On the precision of numerical solutions to the running-coupling Balitsky-Kovchegov equation

Marek Matas
(FJFI-CVUT Prague)
POETIC conference
11.9.2015

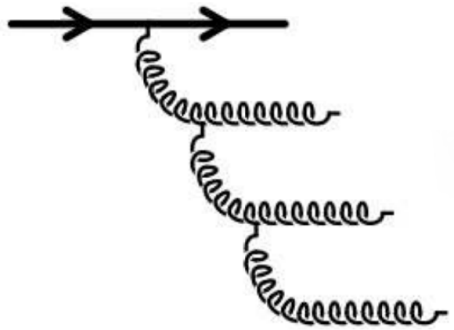
Outline

- **Introduction**
- **Small-x physics**
- **Color Dipole Approach to DIS**
- **Dipole-hadron cross section**
- **The rcBK evolution equation**
- **Solving the rcBK evolution equation**
- **Determining the optimal setup**
- **Conclusion**

Introduction

- One of the most important ways to study the high energy limit of QCD is the Deep Inelastic Scattering at small values of Bjorken-x.
- Experimental results show that the gluon density grows with decreasing of the Bjorken-x. This increase in gluon density is eventually tamed by recombination.
- Balitsky-Kovchegov equation describes such systems and its important to know how the solution depends on the implicit parameters of the numerical solution
- We numerically solved the BK equation and searched for an optimal setup of parameters that would grant us a good running time while still keeping high precision.

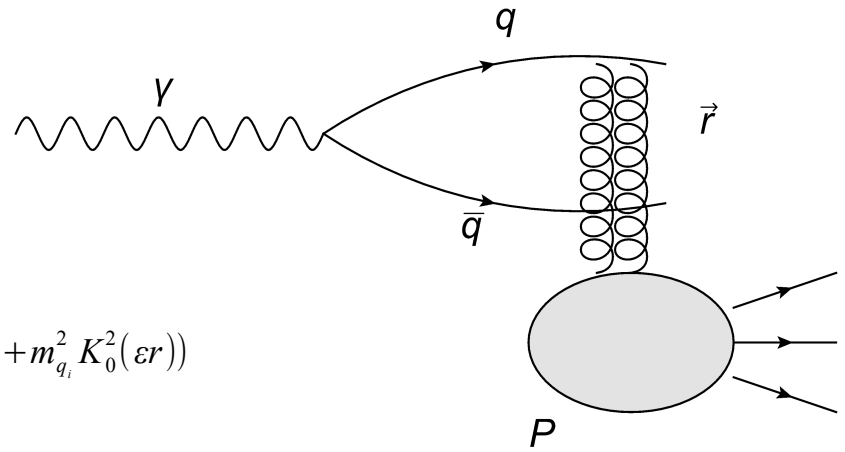
Small-x physics



- Low-x partons are mostly gluons, branching process
- Parton density diverges when x decreases, BFKL evolution equation
- Unitarity of cross section, gluon recombination process
- Dynamical balance between gluon number growth and recombination = saturation
- BK evolution equation describes saturation
- DIS is a useful tool for studying low-x physics

Color dipole approach to DIS

Color dipole approach explains DIS as the interaction of the virtual photon with the target hadron in hadrons rest frame



Photon emitted by electron fluctuates into a quark-antiquark pair:

$$|\Psi_T^i(z,\vec{r},Q^2)^2|=\frac{3\alpha_{em}}{2\pi^2}\sum_f e_{q_i}^2\left((z^2+(1-z)^2)\varepsilon^2 K_1^2(\varepsilon r)+m_{q_i}^2 K_0^2(\varepsilon r)\right)$$

$$|\Psi_L^i(z,\vec{r},Q^2)^2|=\frac{3\alpha_{em}}{2\pi^2}\sum_f e_{q_i}^2\left(4Q^2 z^2(1-z)^2 K_0^2(\varepsilon r)\right)$$

where $\varepsilon^2=z(1-z)Q^2+m_{q_i}^2$ r is the dipole distance, z is the quarks momentum fraction and K_0, K_1 are the MacDonald functions and e_{q_i} is the fraction of charge of quark i.

