Resumming large radiative corrections in the high-energy evolution of the Color Glass Condensate

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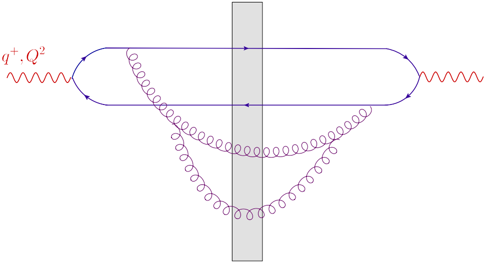
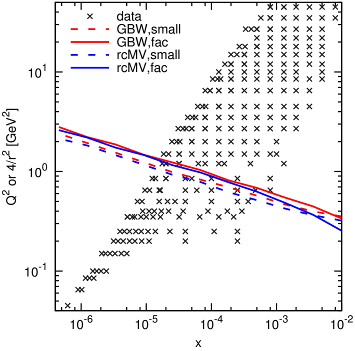




Introduction

- BK and B-JIMWLK equations have recently been promoted to NLO
  - running coupling corrections to BK *(Kovchegov, Weigert; Balitsky, 06)*
  - full NLO version of the BK equation *(Balitsky, Chirilli, 08)*
  - Balitsky hierarchy (Wilson lines) at NLO *(Balitsky, Chirilli, 13)*
  - JIMWLK evolution at NLO *(Kovner, Lublinsky, Mulian, 2013)*

- This evolution encompasses the NLO version of BFKL equation *(Fadin, Kotsky, Lipatov, Camici, Ciafaloni ... 95-98)*

- Large corrections, unstable evolution: double and single collinear logs

- Powerful resummation schemes cured this problem for NLO BFKL *(Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03)*

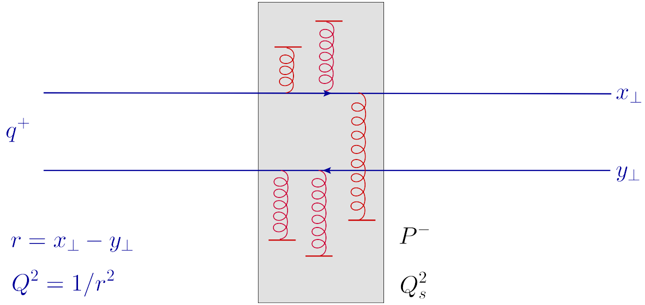
- The collinear logs become important in the dilute/weak scattering regime $\Longrightarrow$ cannot be alleviated by non-linear effects like saturation

- The instabilities are numerically observed for NLO BK as well *(Lappi, Mäntysaari, arXiv:1502.02400)*

Introduction

- The resummation schemes proposed for NLO BFKL cannot be (easily) adapted to the non-linear evolution
  - formulated in Mellin space
  - the non-linear equations are formulated transverse coordinate space
  - multiple scattering in the eikonal approximation
- Resummation of double logs inspired by CCFM *(Beuf, arXiv:1401.0313)*
  - non-local in rapidity, contact with full NLO BK cumbersome
  - a bit ad-hoc: contact with pQCD still unclear
- A fresh look at the Feynman graphs : light-cone perturbation theory *(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1502.05642)*
  - clarifying the double logarithmic approximation in QCD
  - collinearly improved version of BK, local in rapidity
  - easy to use for phenomenology, or to extend to full NLO
- Promising phenomenology: excellent fits to HERA data *(arXiv:1507.03651, 1507.07120)* ... see also next talk by Javier Albacete

Dipole–hadron scattering ($\gamma^* p$, $\gamma^* A$, pA, ...)

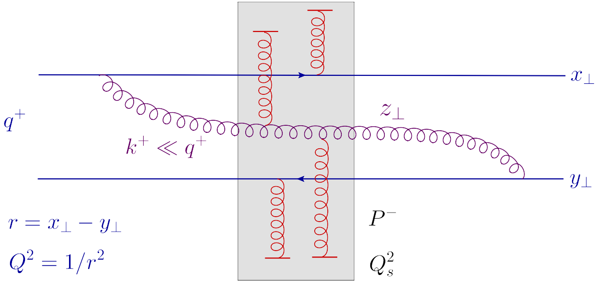


- Dipole ('projectile'): large q^+, transverse resolution $Q^2 = 1/r^2$
- Hadron ('target'): large P^-, high gluon density (Q_s^2)
- Wilson lines : multiple scattering in the eikonal approximation

$$S_{\boldsymbol{xy}} = \frac{1}{N_c} \mathrm{tr}\big(V_{\boldsymbol{x}}^\dagger V_{\boldsymbol{y}}\big)\,, \quad V^\dagger(\boldsymbol{x}) = \mathrm{P}\exp\left\{ ig \int \mathrm{d}x^+ \, A_a^-(x^+, \boldsymbol{x}) t^a \right\}$$

High energy evolution

- Probability $\alpha_s \ln(1/x)$ to radiate a soft gluon: $x \equiv \frac{k^+}{q^+} \ll 1$

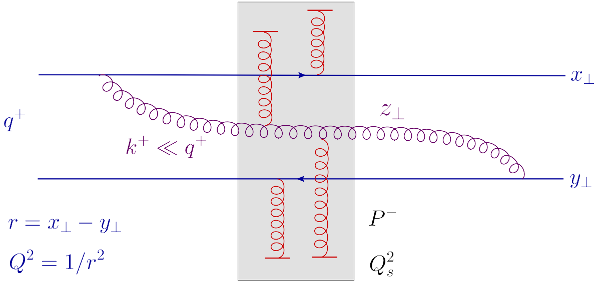


$$\frac{2xq^+}{Q^2} \sim \frac{1}{P^-} \implies x \simeq \frac{s}{Q^2} \qquad (s = 2q^+P^-)$$

- BFKL evolution of the dipole in the background of the dense target
  - non-linear effects due to multiple scattering
  - BK equation, first equation in Balitsky hierarchy

High energy evolution

- Probability $\alpha_s \ln(1/x)$ to radiate a soft gluon: $x \equiv \frac{k^+}{q^+} \ll 1$

