Lattice QCD calculations
of transverse momentum-dependent parton distributions (TMDs)

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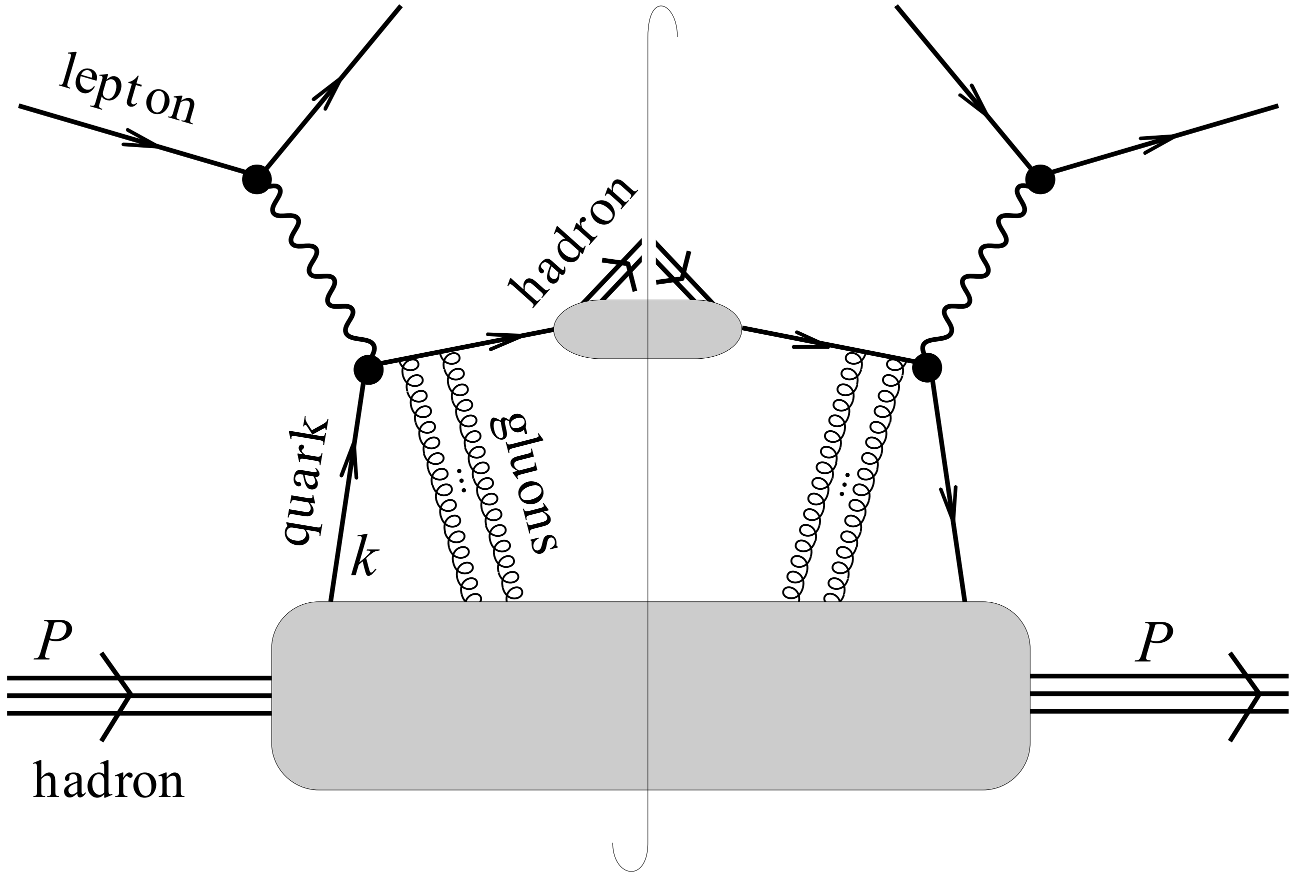
Fundamental TMD correlator

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\ldots) \equiv \frac{1}{2}\langle P,S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0,\ldots,b]\ q(b)\ |P,S\rangle$$

$$\Phi^{[\Gamma]}(x,k_T,P,S,\ldots) \equiv \int \frac{d^2b_T}{(2\pi)^2} \int \frac{d(b\cdot P)}{(2\pi)P^+} \exp\left(ix(b\cdot P) - ib_T\cdot k_T\right) \left.\frac{\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b,P,S,\ldots)}{\widetilde{\mathcal{S}}(b^2,\ldots)}\right|_{b^+=0}$$

- "Soft factor" $\widetilde{\mathcal{S}}$ required to subtract divergences of Wilson line $\mathcal{U}$
- $\widetilde{\mathcal{S}}$ is typically a combination of vacuum expectation values of Wilson line structures
- Here, will consider only ratios in which soft factors cancel

Gauge link structure motivated by SIDIS



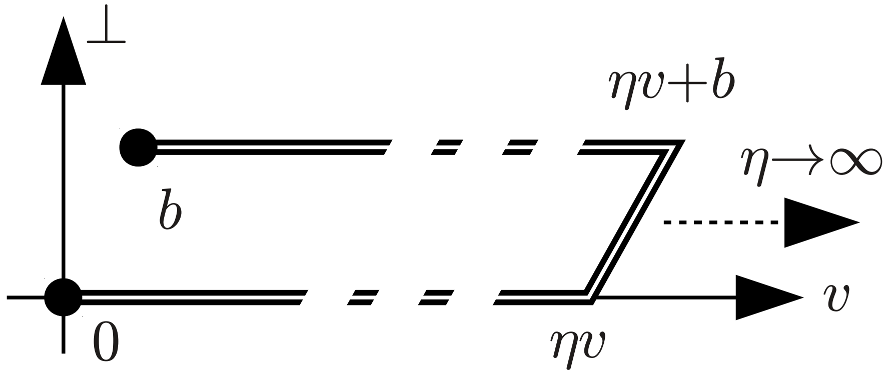
$l + H(P) \longrightarrow l' + h(P_h) + X$

Gauge link structure:

In matrix element $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \equiv$

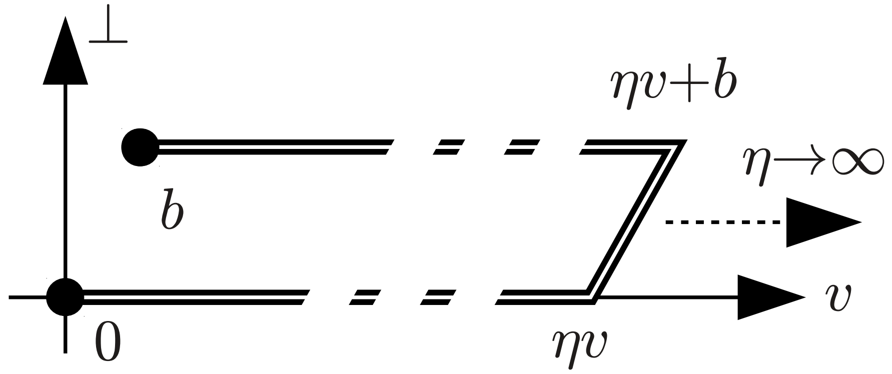
$\frac{1}{2}\langle P, S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \ldots, b]\ q(b)\ |P, S\rangle$

Staple-shaped gauge link $\mathcal{U}[0, \eta v, \eta v + b, b]$



incorporates SIDIS final state effects

Gauge link structure motivated by SIDIS



Beyond tree level: Rapidity divergences suggest taking staple direction slightly off the light cone. Approach of Aybat, Collins, Qiu, Rogers makes v space-like. Parametrize in terms of Collins-Soper parameter

$$\hat{\zeta} \equiv \frac{P\cdot v}{|P||v|}$$

Light-like staple for $\hat{\zeta}\to\infty$. Perturbative evolution equations for large $\hat{\zeta}$.

Fundamental TMD correlator

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S |\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ | P, S \rangle$$

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- "Soft factor" $\widetilde{\mathcal{S}}$ required to subtract divergences of Wilson line $\mathcal{U}$
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Decomposition of Φ into TMDs

All leading twist structures:

$$\Phi^{[\gamma^+]} = f_1 - \left[\frac{\epsilon_{ij} k_i S_j}{m_H} f_{1T}^{\perp}\right] \text{odd}$$

$$\Phi^{[\gamma^+\gamma^5]} = \Lambda g_1 + \frac{k_T \cdot S_T}{m_H} g_{1T}$$

$$\Phi^{[i\sigma^{i+}\gamma^5]} = S_i h_1 + \frac{(2k_i k_j - k_T^2 \delta_{ij}) S_j}{2m_H^2} h_{1T}^{\perp} + \frac{\Lambda k_i}{m_H} h_{1L}^{\perp} + \left[\frac{\epsilon_{ij} k_j}{m_H} h_1^{\perp}\right] \text{odd}$$

TMD Classification

All leading twist structures:

$q \longrightarrow$ H $\downarrow$	U	L	T
U	f_1		$h_1^\perp$
L		g_1	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	h_1 $h_{1T}^\perp$

$\longleftarrow$ Boer-Mulders (T-odd)

$\uparrow$ Sivers (T-odd)

Decomposition of $\widetilde{\Phi}$ into amplitudes

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2}\langle P, S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ |P, S\rangle$$