Dipole-hadron cross section

Assumption - scattering amplitude is not impact parameter dependent, then

$$\sigma^{qq}(r,x)=2\int d\vec{b}\,N(x,r,\vec{b}) \qquad \text{turns into} \qquad \sigma^{qq}(r,x)=\sigma_0 N(x,r)$$

where σ_0 is a parameter that fitted from data.

The final formula for the structure function is then:

$$F_2(x,Q^2)=\frac{Q^2}{4\pi^2\alpha_{em}}\int d\vec{r}\,dz\left(|\Psi_T^i(z,\vec{r},Q^2)^2|+|\Psi_L^i(z,\vec{r},Q^2)^2|\right)\sigma^{qq}(\vec{r},x)$$

The rcBK evolution equation

The Balitsky Kovchegov evolution equation with running coupling is given by integro-differential equation

$$\frac{\partial N(r,Y)}{\partial lnY}=\int d\vec{r}_1 K(\vec{r},\vec{r}_1,\vec{r}_2)\left(N(\vec{r}_1,Y)+N(\vec{r}_2,Y)-N(\vec{r},Y)-N(\vec{r}_1,Y)N(\vec{r}_2,Y)\right)$$

where for the NLO kernel stands

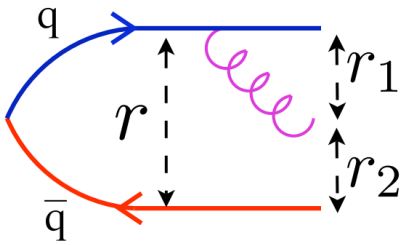
$$K(\vec{r},\vec{r}_1,\vec{r}_2)=\frac{\alpha_s(r^2)N_c}{2\pi}\left(\frac{r^2}{r_1^2 r_2^2}+\frac{1}{r_1^2}\left(\frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)}-1\right)+\frac{1}{r_2^2}\left(\frac{\alpha_s(r_2^2)}{\alpha_s(r_1^2)}-1\right)\right),$$

running coupling is given by equation

$$\alpha_s(r^2)=\frac{12\pi}{\left(11-\frac{2}{3}n_f\right)\ln\left(\frac{4C^2}{r^2\Lambda_{QCD}^2}\right)},\qquad \vec{r}_2=\vec{r}-\vec{r}_1,$$

initial condition was

$$N^{MV}(r)=1-\exp\left(\frac{-\left(r^2Q_{s0}^2\right)^{\gamma}}{4}\ln\left(\frac{1}{r^2\Lambda_{QCD}^2}+e\right)\right)$$



and the constants were taken from(*) to be: $C=2.52,\ \Lambda_{QCD}=0.241\,GeV,\ n_f=N_c=3\quad,\gamma=1.135,Q_{s0}^2=0.165$

* J. L. Albacete at al, Eur.Phys.J. C71 (2011) 1705

Solving the rcBK evolution equation

Up to now does not have an analytic solution

We divide the interval over r into a grid and use Simpson method to integrate it. If r_2 gets of the grid, we linearly interpolate the value of $N(r_2)$.

To speed up the computation process we used following:

- Integration of θ (angle between r_1 and r) over the interval of $< 0; \pi >$ instead of $< 0; 2\pi >$
- Precomputed values of r_2 and kernel
- Simplified Runge-Kutta coefficients

$$Kernel = \int d\vec{r}_1 K(\vec{r}_1, \vec{r}_2, \vec{r}) \qquad Split = \int d\vec{r}_1 K(\vec{r}_1, \vec{r}_2, \vec{r})(N(y, \vec{r}_1) + N(y, \vec{r}_2)) \qquad Recomb = \int d\vec{r}_1 K(\vec{r}_1, \vec{r}_2, \vec{r})(N(y, \vec{r}_1) N(y, \vec{r}_1))$$

