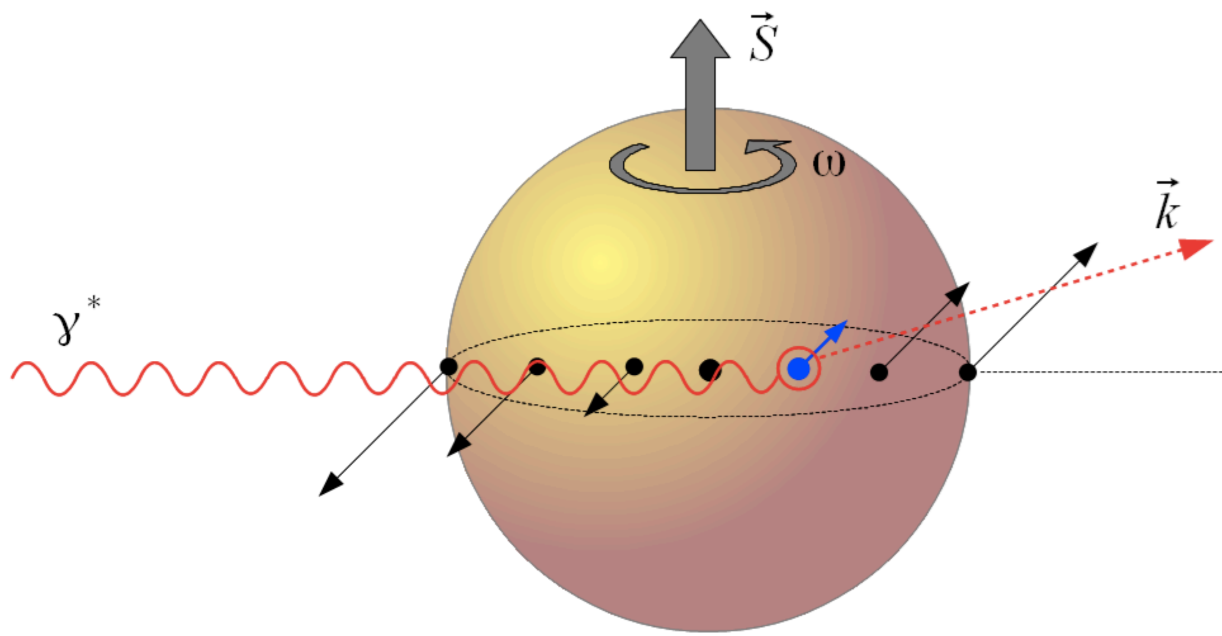
Classical picture of STSA in Drell-Yan

- Think of a transversely polarized proton as a rotating disk with the axis perpendicular to the collision axis
- The proton is not transparent: it has some amount of screening/ shadowing (e.g. gray disk, black disk, etc.)
- Incoming anti-quark (in DY) is more likely to interact near the "front" of the proton: hence, due to the rotation, the outgoing virtual photon is more likely to be produced **left-of-beam**, thus generating STSA.



Classical picture of STSA in SIDIS

- Ditto for SIDIS: except now the incoming virtual photon is more likely to interact near the “back” of the proton, in order for the produced quark to be able to escape out of the proton remnants.
- Owing to the same rotation, the outgoing quark is more likely to be produced **right-of-beam**, thus generating STSA in SIDIS with the **opposite sign** compared to STSA in DY!



Mini-Summary

- It appears that STSA can be easily interpreted as a combination of OAM and some amount of (anti-)shadowing.
- Sign-flip between STSA and DY has a very simple interpretation in this framework too.
- Note the mixing between the nuclear Sivers function and the nucleon unpolarized quark distribution.

Quasi-Classical TMD Mixing

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]

Unpolarized Quark TMD

- We can also calculate the unpolarized quark TMD in the quasi-classical picture at large-x:

$$f_1^A(x, k_T) = \frac{2A\, g_{+-}}{(2\pi)^5} \int d^{2+}p\, d^{2-}b\, d^2r\, d^2k'\, e^{-i(\underline{k}-\underline{k}'-\hat{x}\underline{p})\cdot\underline{r}}\, S_{(r_T, b_T)}^{[\infty^-, b^-]}$$
$$\times \left(W_{unp}(p,b)\, f_1^N(\hat{x}, k'_T) - \tfrac{g_{+-}}{M_A m_N} (P^+ b^-)(\underline{p}\cdot\underline{k}')\, W_{OAM}(p,b)\, f_{1T}^{\perp N}(\hat{x}, k'_T) \right)$$

- Note again the mixing between TMDs: the nuclear unpolarized quark TMD depends on the nucleonic Sivers function $f_{1T}^{\perp N}$!