Decompose in terms of invariant amplitudes; at leading twist,

$$\begin{aligned}
\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} &= \widetilde{A}_{2B} + im_H\epsilon_{ij}b_iS_j\widetilde{A}_{12B} \\
\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+\gamma^5]}_{\text{unsubtr.}} &= -\Lambda\widetilde{A}_{6B} + i[(b\cdot P)\Lambda - m_H(b_T\cdot S_T)]\widetilde{A}_{7B} \\
\frac{1}{2P^+}\widetilde{\Phi}^{[i\sigma^{i+}\gamma^5]}_{\text{unsubtr.}} &= im_H\epsilon_{ij}b_j\widetilde{A}_{4B} - S_i\widetilde{A}_{9B} \\
&\quad -im_H\Lambda b_i\widetilde{A}_{10B} + m_H[(b\cdot P)\Lambda - m_H(b_T\cdot S_T)]b_i\widetilde{A}_{11B}
\end{aligned}$$

(Decompositions analogous to work by Metz et al. in momentum space)

Fourier-transformed TMDs

$$\tilde{f}(x, b_T^2, \ldots) \equiv \int d^2k_T \, \exp(ib_T \cdot k_T) f(x, k_T^2, \ldots)$$

$$\tilde{f}^{(n)}(x, b_T^2, \ldots) \equiv n! \left(-\frac{2}{m_H^2} \partial_{b_T^2} \right)^n \tilde{f}(x, b_T^2, \ldots)$$

In limit $|b_T| \to 0$, recover k_T-moments:

$$\tilde{f}^{(n)}(x, 0, \ldots) \equiv \int d^2k_T \left(\frac{k_T^2}{2m_H^2} \right)^n f(x, k_T^2, \ldots) \equiv f^{(n)}(x)$$

ill-defined for large k_T, so will not attempt to extrapolate to $b_T = 0$, but give results at finite $|b_T|$.

In this study, only consider first x-moments (accessible at $b \cdot P = 0$), rather than scanning range of $b \cdot P$:

$$f^{[1]}(k_T^2, \ldots) \equiv \int_{-1}^{1} dx \, f(x, k_T^2, \ldots)$$

⟶ Bessel-weighted asymmetries (Boer, Gamberg, Musch, Prokudin, JHEP 1110 (2011) 021)

Relation between Fourier-transformed TMDs and invariant amplitudes $\widetilde{A}_i$

Invariant amplitudes directly give selected x-integrated TMDs in Fourier (b_T) space (showing just the ones relevant for Sivers, Boer-Mulders shifts), up to soft factors:

$$\tilde{f}_1^{[1](0)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\widetilde{S}(b^2, \ldots)$$

$$\tilde{f}_{1T}^{\perp[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = -2\widetilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\widetilde{S}(b^2, \ldots)$$

$$\tilde{h}_1^{\perp[1](1)}(b_T^2, \hat{\zeta}, \ldots, \eta v \cdot P) = 2\widetilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)/\widetilde{S}(b^2, \ldots)$$

Generalized shifts

Form ratios in which soft factors, (Γ-independent) multiplicative renormalization factors cancel

Boer-Mulders shift:

$$\langle k_y \rangle_{UT} \equiv m_H \frac{\tilde{h}_1^{\perp [1](1)}}{\tilde{f}_1^{[1](0)}} = \left. \frac{\int dx \int d^2 k_T \, k_y \Phi^{[\gamma^+ + s^j i\sigma^{j+}\gamma^5]}(x, k_T, P, \ldots)}{\int dx \int d^2 k_T \, \Phi^{[\gamma^+ + s^j i\sigma^{j+}\gamma^5]}(x, k_T, P, \ldots)} \right|_{s_T=(1,0)}$$

Average transverse momentum of quarks polarized in the orthogonal transverse ("T") direction in an unpolarized ("U") hadron; normalized to the number of valence quarks. "Dipole moment" in $b_T^2 = 0$ limit, "shift".

Issue: k_T-moments in this ratio singular; generalize to ratio of Fourier-transformed TMDs at *nonzero* b_T^2,

$$\langle k_y \rangle_{UT}(b_T^2, \ldots) \equiv m_H \frac{\tilde{h}_1^{\perp [1](1)}(b_T^2, \ldots)}{\tilde{f}_1^{[1](0)}(b_T^2, \ldots)}$$

(remember singular $b_T \rightarrow 0$ limit corresponds to taking k_T-moment). "Generalized shift".

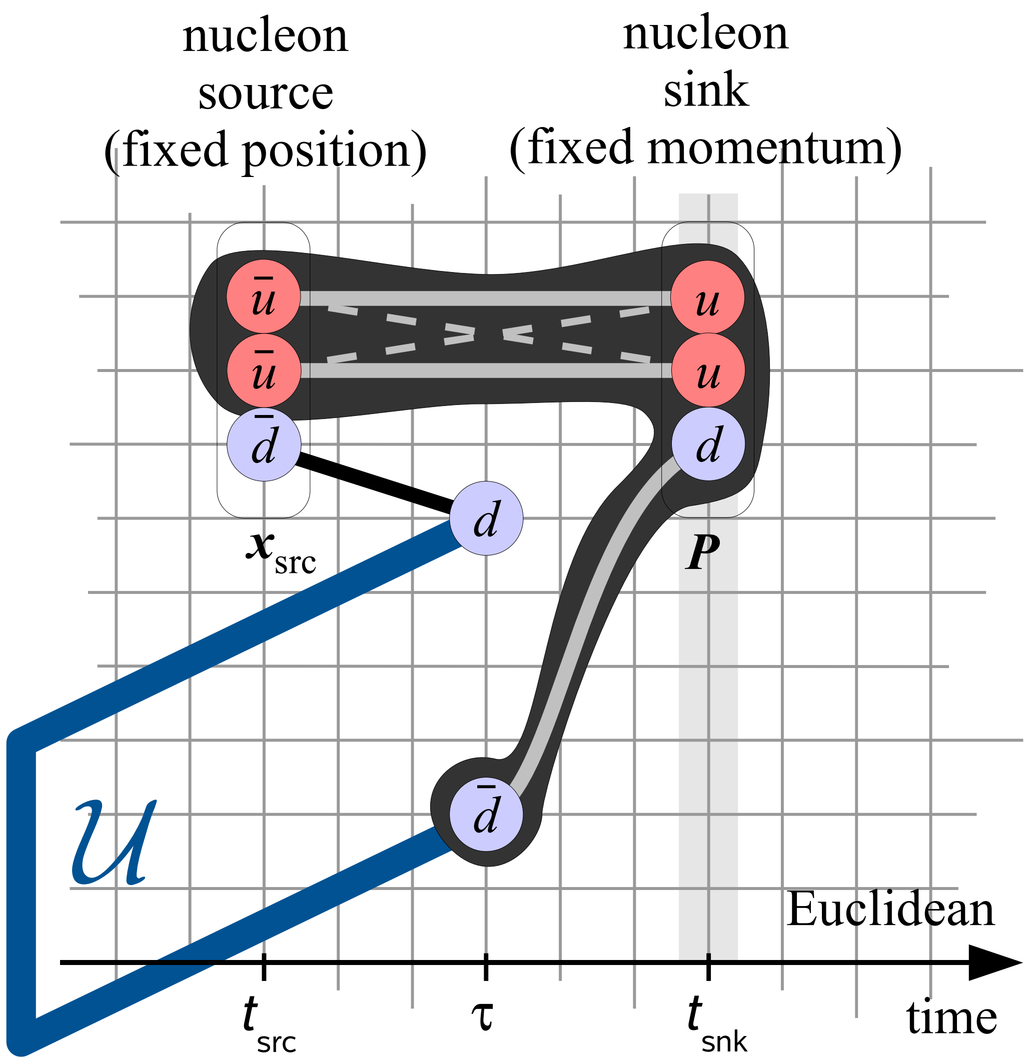
Generalized shifts from amplitudes

Now, can also express this in terms of invariant amplitudes:

$$\langle k_y \rangle_{UT}(b_T^2, \ldots) \equiv m_H \frac{\tilde{h}_1^{\perp[1](1)}(b_T^2, \ldots)}{\tilde{f}_1^{[1](0)}(b_T^2, \ldots)} = m_H \frac{\widetilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$

Analogously, Sivers shift (in a polarized hadron):

$$\langle k_y \rangle_{TU}(b_T^2, \ldots) = -m_H \frac{\widetilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$



Lattice setup

- Evaluate directly $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu)$
  $$\equiv \tfrac{1}{2}\langle P, S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ |P, S\rangle$$
- Euclidean time: Place entire operator at one time slice, i.e., b, ηv purely spatial
- Since generic b, v space-like, no obstacle to boosting system to such a frame!
- Parametrization of correlator in terms of $\widetilde{A}_i$ invariants permits direct translation of results back to original frame; form desired $\widetilde{A}_i$ ratios.
- Use variety of P, b, ηv; here $b \perp P$, $b \perp v$ (lowest x-moment, kinematical choices/constraints)
- Extrapolate $\eta \to \infty$, $\hat{\zeta} \to \infty$ numerically.