Determining the optimal setup

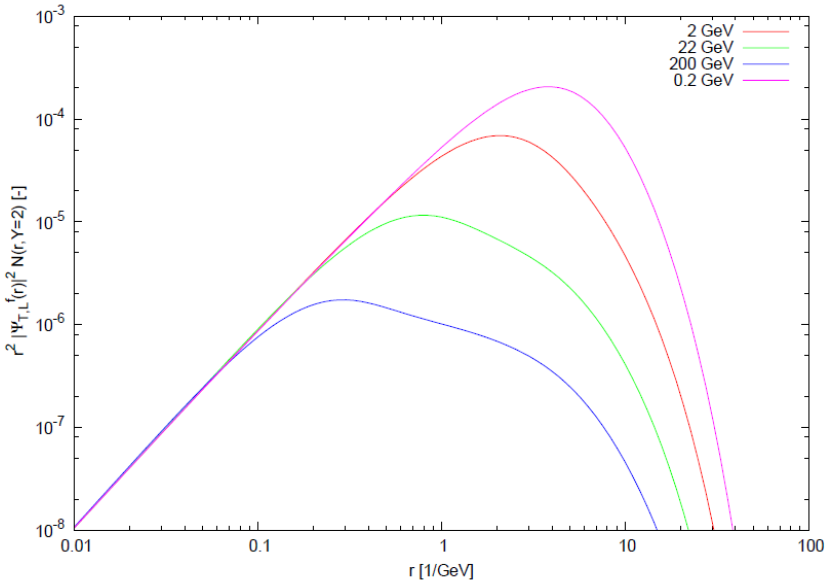
Main parameters of the numerical computation have been varied and the proportional change of the scattering amplitude $D(r,Y)$ has been determined.

Parameters and their default setting was set as:

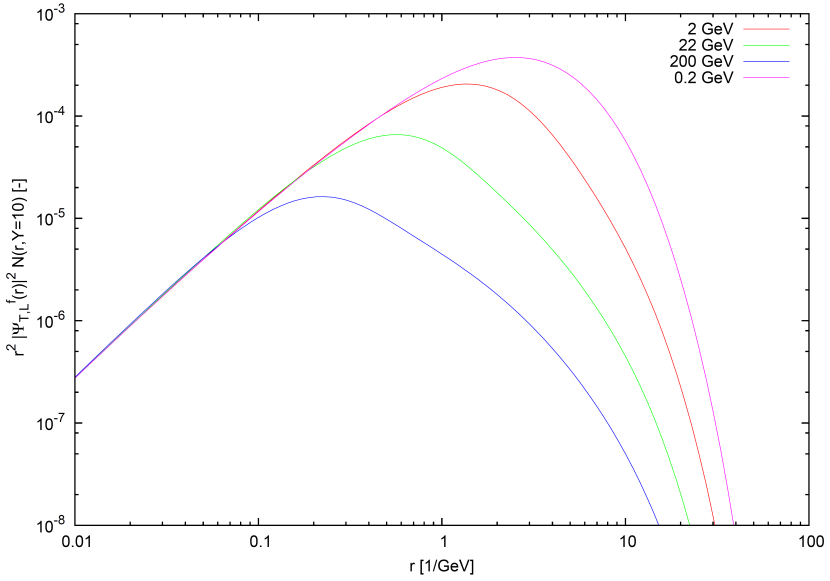
- Runge Kutta method of fourth order
- The step of 0.01 in rapidity in the RK method
- Simpson method for the integration of r_I with 25 steps over one magnitude
- Simpson method for the integration of θ with 10 steps over the interval $< 0; \pi >$