Boer-Mulders Distribution

- One can also calculate the Boer-Mulders function:

$$h_1^{\perp A}(x, k_T) = \frac{2A\, g_{+-}}{(2\pi)^5}\, \frac{M_A}{k_T^2} \int d^{2+}p\, d^{2-}b\, d^2r\, d^2k'\, e^{-i(\underline{k}-\underline{k}'-\hat{x}\underline{p})\cdot\underline{r}}\, S^{[\infty^-,b^-]}_{(r_T,b_T)}$$
$$\times \left(\frac{(\underline{k}\cdot\underline{k}')}{m_N} \left[W_{unp}(p,b)\, h_1^{\perp N}(\hat{x}, k'_T)\right] - \frac{g_{+-}}{M_A}(P^+b^-)(\underline{p}\cdot\underline{k}) \left[W_{OAM}(p,b)\, h_1^N(\hat{x}, k'_T)\right]\right.$$
$$\left. - \frac{g_{+-}}{M_A}(P^+b^-)\frac{k_T'^2}{m_N^2}\left(\frac{(\underline{p}\times\underline{k}')\,(\underline{k}\times\underline{k}')}{k_T'^2} - \frac{1}{2}(\underline{p}\cdot\underline{k})\right)\left[W_{OAM}(p,b)\, h_{1T}^{\perp N}(\hat{x}, k'_T)\right]\right)$$

- Now we have the mixing between the nuclear Boer-Mulders function with the transversity h_1^N and pretzelocity $h_{1T}^{\perp N}$ of the nucleons.

Mini-Summary

- We can calculate any TMD in the quasi-classical approximation, providing initial conditions for TMD evolution.
- TMDs of nucleus and nucleons mix.
- Calculation is done consistently in the high parton density limit of QCD.

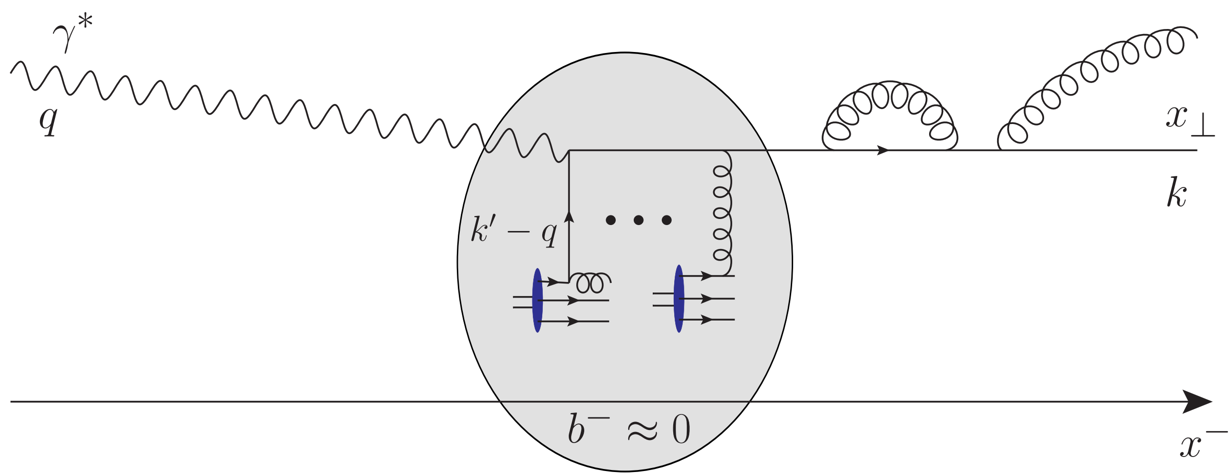
TMD Evolution at Large-x

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]

Evolution Corrections

We work in $A^+ = 0$ gauge (projectile light-cone gauge) and look for emissions giving $\ln s \sim \ln Q^2$.