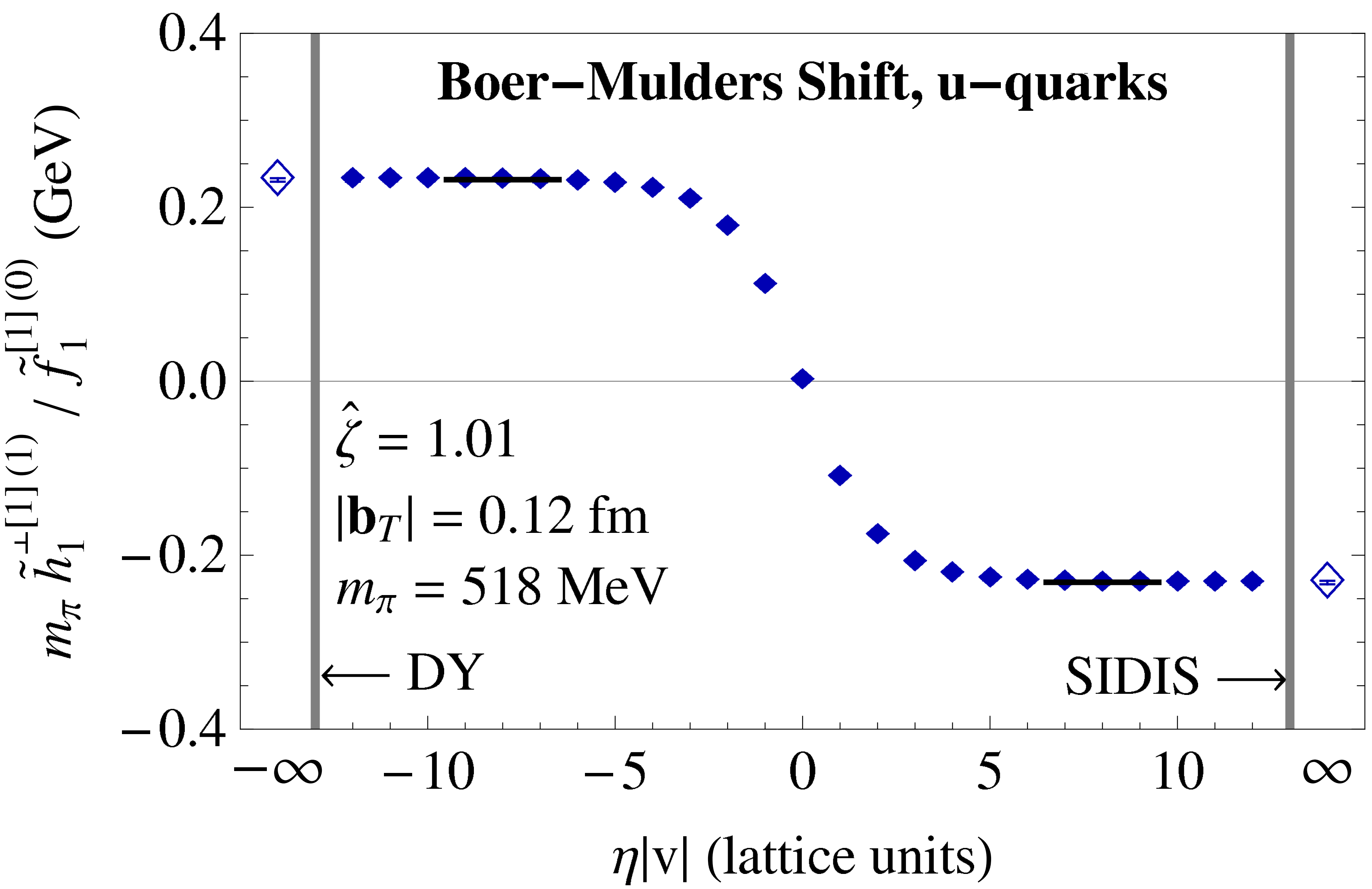
Challenges

- The limit $\hat{\zeta} \to \infty$: Approaching the light cone
- Discretization effects, soft factor cancellation on the lattice in TMD ratios
- Progress toward the physical pion mass

Approaching the light cone (with a pion)

Results: Boer-Mulders shift (pion)

Dependence on staple extent; sequence of panels at different $|b_T|$



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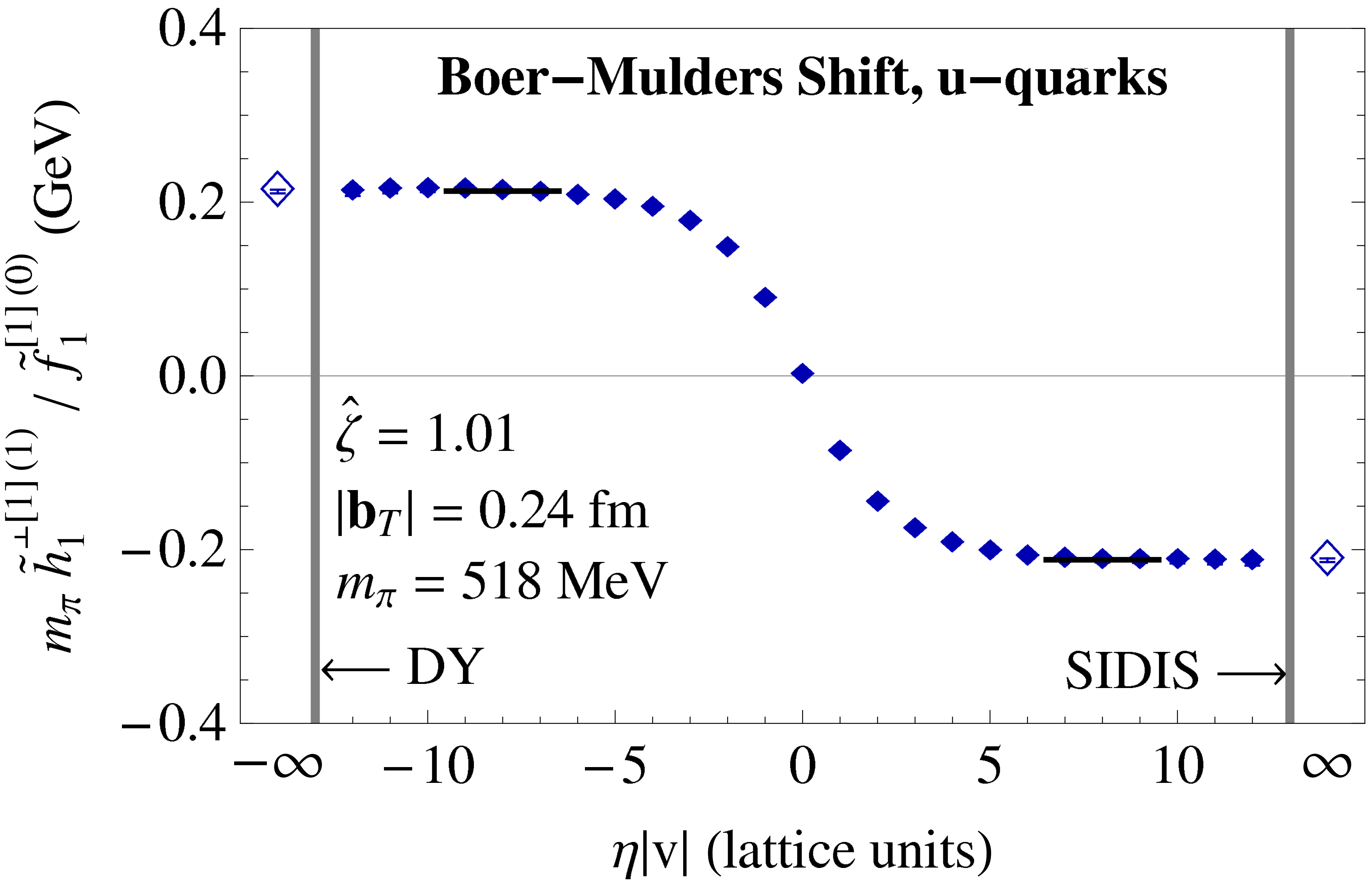

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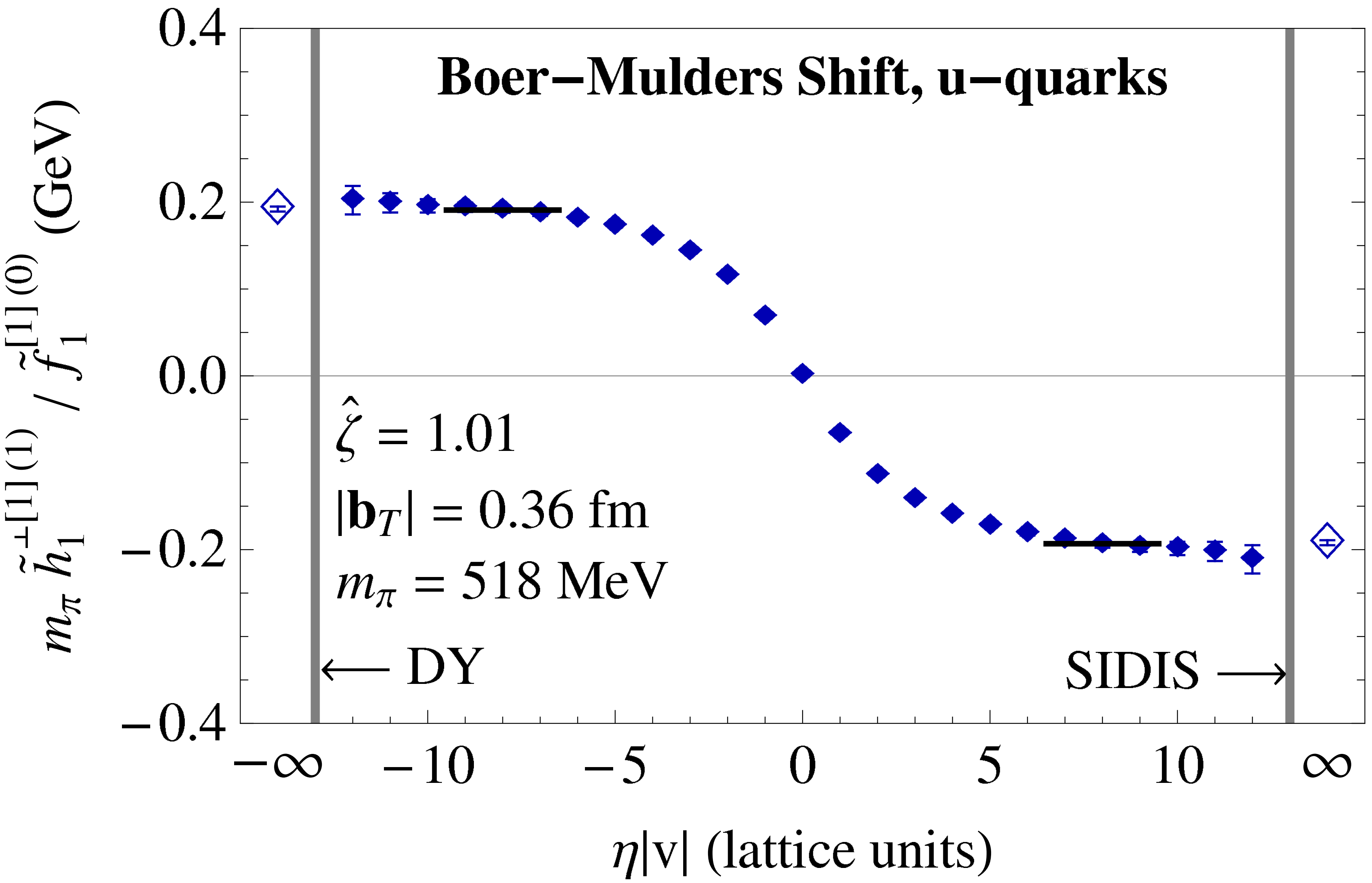