$$D(r,Y)=\frac{|N_{original}(r,Y)-N_{new}(r,Y)|}{N_{original}(r,Y)}$$

Interval of main interest



Y=2

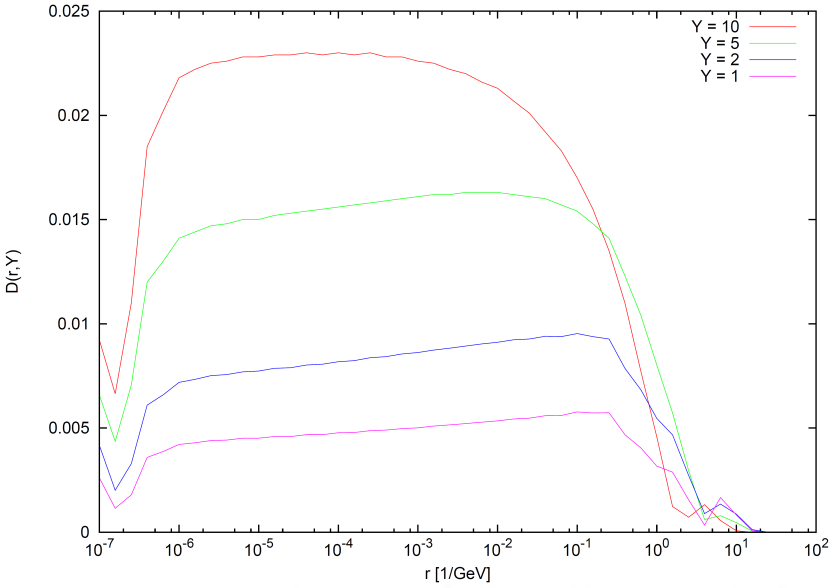


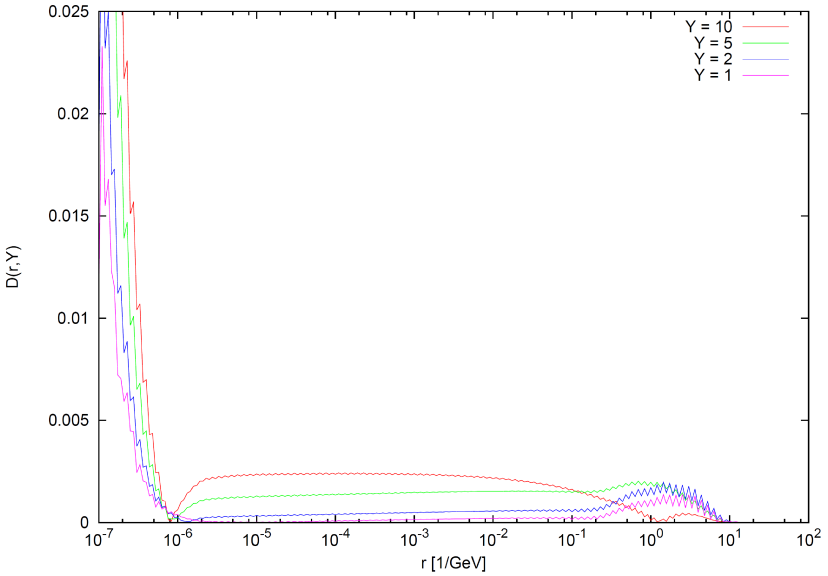
Y=10

Structure function F_2 is obtained by integration of the function shown above over the interval of r