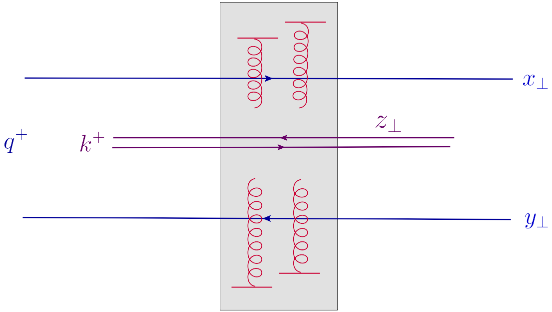


$$\frac{2q^+}{Q^2} \sim \frac{1}{xP^-} \Longrightarrow x \simeq \frac{s}{Q^2} \qquad (s = 2q^+P^-)$$

- JIMWLK evolution of the dense target
  - non-linear effects due to saturation
  - functional evolution equivalent to Balitsky hierarchy

The BK equation *(Balitsky, '96; Kovchegov, '99)*

- Large N_c : the original dipole splits into two new dipoles

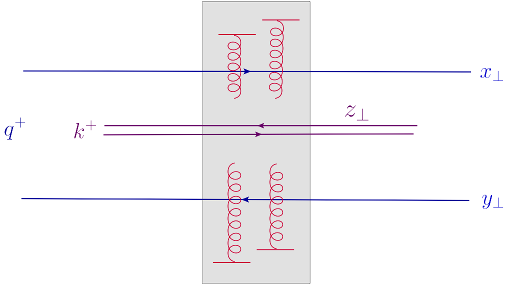


$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right]$$

- 'real term' : daughter dipoles exist at the time of scattering
- 'virtual term' : probability conservation

Deconstructing the BK equation

- Non-linear equation for the dipole scattering amplitude $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

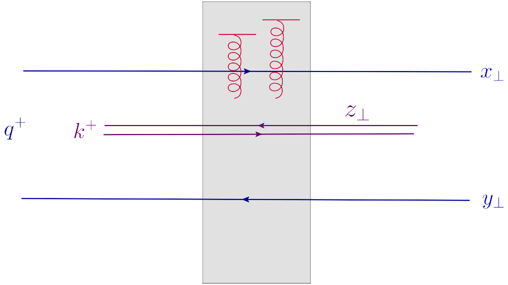


$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}}\right]$$

- respects unitarity bound: $T(Y, r) \leq 1$
- color transparency: $T(Y, r) \propto r^2$ as $r \to 0$
- saturation momentum $Q_s(Y)$: $T(Y, r) = 0.5$ when $r = 2/Q_s(Y)$

Deconstructing the BK equation

- Non-linear equation for the dipole scattering amplitude $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

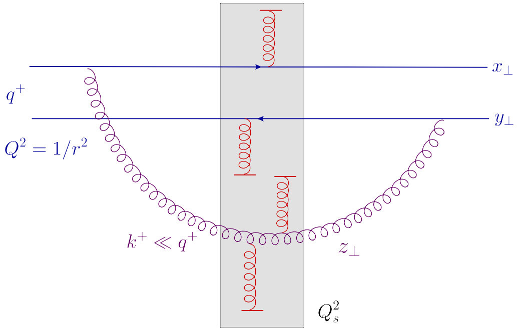


$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}}\right]$$

- Weak scattering (dilute target): $T_{\boldsymbol{xy}} \ll 1 \Rightarrow$ BFKL equation
  - single scattering approximation
  - linear evolution (exponential growth with Y)

Deconstructing the BK equation

- Large transverse separation : $Q^2 = 1/r^2 \gg Q_s^2$



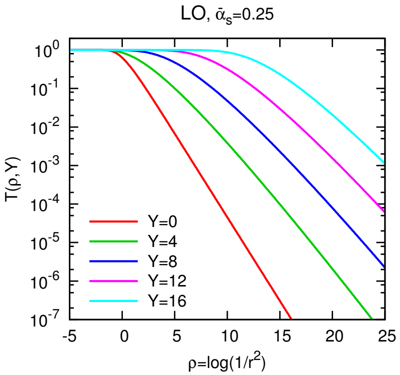
- Large transverse phase-space at $r \ll z_\perp \ll 1/Q_s$

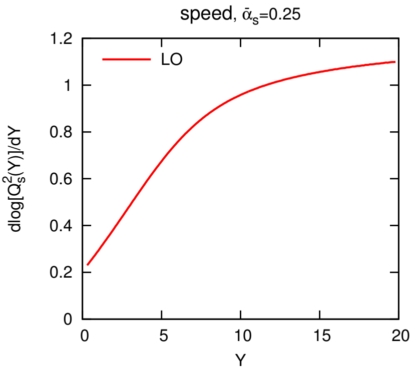
$$\frac{\partial}{\partial Y}\frac{T(r^2)}{r^2} \simeq \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \frac{\mathrm{d}z^2}{z^2}\frac{T(z^2)}{z^2} \implies \Delta T \sim \bar{\alpha}_s Y \ln\frac{Q^2}{Q_s^2}\, T$$

- 'Double-logarithmic approximation' : DLA 1.0

Numerical solutions: LO BK

- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2)$ with increasing Y





- color transparency at large ρ (small r) : $T \propto r^2 = \mathrm{e}^{-\rho}$
- unitarity limit at small ρ (large r) : $T = 1$
- saturation exponent (speed): $\lambda_s \equiv \frac{\mathrm{d}\ln Q_s^2}{\mathrm{d}Y} \simeq 1$ for $Y \gtrsim 10$

BK equation at NLO *Balitsky, Chirilli (arXiv:0710.4330)*

- Very complicated in full generality
- Here: $N_\mathrm{f}=0$, large N_c, tiny fonts

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y}=\frac{\bar\alpha_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}}-S_{\boldsymbol{xy}}\right)\Bigg\{1+$$

$$+\,\bar\alpha_s\left[\bar b\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2-\bar b\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}\right.$$

$$\left.+\,\frac{67}{36}-\frac{\pi^2}{12}-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

$$+\,\frac{\bar\alpha_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}}-S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