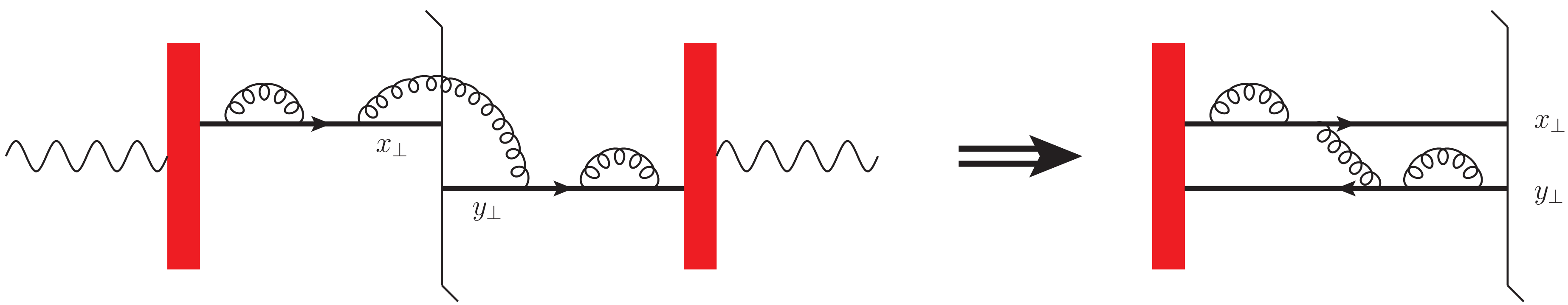
Such emissions and virtual corrections come only from the semi-infinite Wilson line describing the outgoing quark:



How do we sum them up?

Crossing Symmetry

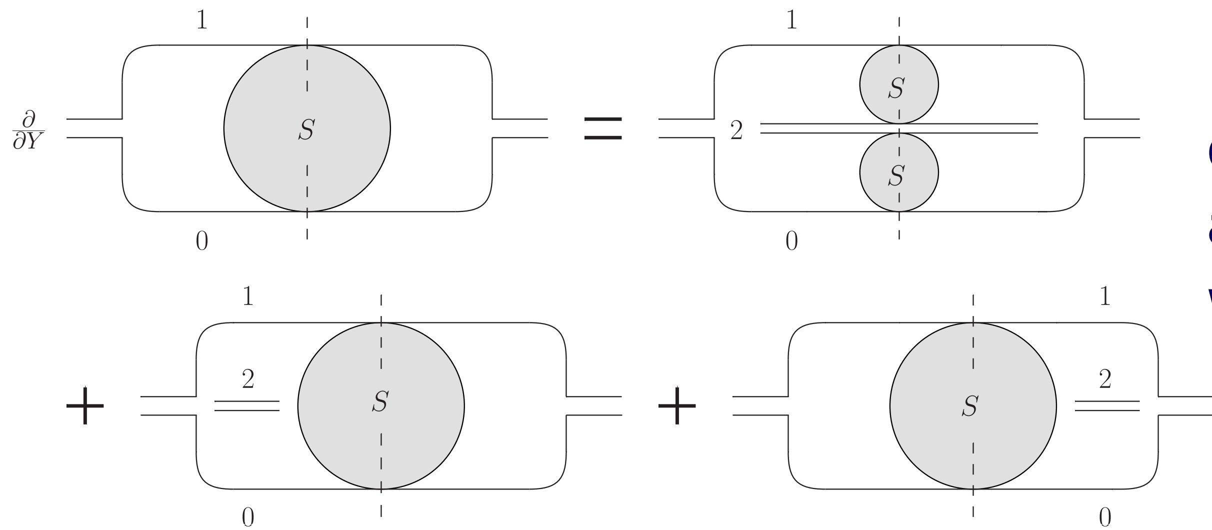
- To sum up evolution corrections for the semi-infinite Wilson line in the amplitude and another one in the cc amplitude, use the "crossing" symmetry to reflect the cc Wilson line into the amplitude (forming the standard SIDIS light-cone staple):



- We end up with a ½ a dipole with, from the standpoint of the dipole evolution, only virtual corrections.

Nonlinear Evolution

To sum up the gluon cascade at large-N_C we write the following equation for the dipole S-matrix:



dashed line = all interactions with the target

$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{01}^2}{x_{02}^2 \, x_{21}^2} \left[S_{\mathbf{x}_0,\mathbf{x}_2}(Y) \, S_{\mathbf{x}_2,\mathbf{x}_1}(Y) - S_{\mathbf{x}_0,\mathbf{x}_1}(Y) \right]$$

Now we only need ½ of the virtual correction (last term on the right).