The interval of major influence is $r \sim (0.1; 20)$

Integration step in *r*



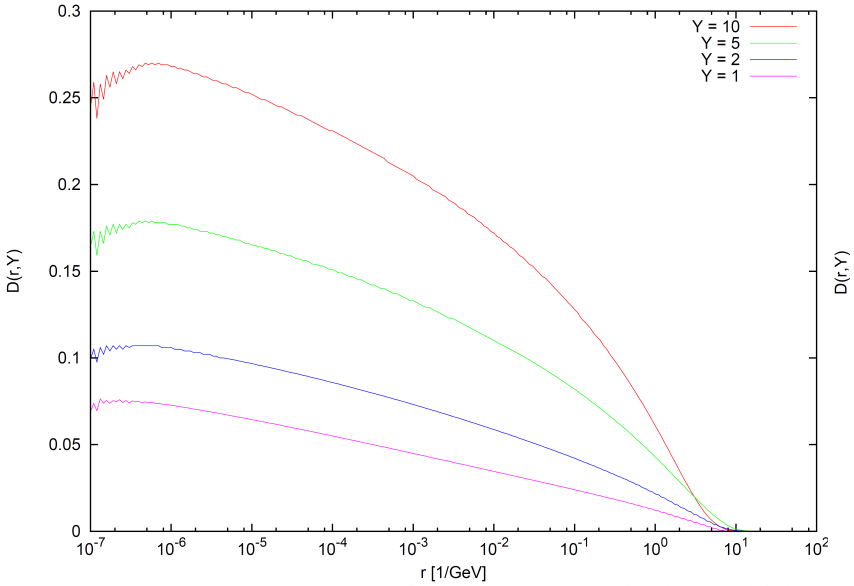


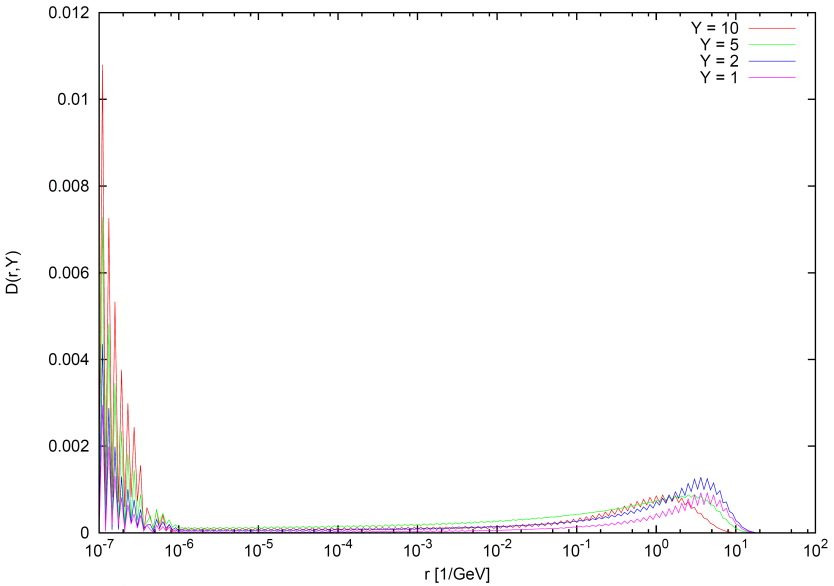
10 steps to 25 steps per order of magnitude of *r*

25 steps to 50 steps per order of magnitude of *r*

Above 2% variation at rapidity 10 when 10 steps are introduced and about 0.25% when 50 steps are introduced