Deconstructing NLO BK

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

$$+\,\bar{\alpha}_s\left[\bar{b}\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}\right.$$

$$\left.+\,\frac{67}{36}-\frac{\pi^2}{12}-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

$$+\,\frac{\bar{\alpha}_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- blue : leading-order (LO) terms
- red : NLO terms enhanced by (double or single) transverse logarithms
- black : pure $\bar{\alpha}_s$ corrections (no logarithms)

Deconstructing NLO BK

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, \left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

$$+\,\bar{\alpha}_s\left[\bar{b}\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}\right.$$

$$\left.+\,\frac{67}{36}-\frac{\pi^2}{12}-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

$$+\,\frac{\bar{\alpha}_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- the running coupling and the double-logarithmic corrections are manifest
- the single logs are still hidden: needs to perform the integral over $\boldsymbol{u}$
- the collinear logs are important only at weak scattering ($T \ll 1$)

Deconstructing NLO BK

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, \left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

$$+\,\bar{\alpha}_s\Bigg[\bar{b}\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

$$+\,\frac{67}{36}-\frac{\pi^2}{12}-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\Bigg]\Bigg\}$$

$$+\,\frac{\bar{\alpha}_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

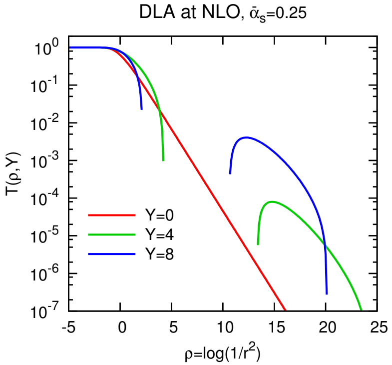
$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

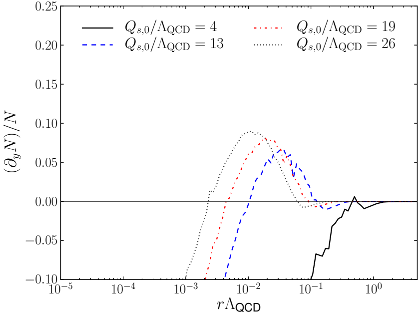
$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- Keeping just the logarithmically enhanced terms $(z_\perp \gg r)$

$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} \simeq \bar{\alpha}_s\int \mathrm{d}z^2\,\frac{r^2}{z^4}\left\{1-\bar{\alpha}_s\left(\frac{1}{2}\ln^2\frac{z^2}{r^2}+\frac{11}{12}\ln\frac{z^2}{r^2}-\bar{b}\,\ln r^2\mu^2\right)\right\}T(z)$$

NLO : unstable numerical solutions





- Left: LO BK + the double collinear logarithm at NLO

  (our calculation, arXiv:1502.05642)

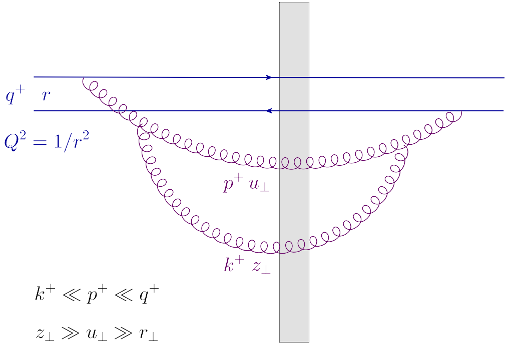
- Right: full NLO BK : evolution speed $(\partial_Y T)/T$

  (Lappi, Mäntysaari, arXiv:1502.02400)

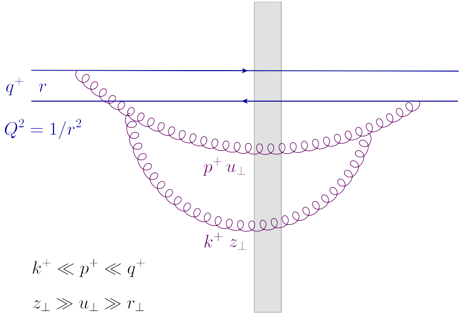
- The main source of instability: the double collinear logarithm

The double collinear logarithm

- Two successive emissions which are strongly ordered in both ...



  - longitudinal momenta : $q^+ \gg p^+ \gg k^+$
  - ... and transverse sizes (or momenta): $r_\perp^2 \ll u_\perp^2 \ll z_\perp^2 \ll 1/Q_s^2$
- "Two iterations of DLA 1.0 $\Longrightarrow \mathcal{O}((\bar{\alpha}_s Y \rho)^2)$" ... not exactly !
  - additional constraint due to time ordering: $\tau_p = p^+ u_\perp^2 \; > \; \tau_k = k^+ z_\perp^2$



$$p^+ u_\perp^2 > k^+ z_\perp^2 \implies \Delta Y \equiv \ln \frac{p^+}{k^+} > \Delta\rho \equiv \ln \frac{z_\perp^2}{u_\perp^2}$$

- Additional restriction on the DLA phase-space ⇒ double collinear logs

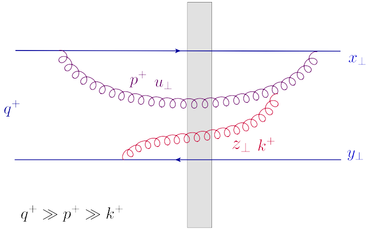
$$Y > \rho \implies \bar{\alpha}_s Y \rho \longrightarrow \bar{\alpha}_s (Y - \rho)\rho = \bar{\alpha}_s Y \rho - \bar{\alpha}_s \rho^2$$

- Time ordering enters perturbation theory via energy denominators

Time-ordered (light-cone) perturbation theory

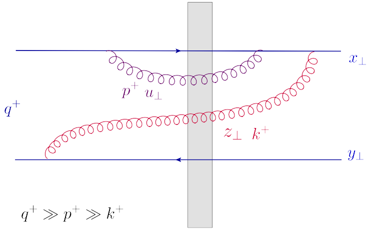
- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions

- time ordered graphs



$$\frac{\tau_p}{\tau_p + \tau_k} \simeq \Theta(\tau_p - \tau_k)$$

- anti time-ordered graphs

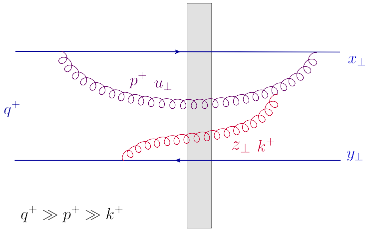


$$\frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)$$

- The time (x^+) integrals yield energy denominators
- Integrate out the harder gluon $(p^+, u_\perp)$ to DLA : $r_\perp \ll u_\perp \ll z_\perp$