TMD Evolution at large-x

- Keeping only (a half of) the virtual corrections of the dipole evolution we write

$$\partial_Y S_{xy}[\infty^-, b^-](Y) = -\frac{\alpha_s\, C_F}{2\,\pi^2} \int d^2 z_\perp\, \frac{(\underline{x}-\underline{y})^2}{(\underline{x}-\underline{z})^2\,(\underline{z}-\underline{y})^2}\, S_{xy}[\infty^-, b^-](Y)$$

where rapidity is ($s \approx Q^2$)

$$Y = \ln[s\,(\underline{x}-\underline{y})^2] \approx \ln[Q^2\,(\underline{x}-\underline{y})^2]$$

- The z-integral is UV divergent, and has to be regulated by 1/Q. The solution is

$$S_{xy}[\infty^-, b^-](Q^2) = \exp\left(- \int\limits_{Q_0^2}^{Q^2} \frac{d\mu^2}{\mu^2}\, \frac{\alpha_s\, C_F}{\pi}\, \ln[\mu^2\,(\underline{x}-\underline{y})^2] \right) S_{xy}[\infty^-, b^-](Q_0^2)$$

with the initial condition at Q_0 given by the quasi-classical expression (see above).

Sudakov Form-Factor

- We reproduced the standard Sudakov form-factor, which is characteristic of the CSS evolution.

$$S_{xy}[\infty^-, b^-](Q^2) = \exp\left(- \int\limits_{Q_0^2}^{Q^2} \frac{d\mu^2}{\mu^2} \, \frac{\alpha_s \, C_F}{\pi} \, \ln[\mu^2 \, (\underline{x} - \underline{y})^2] \right) \, S_{xy}[\infty^-, b^-](Q_0^2)$$

- To calculate any TMD of an unpolarized nucleus, simply 'evolve' the classical semi-infinite dipole amplitude using the form-factor above, and use it in the quark-quark correlator to extract all TMDs

$$\Phi^A(x, \underline{k}; P; Q^2) = \frac{2A \, g_{+-}}{(2\pi)^5} \int d^{2+}p \, d^{2-}b \, d^2 r \, d^2 k' \, e^{-i(\underline{k} - \underline{k}' - \hat{x}\underline{p}) \cdot \underline{r}}$$

$$\times \left(W_{unp}(p, b; P) \, \phi_{unp}(\hat{x}, \underline{k}'; p; Q_0^2) - \hat{W}_{pol,\,\mu}(p, b; P) \hat{\phi}^{\mu}_{pol}(\hat{x}, \underline{k}'; p; Q_0^2) \right) \, S^{[\infty^-, b^-]}_{(r_T, b_T)}(Q^2)$$

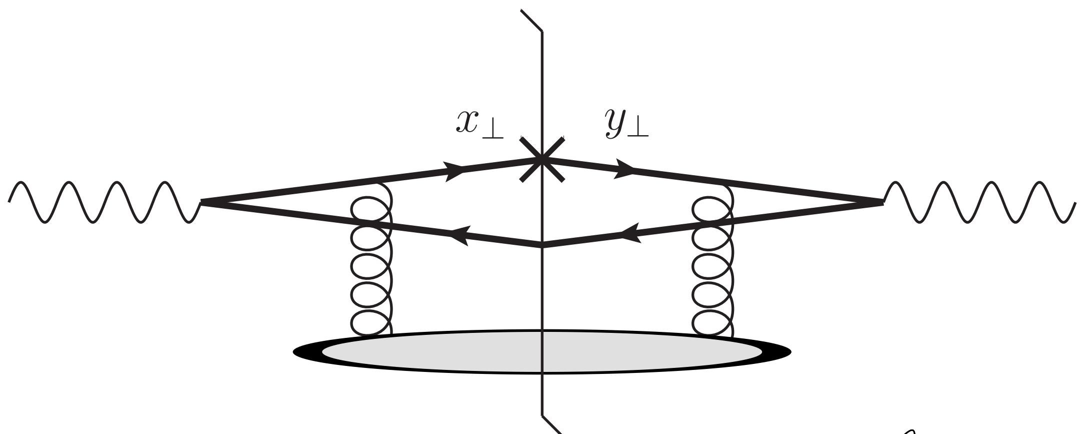
- For gluon TMD evolution simply replace the Casimir, C_F ➔ N_c in S.