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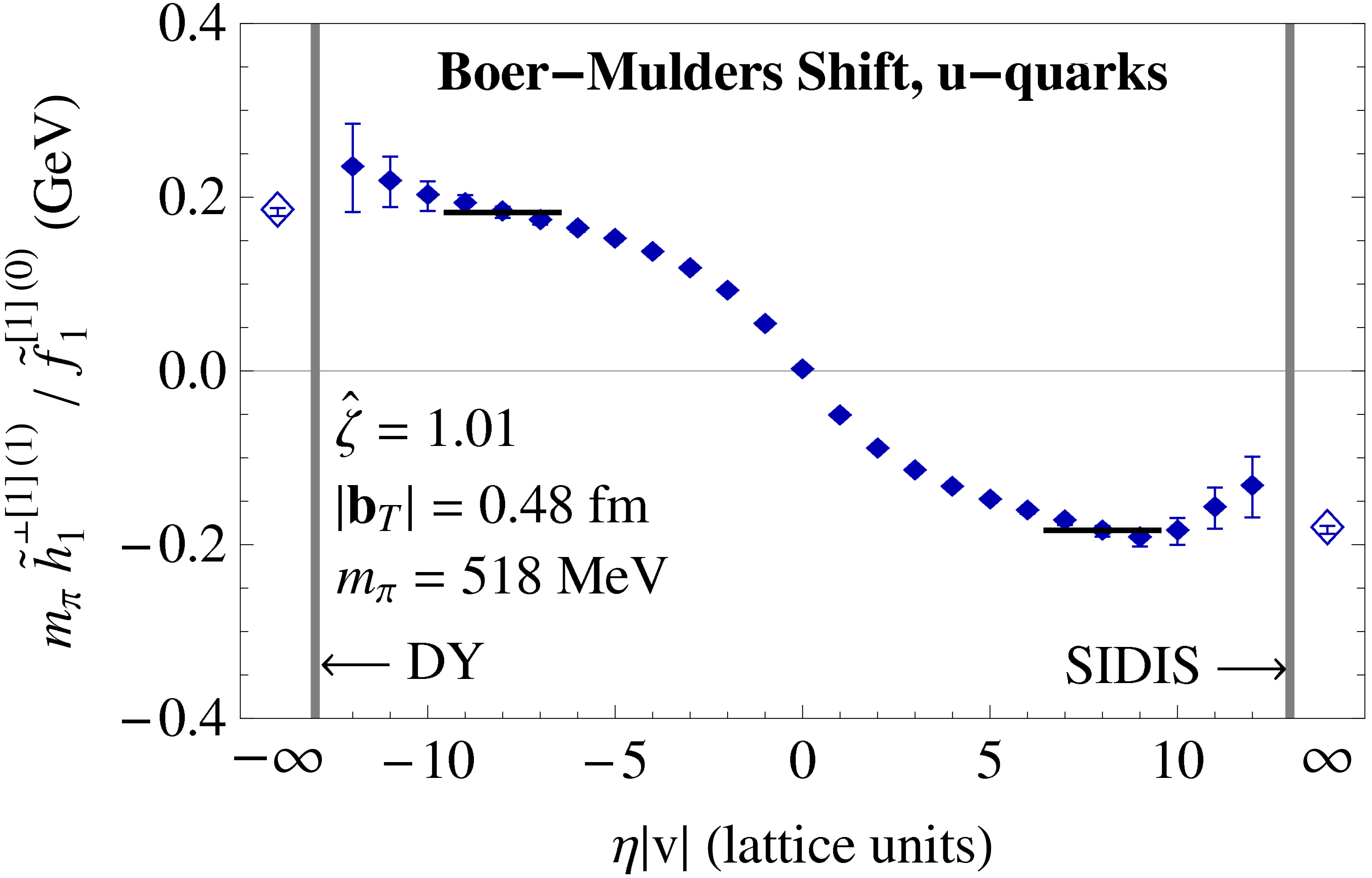