Integration step in θ



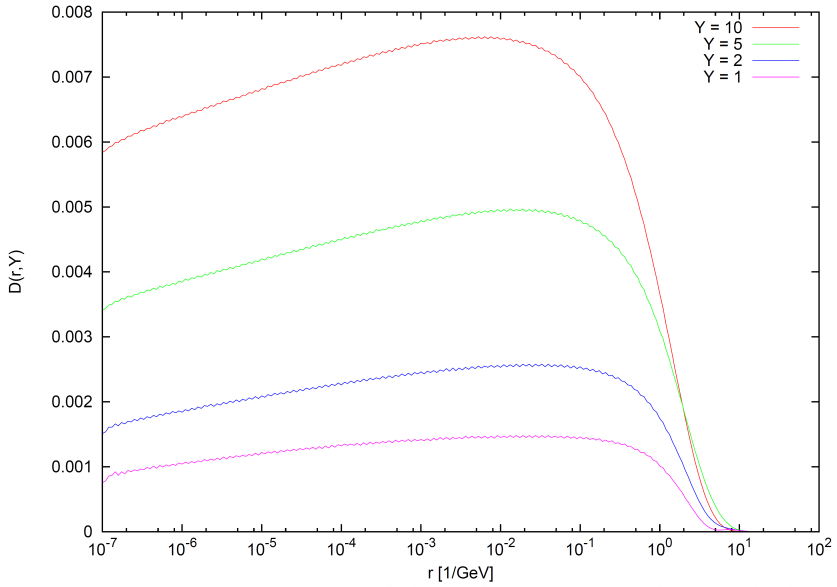


5 steps to 10 steps per $< 0; \pi >$ in θ

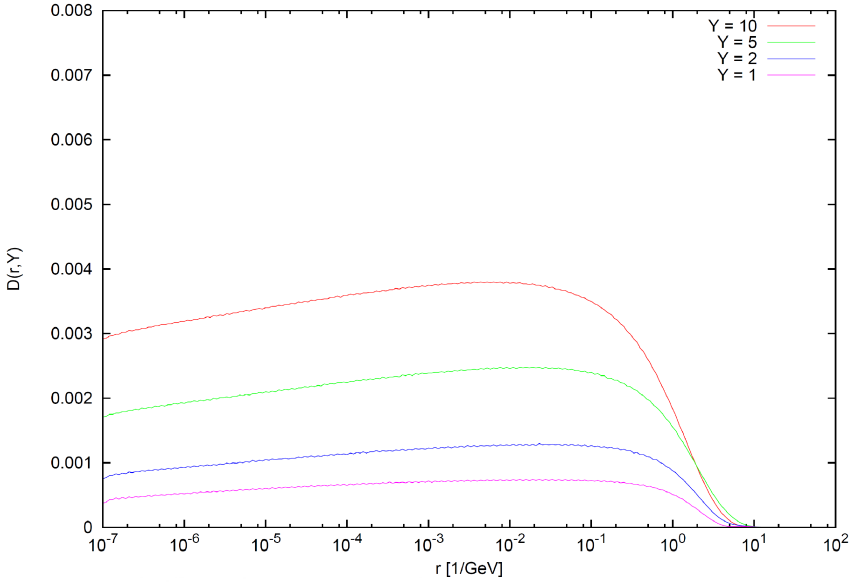
10 steps to 20 steps per $< 0; \pi >$ in θ

Up to 25% variation when 5 steps are introduced and below 0.1 % when 20 steps are introduced

The Runge-Kutta method



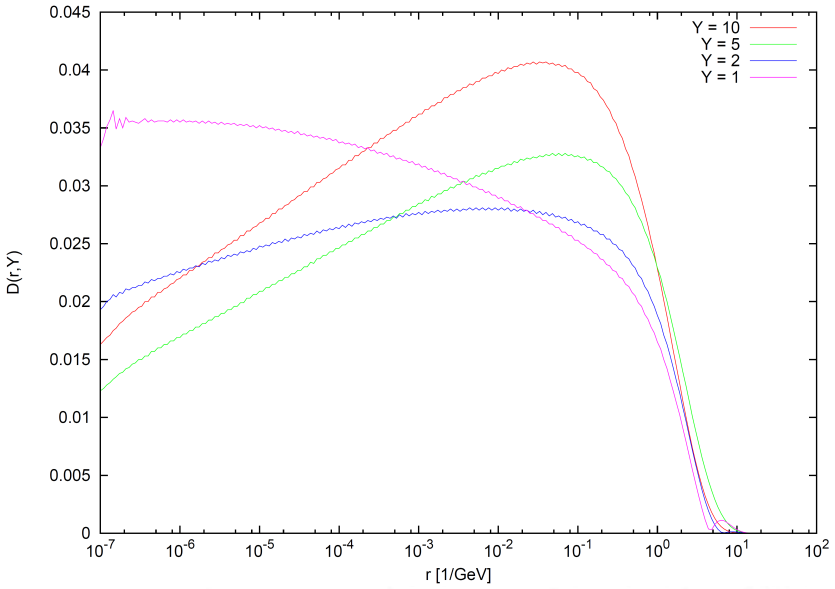
1st order to 2nd order of RK method



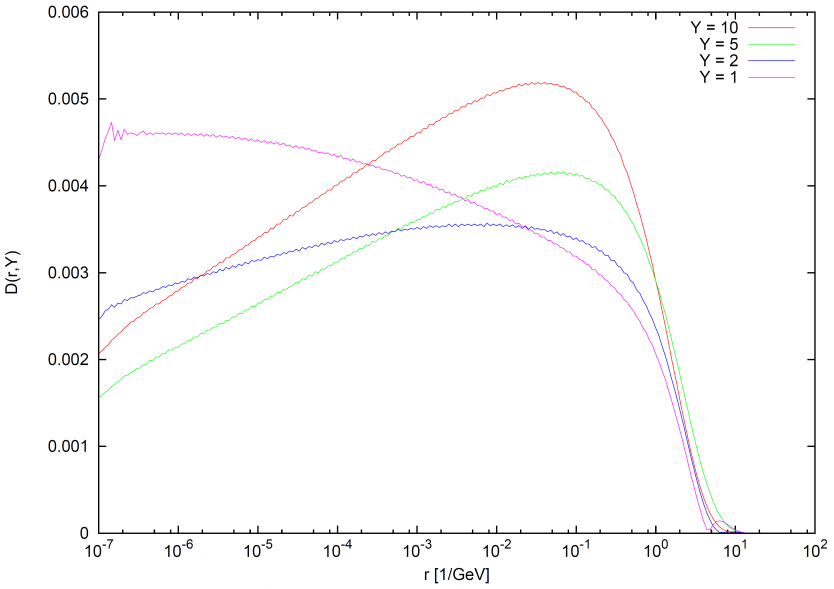
2nd order to 4th order of RK method

Up to 0.7% variation at Y=10 comparing 1st to 2nd order result, half of it between 2nd and 4th order