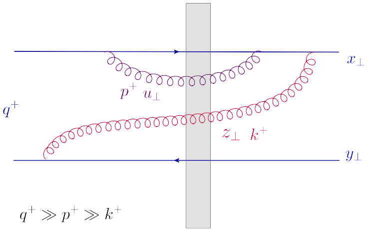
- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions

- time ordered graphs



$$\frac{\tau_p}{\tau_p+\tau_k} \simeq \Theta(\tau_p - \tau_k)$$

- anti time-ordered graphs

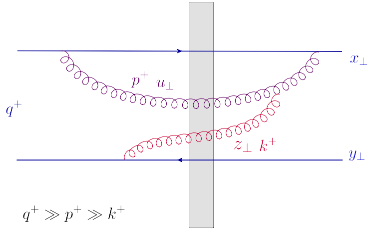


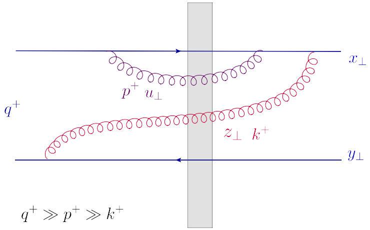
$$\frac{\tau_k}{\tau_p+\tau_k} \simeq \Theta(\tau_k - \tau_p)$$

$$\text{TO}\;:\quad \bar{\alpha}_s \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+}\,\Theta(p^+u^2 - k^+z^2) = \bar{\alpha}_s Y\rho - \frac{\bar{\alpha}_s \rho^2}{2}$$

- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions

- time ordered graphs
- anti time-ordered graphs





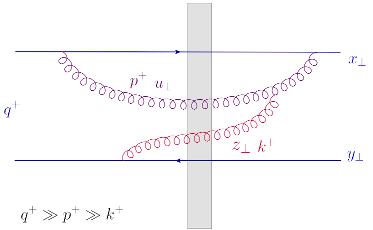
$$\frac{\tau_p}{\tau_p+\tau_k} \simeq \Theta(\tau_p-\tau_k) \qquad\qquad \frac{\tau_k}{\tau_p+\tau_k} \simeq \Theta(\tau_k-\tau_p)$$

$$\text{ATO}: \quad \bar{\alpha}_s \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+}\,\Theta(k^+z^2 - p^+u^2) = \frac{\bar{\alpha}_s \rho^2}{2}$$

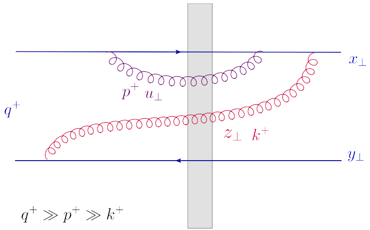
Time-ordered (light-cone) perturbation theory

- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$

- time ordered graphs



- anti time-ordered graphs



- TO graphs generate the expected LLA contributions: $\bar{\alpha}_s Y \rho$
- both TO and ATO graphs generate double collinear logs $\bar{\alpha}_s \rho^2$
- the latter precisely cancel in the sum of all the ATO graphs
- net result: the double-collinear logs come from TO graphs alone

(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1502.05642)

DLA 2.0

- Enforce time-ordering within DLA 1.0

$$\frac{\partial T(Y,r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{\mathrm{d}z^2}{z^2}\,\frac{r^2}{z^2}\,T(Y,z^2)$$

DLA 2.0

- Enforce time-ordering within DLA 1.0

$$\frac{\partial T(Y,\rho)}{\partial Y} = \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, T(Y,\rho_1)$$

  - introduce logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2)$

DLA 2.0

- Enforce time-ordering within DLA 1.0

$$\frac{\partial T(Y,\rho)}{\partial Y} = \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, T(Y - \rho + \rho_1, \rho_1)$$

  - introduce logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2)$
  - introduce time-ordering $\Longrightarrow$ non-local in Y
  - resums powers of $\bar{\alpha}_s Y \rho$ and $\bar{\alpha}_s \rho^2$ to all orders

DLA 2.0

- Enforce time-ordering within DLA 1.0

$$\frac{\partial T(Y,\rho)}{\partial Y} = \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, T(Y-\rho+\rho_1, \rho_1)$$

  - introduce logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2)$
  - introduce time-ordering $\Longrightarrow$ non-local in Y
  - resums powers of $\bar{\alpha}_s Y \rho$ and $\bar{\alpha}_s \rho^2$ to all orders

- The importance of time-ordering has since long been recognized
  - coherence effects, kinematical constraint, choice of rapidity scale ...

    Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)

- The precise relation to Feynman graphs had not been studied
- So far, no change in the kernel: all double-logs come from non-locality

DLA 2.0

- Enforce time-ordering within DLA 1.0

$$\frac{\partial T(Y,\rho)}{\partial Y} = \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, T(Y-\rho+\rho_1, \rho_1)$$

  - introduce logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2)$
  - introduce time-ordering $\Longrightarrow$ non-local in Y
  - resums powers of $\bar{\alpha}_s Y \rho$ and $\bar{\alpha}_s \rho^2$ to all orders

- The importance of time-ordering has since long been recognized
  - coherence effects, kinematical constraint, choice of rapidity scale ...

    Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)

- The precise relation to Feynman graphs had not been studied
- Equivalently: a local equation but with an all-order resummed kernel

Getting local

$$T(Y,\rho) = T(0,\rho) + \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \int_0^{Y-\rho+\rho_1} \mathrm{d}Y_1 \, T(Y_1,\rho_1)$$

- For $Y \geq \rho$, the solution $T(Y,\rho)$ to the above equation coincides with the solution $\tilde{T}(Y,\rho)$ to the following problem:

$$\tilde{T}(Y,\rho) = \tilde{T}(0,\rho) + \bar{\alpha}_s \int_0^{Y} \mathrm{d}Y_1 \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, \mathcal{K}_{\mathrm{DLA}}(\rho-\rho_1) \tilde{T}(Y_1,\rho_1)$$

with the following, all-orders, kernel: *(see also Sabio Vera, 2005)*

$$\mathcal{K}_{\mathrm{DLA}}(\rho) = \frac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s \rho^2}\big)}{\sqrt{\bar{\alpha}_s \rho^2}} = 1 - \frac{\bar{\alpha}_s \rho^2}{2} + \frac{(\bar{\alpha}_s \rho^2)^2}{12} + \cdots$$

... and the following, all-orders, initial condition:

$$\tilde{T}(0,\rho) = T(0,\rho) - \sqrt{\bar{\alpha}_s} \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, \mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s(\rho-\rho_1)^2}\big) T(0,\rho_1)$$

Extending to single-logs/BFKL/BK

- Recall the NLO equation with all the transverse logs

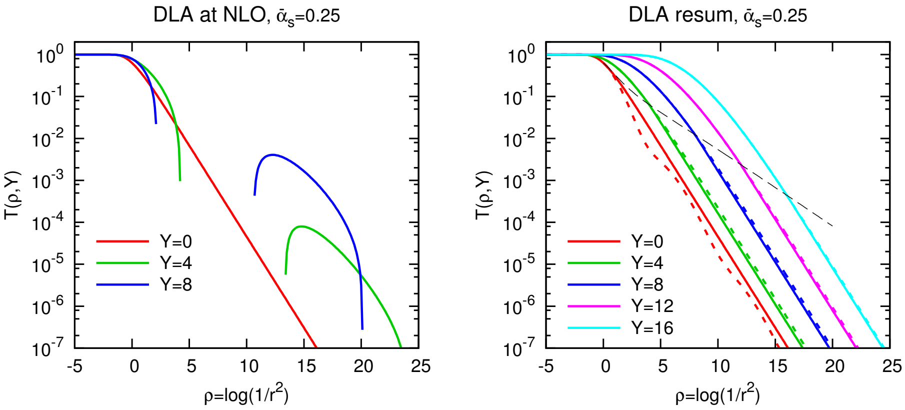
$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} = \bar{\alpha}_s \int \mathrm{d}z^2 \, \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left(\frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} - \bar{b} \, \ln r^2 \mu^2 \right) \right\} T(z)$$

  - the double-logarithm is already included within $\mathcal{K}_{\mathrm{DLA}}(\rho)$ ✓
  - the collinear single-log is part of the DGLAP anomalous dimension ✓
  - the running coupling log is resummed by replacing $\bar{\alpha}_s \to \bar{\alpha}_s(r_{\min})$ ✓

$$\frac{\mathrm{d}\tilde{T}_{\boldsymbol{xy}}}{\mathrm{d}Y} = \int \frac{\mathrm{d}^2\boldsymbol{z}}{2\pi} \, \bar{\alpha}_s(r_{\min}) \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left(\tilde{T}_{\boldsymbol{xz}} + \tilde{T}_{\boldsymbol{zy}} - \tilde{T}_{\boldsymbol{xy}} - \tilde{T}_{\boldsymbol{xz}}\tilde{T}_{\boldsymbol{zy}} \right)$$
$$\times \left[\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{\min\{(\boldsymbol{x}-\boldsymbol{z})^2, (\boldsymbol{y}-\boldsymbol{z})^2\}} \right]^{\pm\bar{\alpha}_s A_1} \mathcal{K}_{\mathrm{DLA}}\left(\bar{\rho}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\right)$$

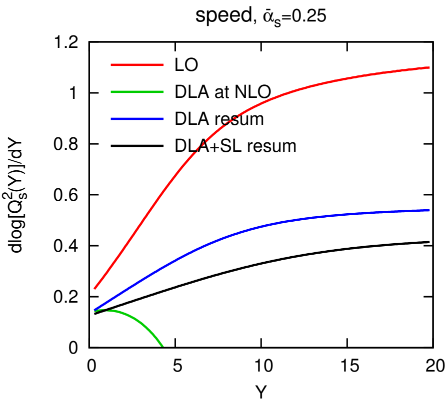
$$A_1 \equiv \frac{11}{12}, \qquad \bar{\rho}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

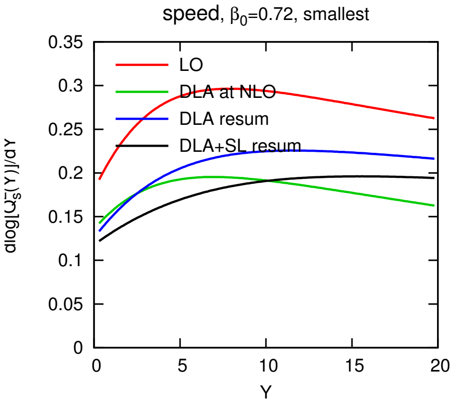
Numerical solutions: saturation front



- Fixed coupling $\bar{\alpha}_s = 0.25$, double collinear logs alone
  - left: expanded to NLO
  - right: resummed to all orders
- The resummation stabilizes & slows down the evolution

Numerical solutions: saturation exponent





- Fixed coupling
  - LO: $\lambda_s \simeq 4.88\bar{\alpha}_s \simeq 1$
  - resummed DL: $\lambda_s \simeq 0.5$
  - DL + SL: $\lambda_s \simeq 0.4$
- Running coupling
  - LO: $\lambda_s = 0.25 \div 0.30$
  - DL + SL: $\lambda_s \simeq 0.2$
  - better convergence

Fitting the HERA data (1)

(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1507.03651)

- Numerical solutions to the collinearly-improved rc BK equation with initial conditions (at $x_0 = 0.01$) which involve free parameters
  - Golec-Biernat Wüsthoff 'saturation model'
  - McLerran-Venugopalan model with fixed coupling
  - McLerran-Venugopalan model with running coupling (rcMV)
- One loop running coupling with various prescriptions :
  - smallest dipole size
  - Balitsky prescription (2006)
  - fastest apparent convergence (Grünberg, 84)
- 3 light quarks + charm quark, all treated on the same footing
  - good quality fits for $m_{u,d,s} = 0 \div 140$ MeV and $m_c = 1.3$ or 1.4 GeV
- 4 (GBW) or 5 (MV) free parameters, including the proton radius