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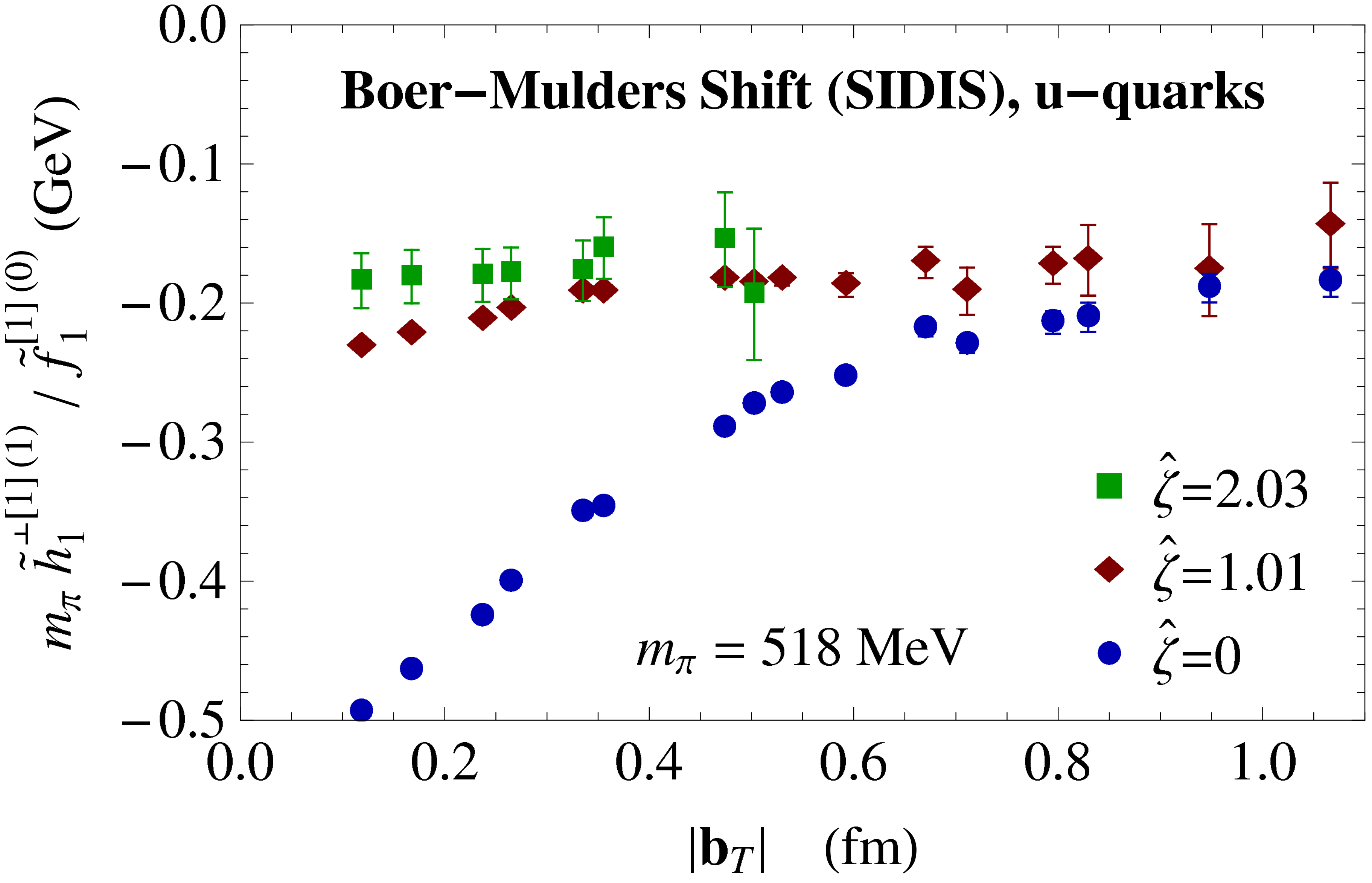
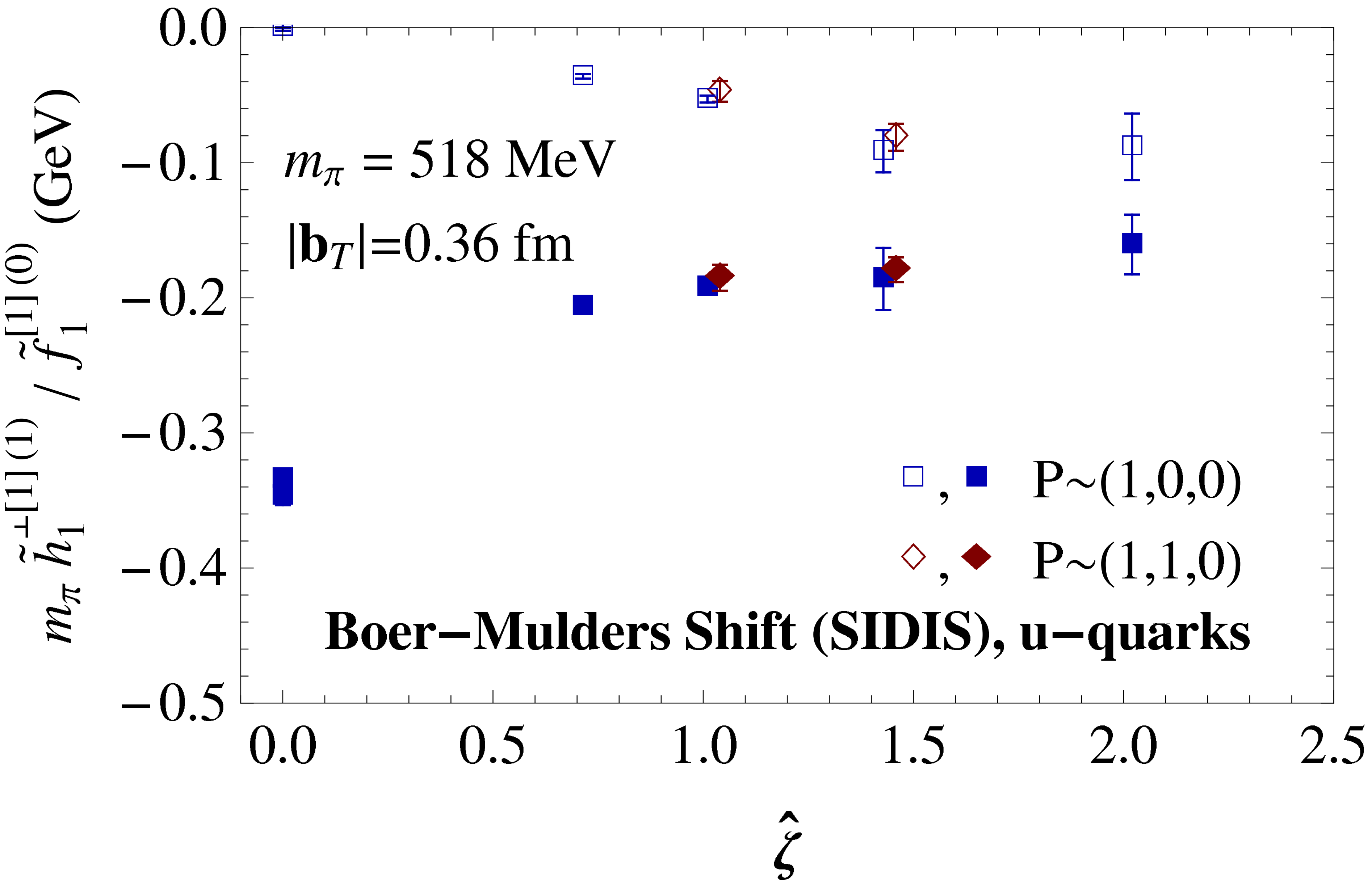
Dependence of SIDIS limit on $|b_T|$



Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on $\hat{\zeta}$; open symbols: contribution $\widetilde{A}_4$ only



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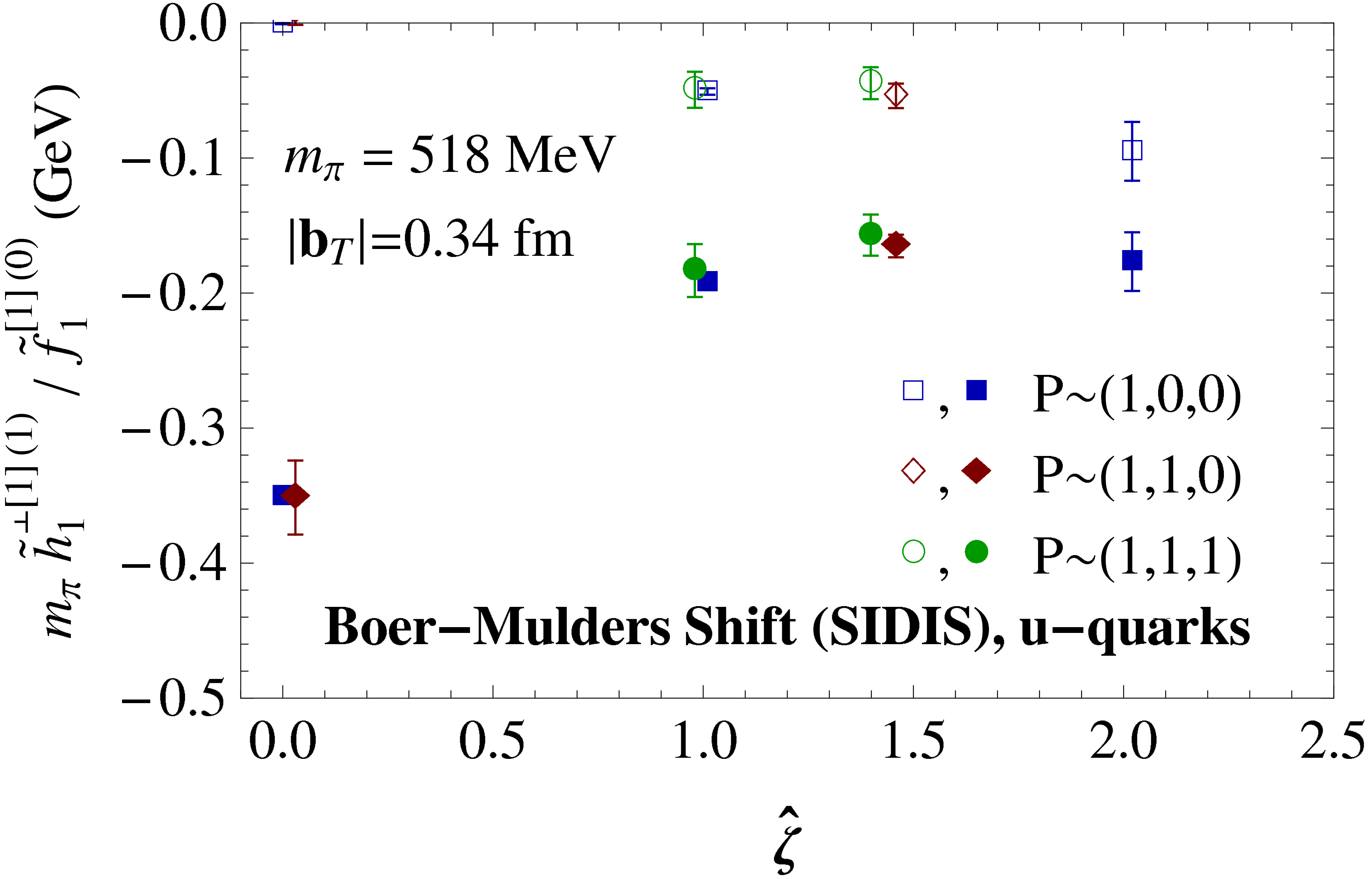
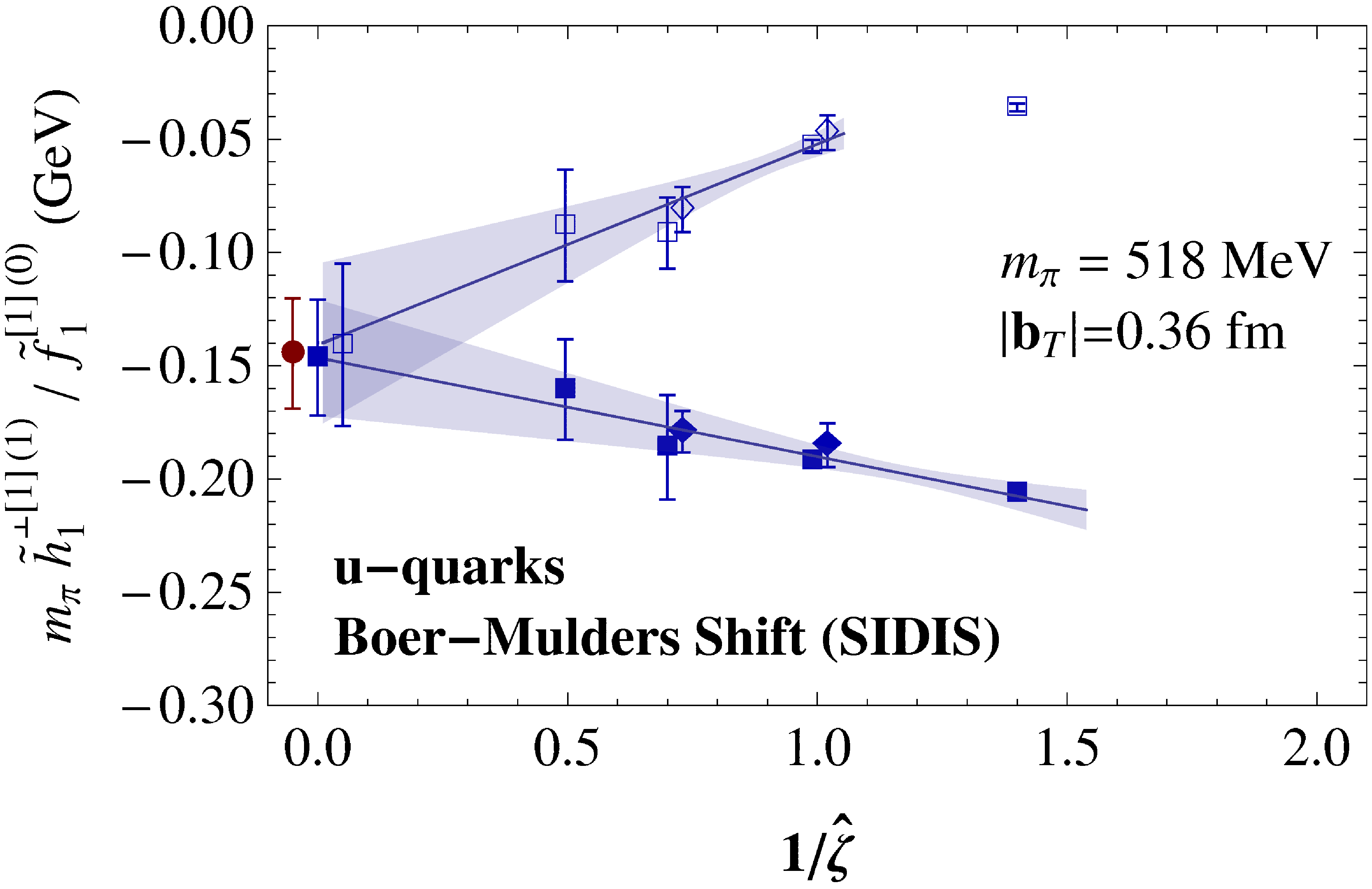


Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on $\hat{\zeta}$; fit function $a + b/\hat{\zeta}$



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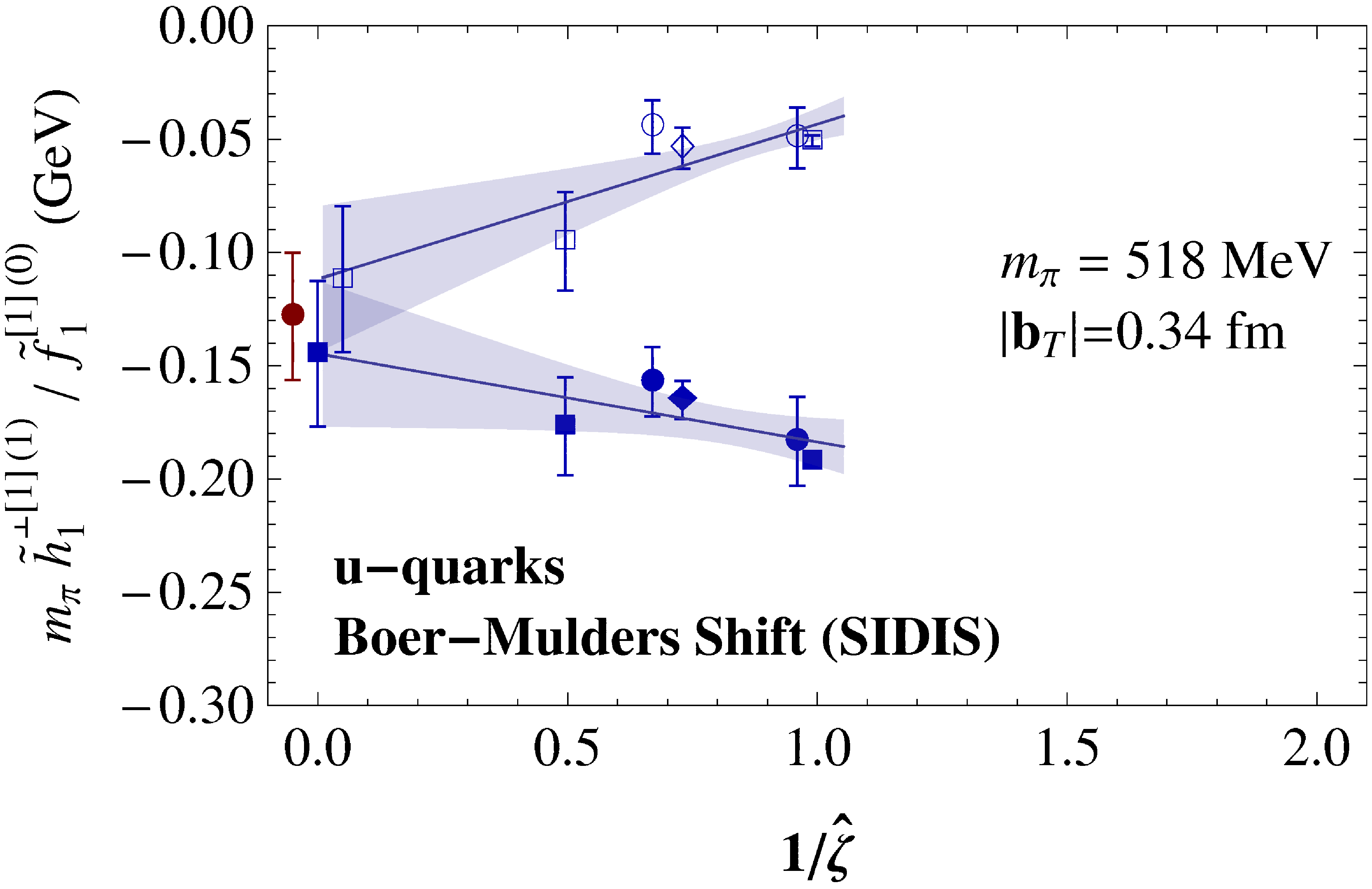


Discretization effects:

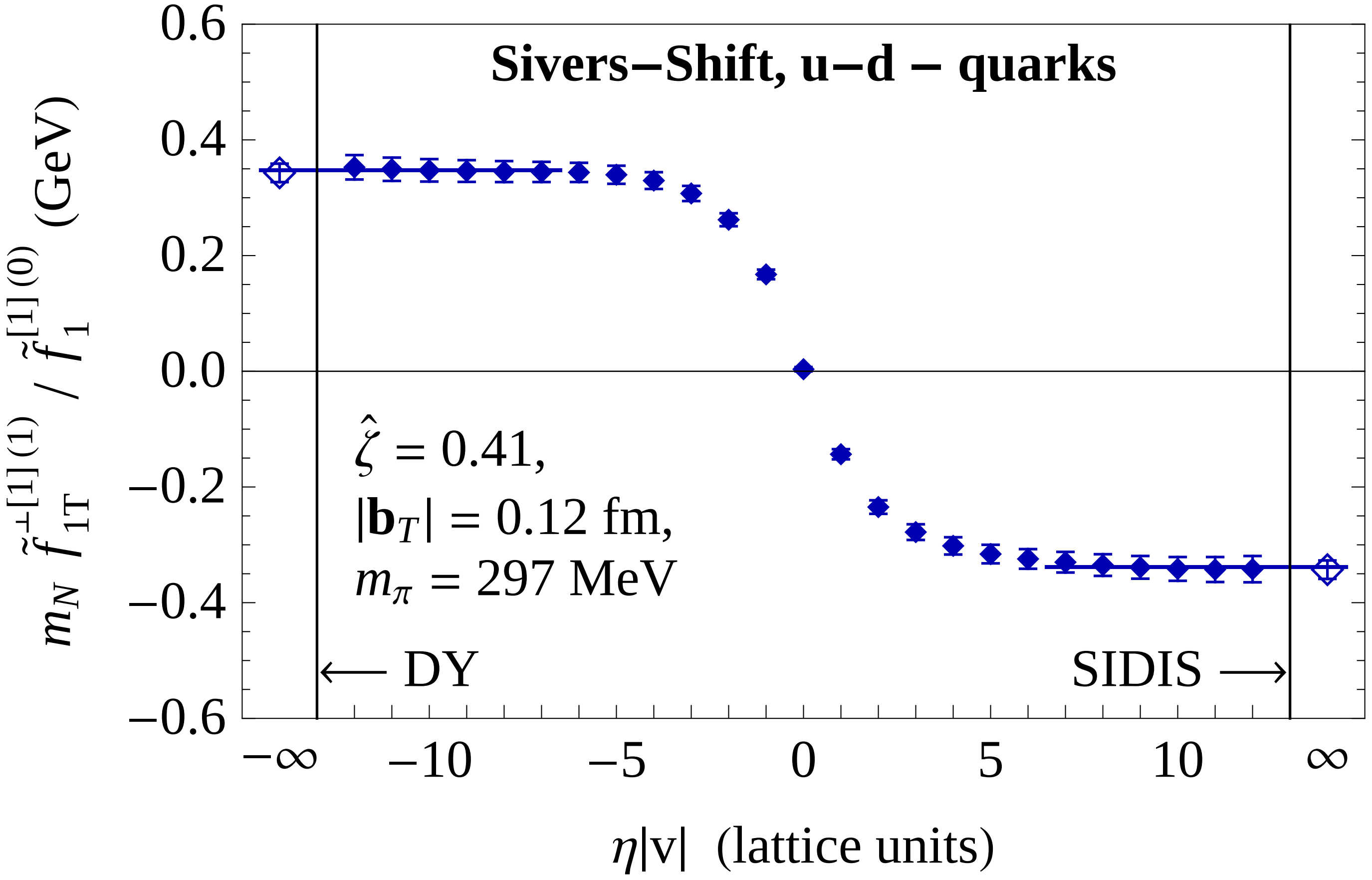
Comparison of

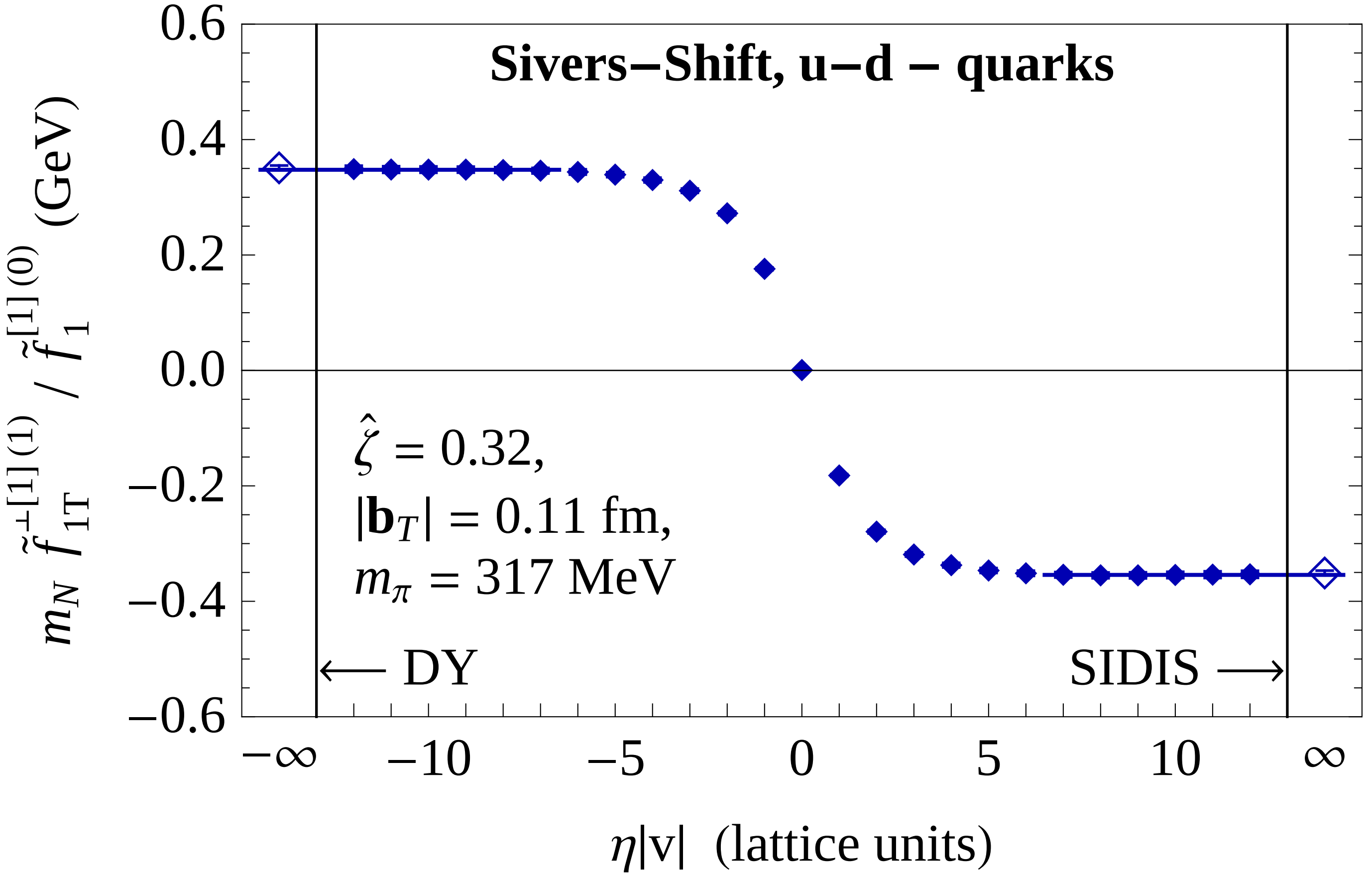
RBC/UKQCD DWF ensemble ($m_\pi = 297\,$MeV, $a = 0.084\,$fm)

with clover ensemble ($m_\pi = 317\,$MeV, $a = 0.114\,$fm)
produced by K. Orginos and JLab collaborators

Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$





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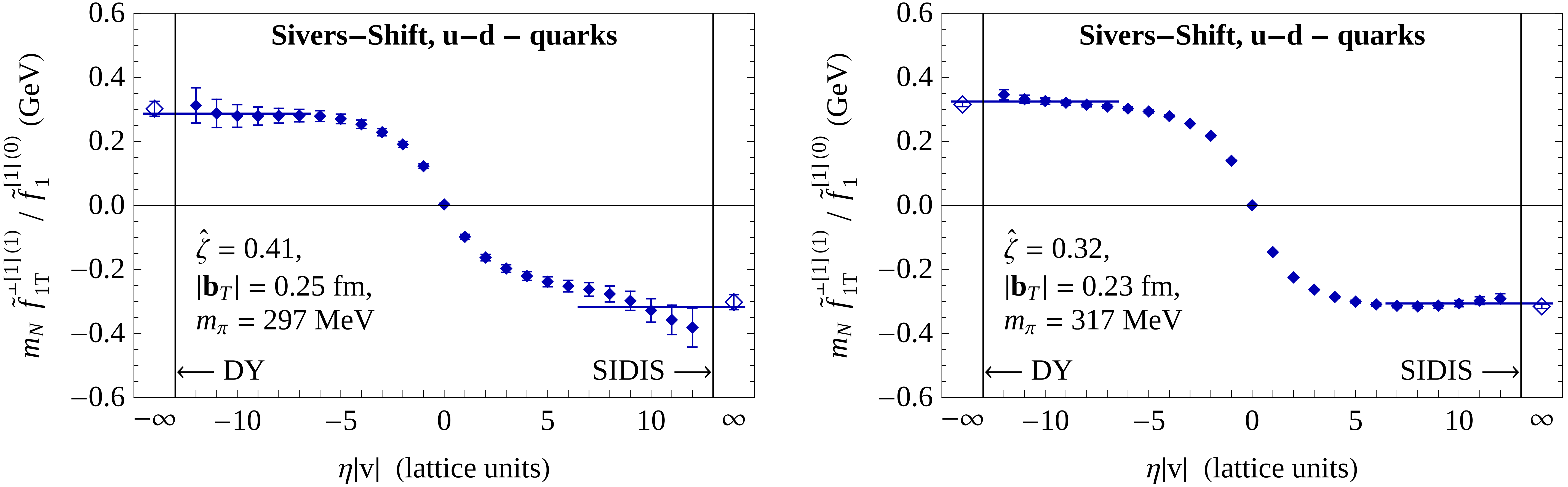