Quark TMD Evolution: small-x, unpolarized nucleus

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]

Quark Production in SIDIS at Small-x

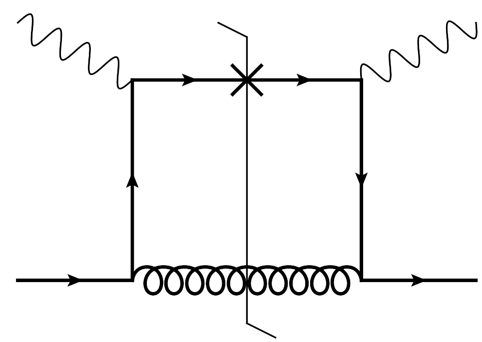
- To find unpolarized target TMDs at small-x it is convenient to start by considering the quark production cross section for SIDIS on an **unpolarized** nucleus.
- The dominant process is different, even at the lowest order:



- Compared to the standard LO process, the one above comes in with an extra factor of

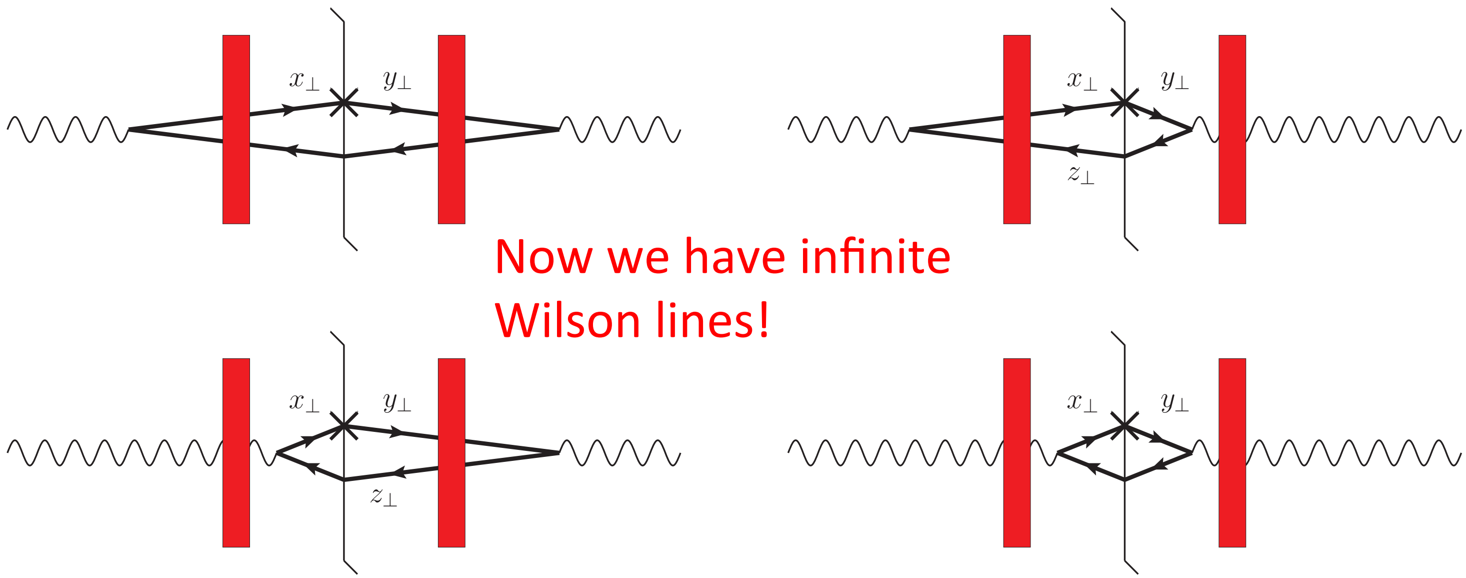
$$\sim \frac{\alpha_s}{x}$$

and is dominant at very low x.



SIDIS to All Orders

- SIDIS process can now be easily generalized to include all-order interactions with the shock waves:



- The SIDIS cross section is

$$\frac{d\sigma_{T,L}^{SIDIS}}{d^2 k_T} = \int\limits_0^1 \frac{dz}{z\,(1-z)} \int \frac{d^2 x_\perp \, d^2 y_\perp \, d^2 z_\perp}{2(2\pi)^3} \, e^{-i\underline{k}\cdot(\underline{x}-\underline{y})} \, \Psi_{T,L}^{\gamma^* \to q\bar{q}}(\underline{x}-\underline{z}, z) \, \left[\Psi_{T,L}^{\gamma^* \to q\bar{q}}(\underline{y}-\underline{z}, z)\right]^*$$

$$\times \left[S_{x,y}^{[+\infty,-\infty]} - S_{x,z}^{[+\infty,-\infty]} - S_{z,y}^{[+\infty,-\infty]} + 1\right]$$