Fitting the HERA data (2)

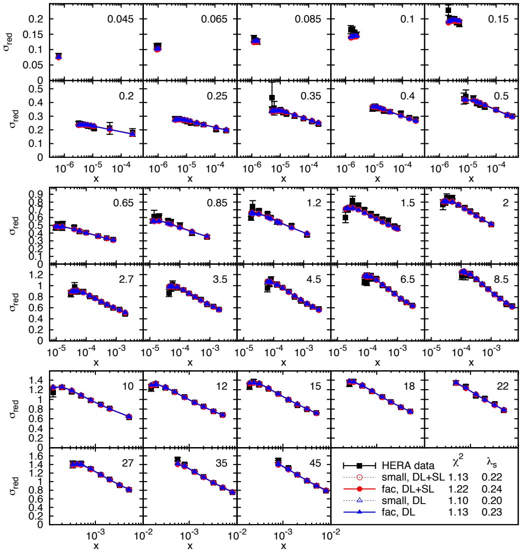
- The most recent HERA data for the reduced photon-proton cross-section (combined analysis by ZEUS and H1)
  - small Bjoerken $x \leq 0.01$
  - $Q^2 < Q^2_{\max}$ with $Q^2_{\max} = 50 \div 400$ GeV^2
- Good quality fits: χ^2 per point around 1.1-1.2
- Very discriminatory: the fits favor
  - rcMV initial condition (pQCD + saturation)
  - physical prescriptions for RC: smallest-dipole, FAC
  - physical values for the free parameters
- Very discriminatory: the fits disfavor
  - fixed coupling MV, GBW at high Q^2
  - Balitsky prescriptions for RC
  - 'anomalous dimension' $\gamma > 1$ in the initial condition

The Fit in tables

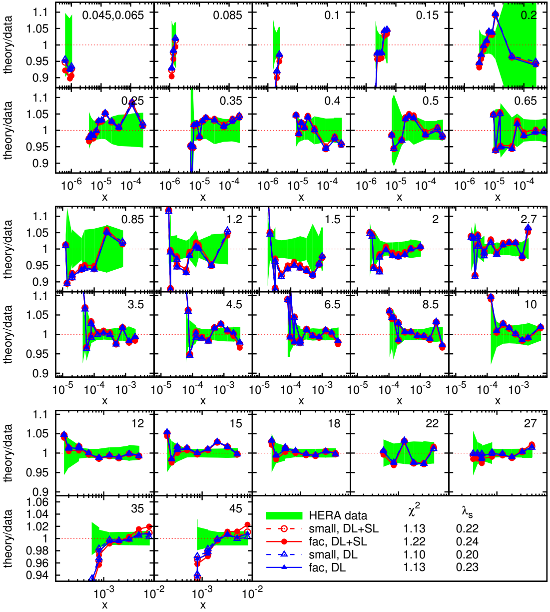
init cdt.	RC schm	sing. logs	χ^2 per data point			parameters					
			$\sigma_{\rm red}$	$\sigma^{cc}_{\rm red}$	F_L	R_p[fm]	Q_0[GeV]	C_α	p	$C_{\rm MV}$
GBW	small	yes	1.135	0.552	0.596	0.699	0.428	2.358	2.802	-
GBW	fac	yes	1.262	0.626	0.602	0.671	0.460	0.479	1.148	-
rcMV	small	yes	1.126	0.578	0.592	0.711	0.530	2.714	0.456	0.896
rcMV	fac	yes	1.222	0.658	0.595	0.681	0.566	0.517	0.535	1.550
GBW	small	no	1.121	0.597	0.597	0.716	0.414	6.428	4.000	-
GBW	fac	no	1.164	0.609	0.594	0.697	0.429	1.195	4.000	-
rcMV	small	no	1.097	0.557	0.593	0.723	0.497	7.393	0.477	0.816
rcMV	fac	no	1.128	0.573	0.591	0.703	0.526	1.386	0.502	1.015

init cdt.	RC schm	sing. logs	χ^2/npts for $Q^2_{\rm max}$			
			50	100	200	400
GBW	small	yes	1.135	1.172	1.355	1.537
GBW	fac	yes	1.262	1.360	1.654	1.899
rcMV	small	yes	1.126	1.172	1.167	1.158
rcMV	fac	yes	1.222	1.299	1.321	1.317
GBW	small	no	1.121	1.131	1.317	1.487
GBW	fac	no	1.164	1.203	1.421	1.622
rcMV	small	no	1.097	1.128	1.095	1.078
rcMV	fac	no	1.128	1.177	1.150	1.131

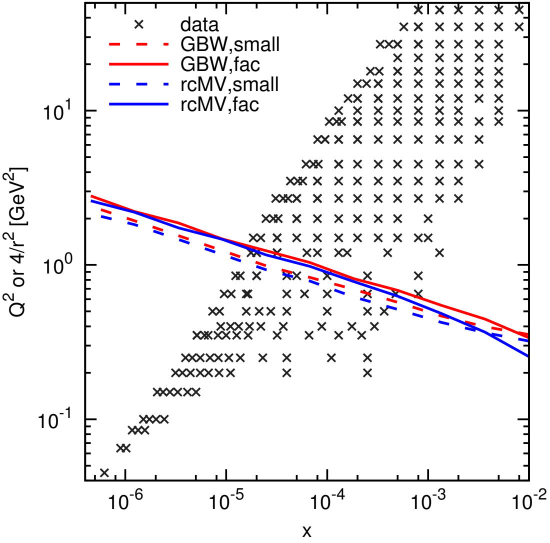
The Fit in plots: rcMV initial condition



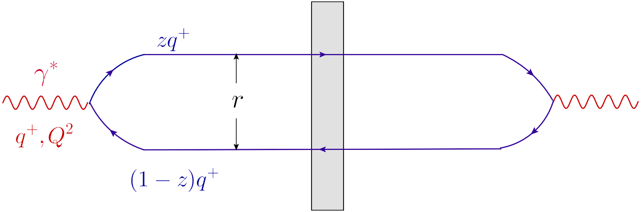
The Fit in plots: rcMV initial condition