Step in rapidity in the RK metod



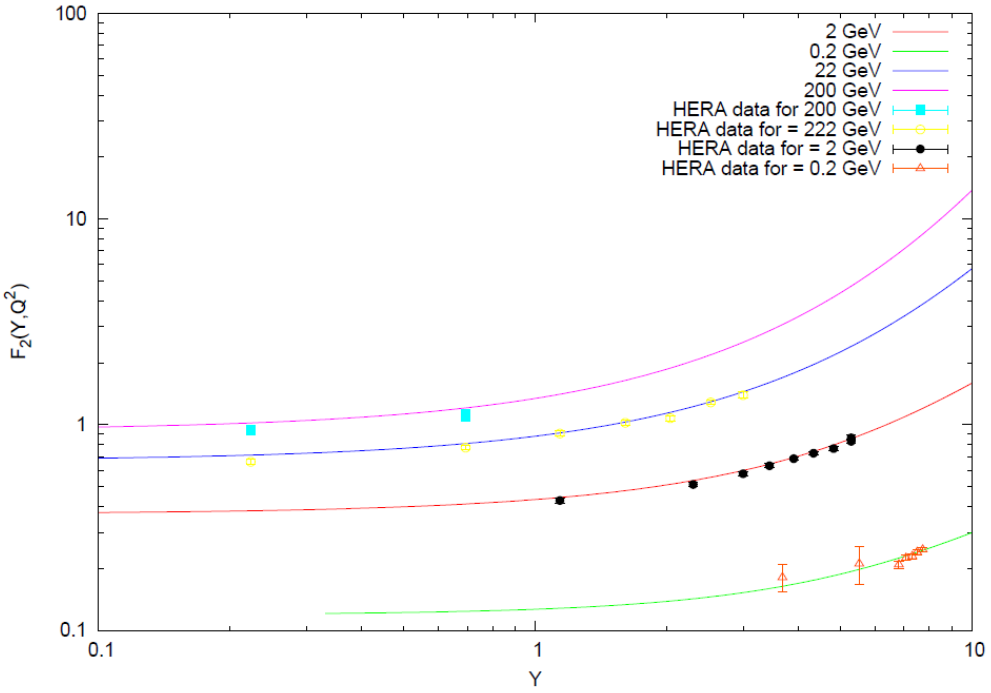
Step of 0.05 to 0.01 in Y



Step of 0.01 to 0.005 in Y

Up to 4% variation when a step of 0.05 in the RK method is introduced and a variation below 0.6% when a step of 0.005 is used.

Computed F_2 structure function



- Running time was 1m 38s on personal laptop for a set scale.
- Mean square error was below 1.5% of the experimental value.

H1 and ZEUS collaborations: Combined measurement and QCD analysis of the inclusive ep scattering cross sections at HERA JHEP01 (2010) 109, [arXiv: 0911.0884].

Conclusion

RcBK equation has been solved with NLO kernel for various scales, sensitivity on numerical parameters has been studied in large rapidity range and optimal setup has been suggested

Solutions of the BK evolution equation describe well a large amount of F_2 structure function data measured by HERA

Parameters have great influence on the accuracy of the result. Reasonable middle ground between precision and running time has to be found.

The determined parameters with a reasonable error/computation time ratio are:

- 25 steps in a magnitude of r
- 10 steps in θ over the interval $< 0; \pi >$
- Runge-Kutta method of fourth order
- A step of 0.01 in rapidity for the Runge-Kutta method

Outlook

I will study the case with explicit b dependence and do the optimization of numerical parameters as a next step for my Diploma Thesis.

Thank you for your attention