Quark TMD Evolution at Small-x

- Taking the large-Q^2 limit of the SIDIS cross section we can extract the unpolarized quark TMD out of it:

$$f_1^A(x, k_T) = \frac{2\,N_c}{\pi^3\,x} \int \frac{d^2x_\perp\, d^2y_\perp\, d^2z_\perp}{2(2\pi)^3}\, e^{-i\underline{k}\cdot(\underline{x}-\underline{y})}\, \frac{\underline{x}-\underline{z}}{|\underline{x}-\underline{z}|^2} \cdot \frac{\underline{y}-\underline{z}}{|\underline{y}-\underline{z}|^2}$$

$$\times \frac{|\underline{x}-\underline{z}|^4 - |\underline{y}-\underline{z}|^4 - 2\,|\underline{x}-\underline{z}|^2\,|\underline{y}-\underline{z}|^2\, \ln\frac{|\underline{x}-\underline{z}|^2}{|\underline{y}-\underline{z}|^2}}{(|\underline{x}-\underline{z}|^2 - |\underline{y}-\underline{z}|^2)^3} \left[S_{x,y}^{[+\infty,-\infty]} - S_{x,z}^{[+\infty,-\infty]} - S_{z,y}^{[+\infty,-\infty]} + 1\right]$$

- Since the Wilson lines are now infinite, we have infinite dipoles, whose evolution is given by the BK equation at large-N_c:

$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s\,N_c}{2\pi^2} \int d^2x_2\, \frac{x_{01}^2}{x_{02}^2\,x_{21}^2}\, \left[S_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, S_{\mathbf{x}_2,\mathbf{x}_1}(Y) - S_{\mathbf{x}_0,\mathbf{x}_1}(Y)\right]$$

- Initial conditons are given by the quasi-classical Glauber-Mueller formula again.

For unpolarized <u>gluon</u> TMD at low-x
see F. Dominguez et al, ‘11; Balitsky & Tarasov ‘15 (next talk)

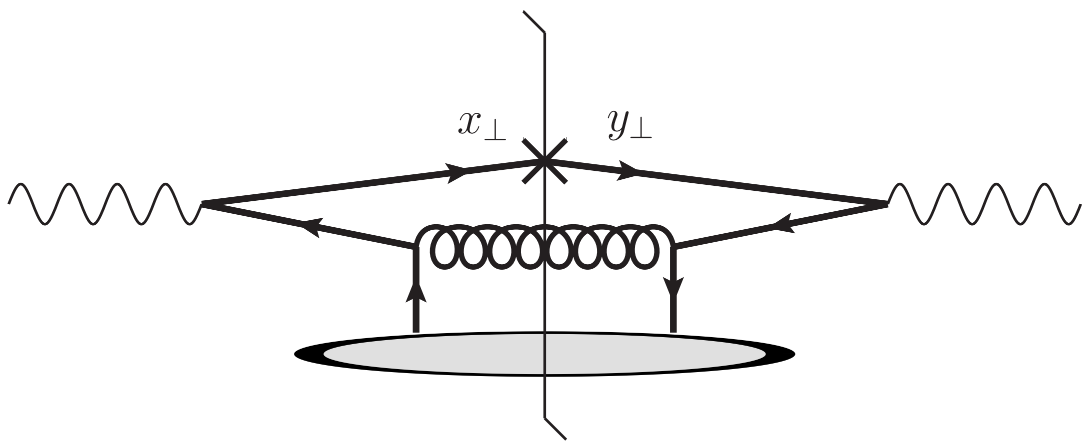
Polarized Quark TMD Evolution at Small-x: an Outline

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, R. Venugopalan, in preparation

Target Spin-Dependent SIDIS

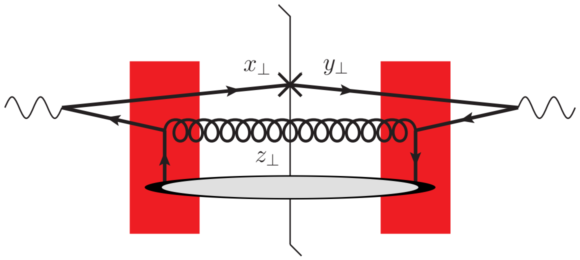
To transfer spin information between the polarized target and the produced quark we either need to exchange quarks in the t-channel, or non-eikonal gluons.