The Fit in plots: rcMV initial condition



- Saturation line $Q_s^2(x)$ together with the experimental data points



$$\sigma_{\gamma^* p}(Q^2, x) = 2\pi R_p^2 \sum_f \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \left|\Psi_f(r, z; Q^2)\right|^2 T(r, x)$$

$$x = \frac{Q^2}{2P \cdot q} \simeq \frac{Q^2}{s} \ll 1 \quad \text{(Bjoerken' } x\text{)}$$

- $T(r, x)$: scattering amplitude for a $q\bar{q}$ color dipole with transverse size r
  - $r^2 \sim 1/Q^2$: the resolution of the dipole in the transverse plane
  - x : longitudinal fraction of a gluon from the target that scatters

Fitting the HERA data: initial conditions

- Use numerical solutions to collinearly-improved running-coupling BK equation using initial conditions which involve free parameters
  - a similar strategy as for the DGLAP fits
- Various choices for the initial condition at $x_0 = 0.01$:

$$\text{GBW :} \qquad T(Y_0, r) = \left\{1 - \exp\left[-\left(\frac{r^2 Q_0^2}{4}\right)^p\right]\right\}^{1/p}$$

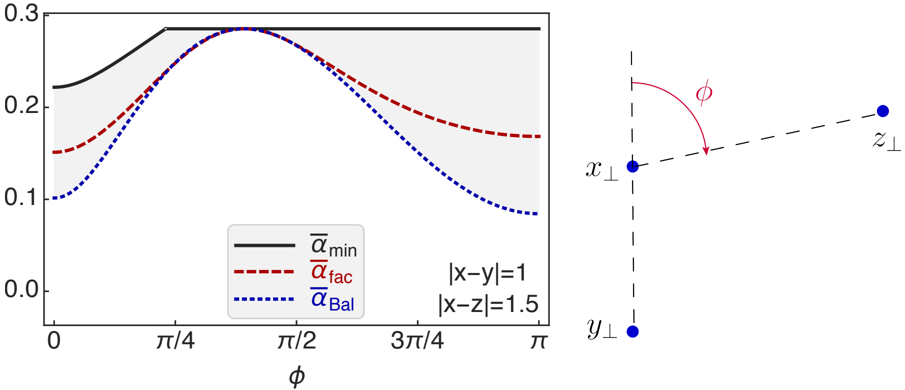
$$\text{rcMV :} \quad T(Y_0, r) = \left\{1 - \exp\left[-\left(\frac{r^2 Q_0^2}{4}\,\bar{\alpha}_s(C_{\rm MV} r)\left[1 + \ln\left(\frac{\bar{\alpha}_{\rm sat}}{\bar{\alpha}_s(C_{\rm MV} r)}\right)\right]\right)^p\right]\right\}^{1/p}$$

- One loop running coupling with scale $\mu = 2C_\alpha / r$:

$$\alpha_s(r) = \frac{1}{b_0 \ln\left[4C_\alpha^2/(r^2\Lambda^2)\right]}, \quad \text{with} \quad r = \min\{|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{x}-\boldsymbol{z}|, |\boldsymbol{y}-\boldsymbol{z}|\}$$

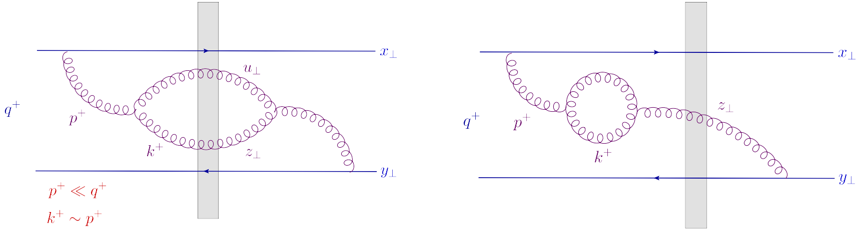
- Up to 5 free parameters: R_p (proton radius), Q_0, p, C_α, ($C_{\rm MV}$)

Prescriptions for running coupling



Next-to-leading order

- Any effect of $\mathcal{O}(\bar{\alpha}_s^2 Y) \Longrightarrow \mathcal{O}(\bar{\alpha}_s)$ correction to the BFKL kernel



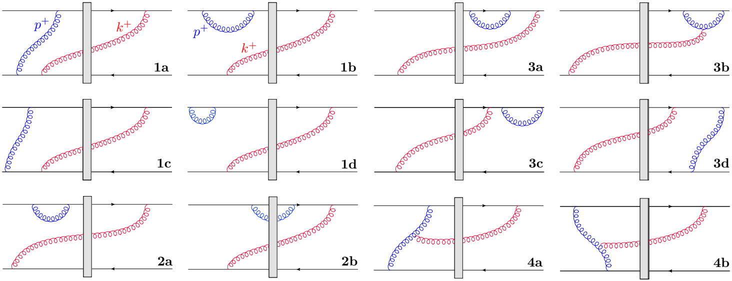
- The prototype: two successive emissions, one soft and one non-soft
- The maximal contribution thus expected: 'BFKL' $\times$ 'DGLAP'

  $(\bar{\alpha}_s Y \rho) \times (\bar{\alpha}_s \rho) \sim \bar{\alpha}_s^2 Y \rho^2$, with $\rho \equiv \ln(Q^2/Q_s^2)$
- ... but in fact one finds an even larger effect $\mathcal{O}(\bar{\alpha}_s^2 Y \rho^3)$

  $\Longrightarrow \mathcal{O}(\bar{\alpha}_s \rho^2)$ correction to the BFKL kernel : 'double collinear log'

  (Fadin, Kotsky, Lipatov, Camici, Ciafaloni ... 95-98; Balitsky, Chirilli, 07)

The Anti-Time-Ordered graphs

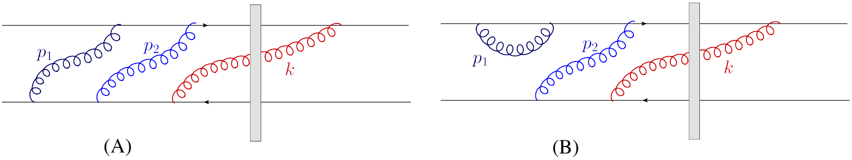


$$\frac{\tau_p}{\tau_p + \tau_k} \longrightarrow \frac{\tau_k}{\tau_p + \tau_k} \simeq \Theta(\tau_k - \tau_p)$$

- The softer gluon k^+ lives longer than the harder one p^+
- The DLA terms exactly cancel in the sum of all the ATO graphs
  - IR logs cancel between vertex (1a) and self-energy (1b) corrections
  - ‘virtual’ (2a) cancel against ‘real’ (2b) since hard gluon is not measured

More gluons

- 3 successive emissions strictly anti-time-ordered



$$q^+ \gg p_1^+ \gg p_2^+ \gg k^+ \,; \qquad r \ll u_1 \ll u_2 \ll z \,; \qquad \tau_1 \ll \tau_2 \ll \tau_k$$

- Integrating gluon 1 for fixed 2 and 3 $\Rightarrow$ double logs for single graphs

$$\bar{\alpha}_s \int_{r^2}^{u_2^2} \frac{\mathrm{d}u_1^2}{u_1^2} \int_{p_2^+}^{q^+} \frac{\mathrm{d}p_1^+}{p_1^+} \,\Theta(p_2^+ u_2^2 - p_1^+ u_1^2) \;=\; \frac{\bar{\alpha}_s}{2} \ln^2 \frac{u_2^2}{r^2}$$

- ... but they cancel between graphs A ('vertex') and B ('self-energy')

Adding the single transverse logarithms

- Recall the NLO equation with all the single logs

$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} = \bar{\alpha}_s \int \mathrm{d}z^2\, \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left(\frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} - \bar{b}\, \ln r^2 \mu^2 \right) \right\} T(z)$$

- The double-logarithm is already included within $\mathcal{K}_{\rm DLA}(\rho)$ ✓
- The collinear single-log comes from the DGLAP regime:
  - one soft $(x \ll 1)$ emission + one non-soft $(x \sim 1)$ one
  - coefficient $A_1 = 11/12$ related to the DGLAP anomalous dimension:

$$\gamma(\omega) = \int_0^1 \mathrm{d}z\, z^{\omega} \left[P_{\rm gg}(z) + \frac{C_{\rm F}}{N_{\rm c}}\, P_{\rm qg}(z) \right] = \frac{1}{\omega} - A_1 + \mathcal{O}\left(\omega, \frac{N_{\rm f}}{N_{\rm c}^3} \right)$$

  - can be resummed by including the A_1 piece of $\gamma(\omega)$ into $\mathcal{K}_{\rm DLA}(\rho)$ ✓
- The running coupling log is resummed by replacing $\bar{\alpha}_s \to \bar{\alpha}_s(r^2)$ ✓

Solving DLA 2.0

- Easier: the 'unintegrated gluon distribution' $f(Y,\rho)$ of the dipole:
  - same equation but initial condition $f(0,\rho) = \delta(\rho)$
  - in turn, this determines the dipole amplitude via:

$$\mathcal{A}(Y,\rho) = \int_0^{\rho} \mathrm{d}\rho_1\, f(Y,\rho-\rho_1)\mathcal{A}(0,\rho_1)$$

  - straightforward to solve via iterations:

$$f(Y,\rho) = \delta(\rho) + \Theta(Y-\rho) \sum_{k=1}^{\infty} \frac{\bar{\alpha}_s^k (Y-\rho)^k \rho^{k-1}}{k!(k-1)!}$$

  - the sum is recognized as a modified Bessel function I_1
- The function $f(Y,\rho)$ admits an analytic continuation $\tilde{f}(Y,\rho)$, which obeys an evolution equation local in Y ...
  - .. but with resummed kernel and initial conditions !

Getting local

- $f(Y,\rho) = \Theta(Y-\rho)\tilde{f}(Y,\rho)$, with $\tilde{f}(Y,\rho)$ defined by

$$\tilde{f}(Y,\rho) \equiv \int_{\frac{1}{2}-\mathrm{i}\infty}^{\frac{1}{2}+\mathrm{i}\infty} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \exp\left[\frac{\bar{\alpha}_s}{1-\xi}(Y-\rho) + (1-\xi)\rho\right]$$

for any positive Y and ρ (including $Y < \rho$).

- Mellin representation up to a change of variables: $\gamma = \xi + \bar{\alpha}_s/(1-\xi)$

$$\tilde{f}(Y,\rho) = \int_{\frac{1}{2}-\mathrm{i}\infty}^{\frac{1}{2}+\mathrm{i}\infty} \frac{\mathrm{d}\gamma}{2\pi\mathrm{i}} J(\gamma) \exp\left[\bar{\alpha}_s \chi_{\mathrm{DLA}}(\gamma) Y + (1-\gamma)\rho\right]$$

- Characteristic function $\chi_{\mathrm{DLA}}(\gamma)$ and Jacobian $J(\gamma) = 1 - \bar{\alpha}_s\, \chi'_{\mathrm{DLA}}(\gamma)$

$$\bar{\alpha}_s \chi_{\mathrm{DLA}}(\gamma) = \frac{1}{2}\left[-(1-\gamma) + \sqrt{(1-\gamma)^2 + 4\bar{\alpha}_s}\right]$$

Getting local

- $f(Y,\rho) = \Theta(Y-\rho)\tilde{f}(Y,\rho)$, with $\tilde{f}(Y,\rho)$ defined by

$$\tilde{f}(Y,\rho) \equiv \int_{\frac{1}{2}-\mathrm{i}\infty}^{\frac{1}{2}+\mathrm{i}\infty} \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \exp\left[\frac{\bar{\alpha}_s}{1-\xi}(Y-\rho) + (1-\xi)\rho\right]$$

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- Characteristic function $\chi_{\mathrm{DLA}}(\gamma)$ and Jacobian $J(\gamma) = 1 - \bar{\alpha}_s\, \chi'_{\mathrm{DLA}}(\gamma)$

$$\bar{\alpha}_s \chi_{\mathrm{DLA}}(\gamma) = \frac{\bar{\alpha}_s}{(1-\gamma)} - \frac{\bar{\alpha}_s^2}{(1-\gamma)^3} + \frac{2\bar{\alpha}_s^3}{(1-\gamma)^5} + \cdots$$