Here's an example of the quark exchange:

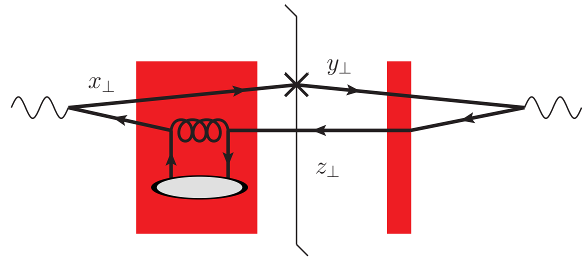


Target Spin-Dependent SIDIS

It is straightforward to include multiple shock wave interactions into the polarized SIDIS cross section:

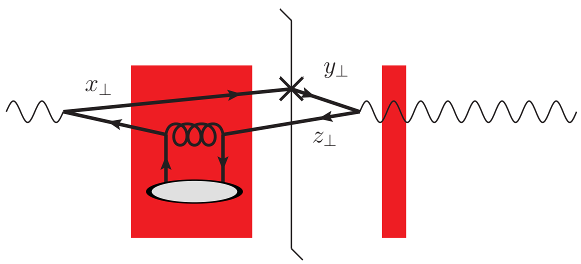


$+$



$+$ c.c. $=0$

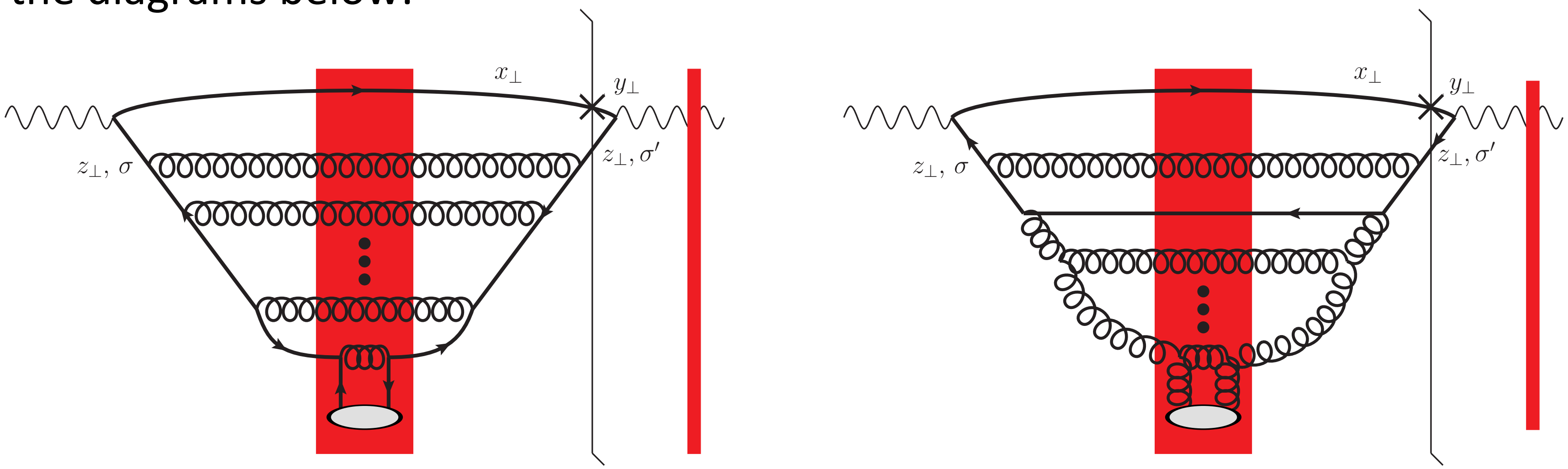
$+$



$+$ c.c. = the answer

Small-x Polarized-Quark TMD Evolution

Evolution corrections can be included into the polarized TMDs using the diagrams below:



Interestingly the quark and non-eikonal gluon ladders mix (see the right panel), resulting in a more complicated evolution equation.

Small-x Polarized-Quark TMD Evolution

- A generic quark-spin dependent TMD can be written as (target spin is Σ, tagged quark spin is λ)

$$f_A(x, k_T; \lambda, \Sigma) = - \int \frac{d^2 x_\perp \, d^2 y_\perp \, d^2 z_\perp}{2(2\pi)^3} \, e^{-i \underline{k} \cdot (\underline{x} - \underline{y})} \sum_{\sigma, \sigma'} f_{\sigma\sigma'}(\underline{x} - \underline{z}, \underline{y} - \underline{z}; \lambda) \times \left[R^{\sigma\,\sigma'}_{x,z}(Y) + R^{\sigma\,\sigma'\,*}_{y,z}(Y) \right]$$

  where R is the QCD Reggeon amplitude and $f_{\sigma\,\sigma'}$ can be calculated from the light-cone wave function.

- R obeys the equation like this: (Itakura, YK, McLerran, Teaney '03), (no gluon ladders ➔ approximation for TMD evolution!)

$$\alpha_s \, \ln^2 s \sim 1$$

$$R^{\sigma\,\sigma'}_{x,z}(Y) = r^{\sigma\,\sigma'}_{x,z}(Y) + \frac{\alpha_s \, C_F}{2\pi^2} \int \frac{d^2 w_\perp}{w_\perp^2} \int\limits_{Y_i}^{\min\{Y, Y - \ln(z_\perp^2 / w_\perp^2)\}} dy \, R^{\sigma\,\sigma'}_{z,w}(y) \, S^{[+\infty, -\infty]}_{x,z}(y)$$

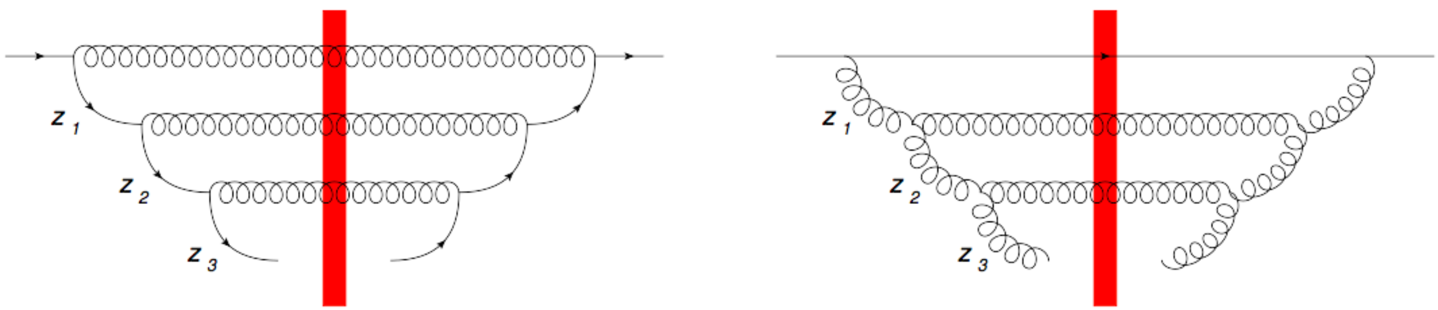
Helicity TMDs

- Summing up mixing quark and gluon ladders only yields

$$g_{1L} \sim \left(\frac{1}{x}\right)^{\omega_+}$$

with

$$\omega_+ = \sqrt{\frac{\alpha_s}{2\pi N_c}} \sqrt{9\,N_c^2 - 1 + \sqrt{(1+7\,N_c^2)^2 + 16\,N_c\,N_f\,(1-N_c^2)}}$$



- The number are encouraging (α_s=0.3, N_c=N_f=3): $g_{1L} \sim \left(\frac{1}{x}\right)^{1.46}$

- But: need to include the non-ladder graphs.

What to Expect

- Without saturation effects, similar evolution for the g_1 structure function was considered by Bartels, Ermolaev and Ryskin in '96.

- Including the mixing of quark and gluon ladders, they obtained

$$R^{\sigma\,\sigma'} \sim x^{-z_s\sqrt{\frac{\alpha_s\,N_c}{2\pi}}}$$

  with z_s = 3.45 for 4 quark flavors.

- The power is large and negative, and can easily become large enough to make the net power of x smaller than −1 for the realistic strong coupling of the order of α_s = 0.2 − 0.3, resulting in polarized TMDs which actually grow with decreasing x fast enough for the integral of the TMDs over the low-x region to be (potentially) large.

- Can this solve the spin puzzle? To be continued…

Conclusions

- Quasi-classics:
  - We can construct any TMD in the quasi-classical approximation: we have calculated the unpolarized quark TMD, Sivers and Boer-Mulders distribution
  - We observed mixing between the nucleon and nuclear TMDs due to spin-orbit coupling.

- TMD evolution:
  - Large-x, both quark and gluon TMDs, polarized and unpolarized target: we reproduced the standard Sudakov form-factor (cf. CSS evolution).
  - Small-x quark TMDs of the unpolarized target: we showed that they evolve with the BK/JIMWLK evolution.
  - Small-x polarized-target TMDs appear to evolve with the QCD Reggeon-like evolution. (D. Pitonyak, M. Sievert, R. Venugopalan, YK, in preparation).

Outlook

- Quasi-classical expressions provide initial conditions which can be used in a global fit of all the TMD data.

- All is ready for a global fit of small-x evolved unpolarized-target TMDs.

- The missing part is the polarized-target TMD evolution.