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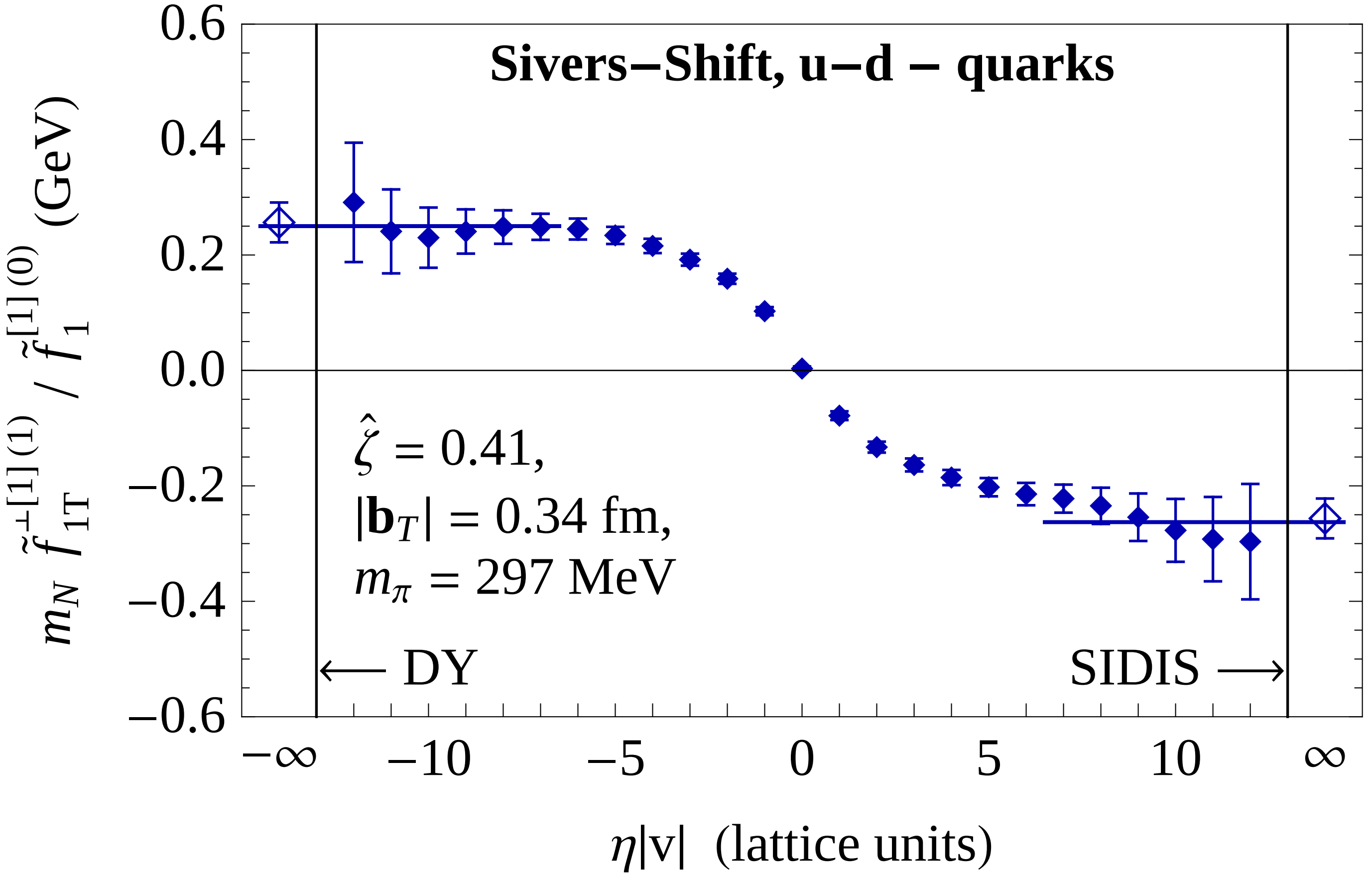
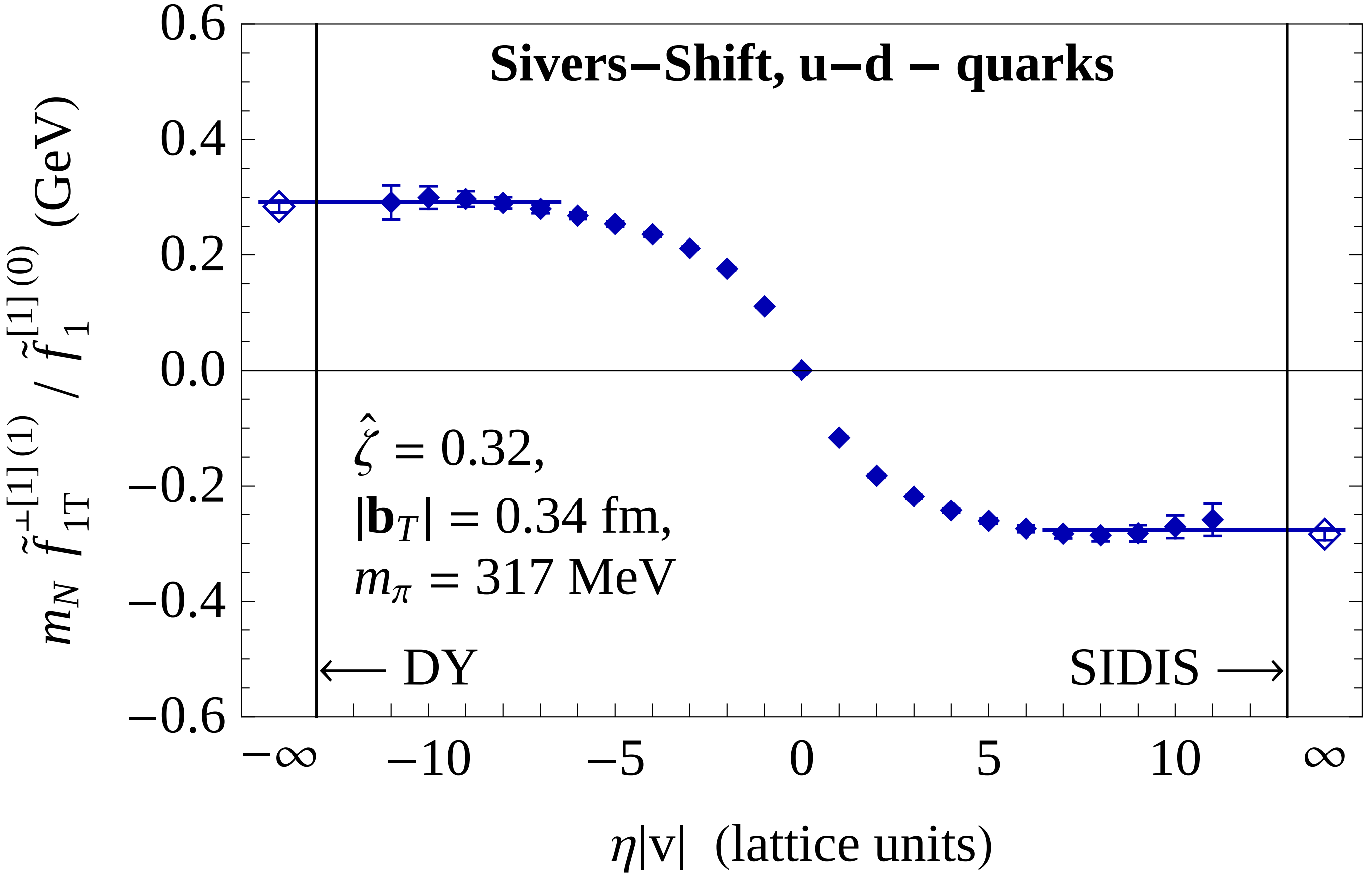
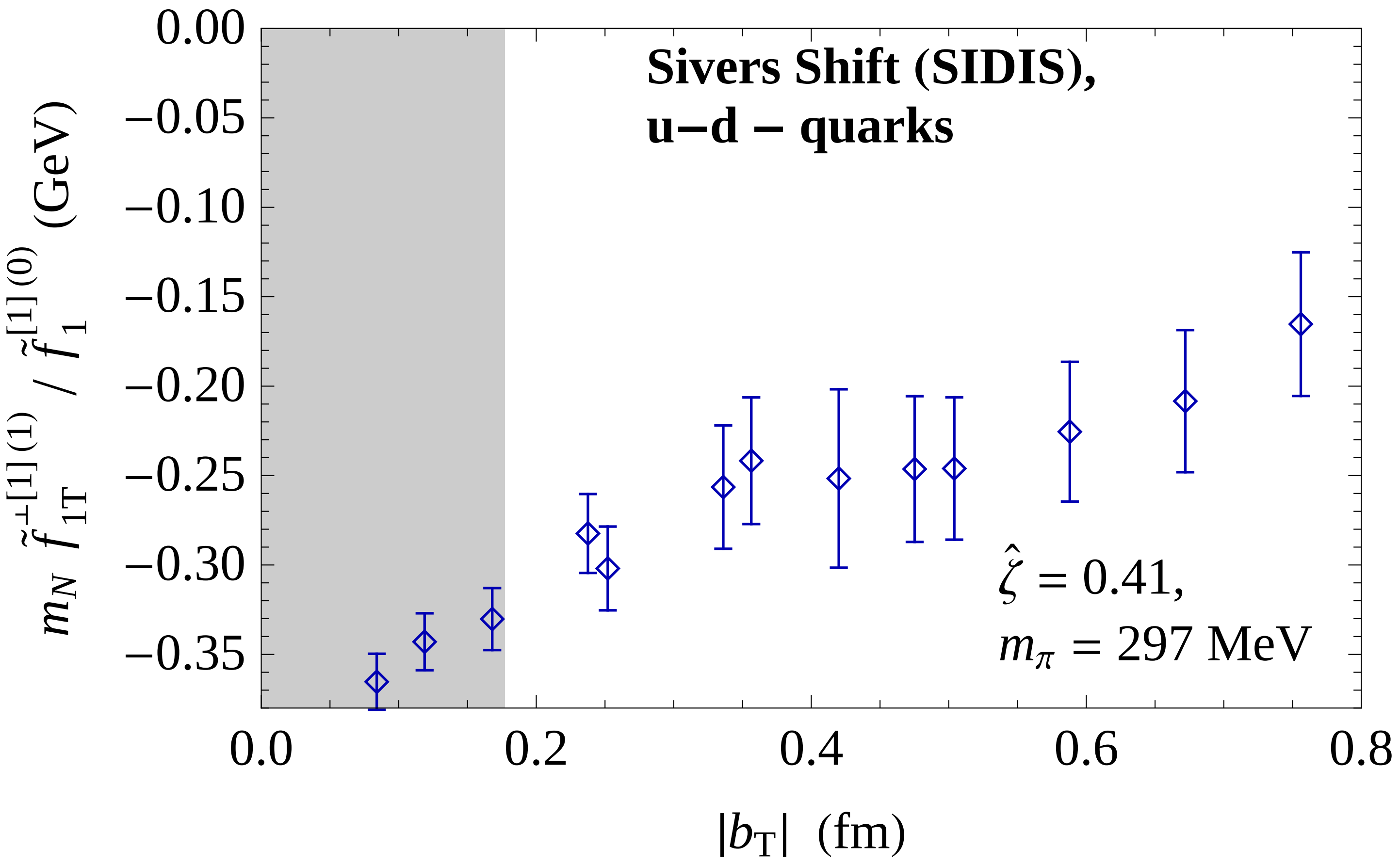
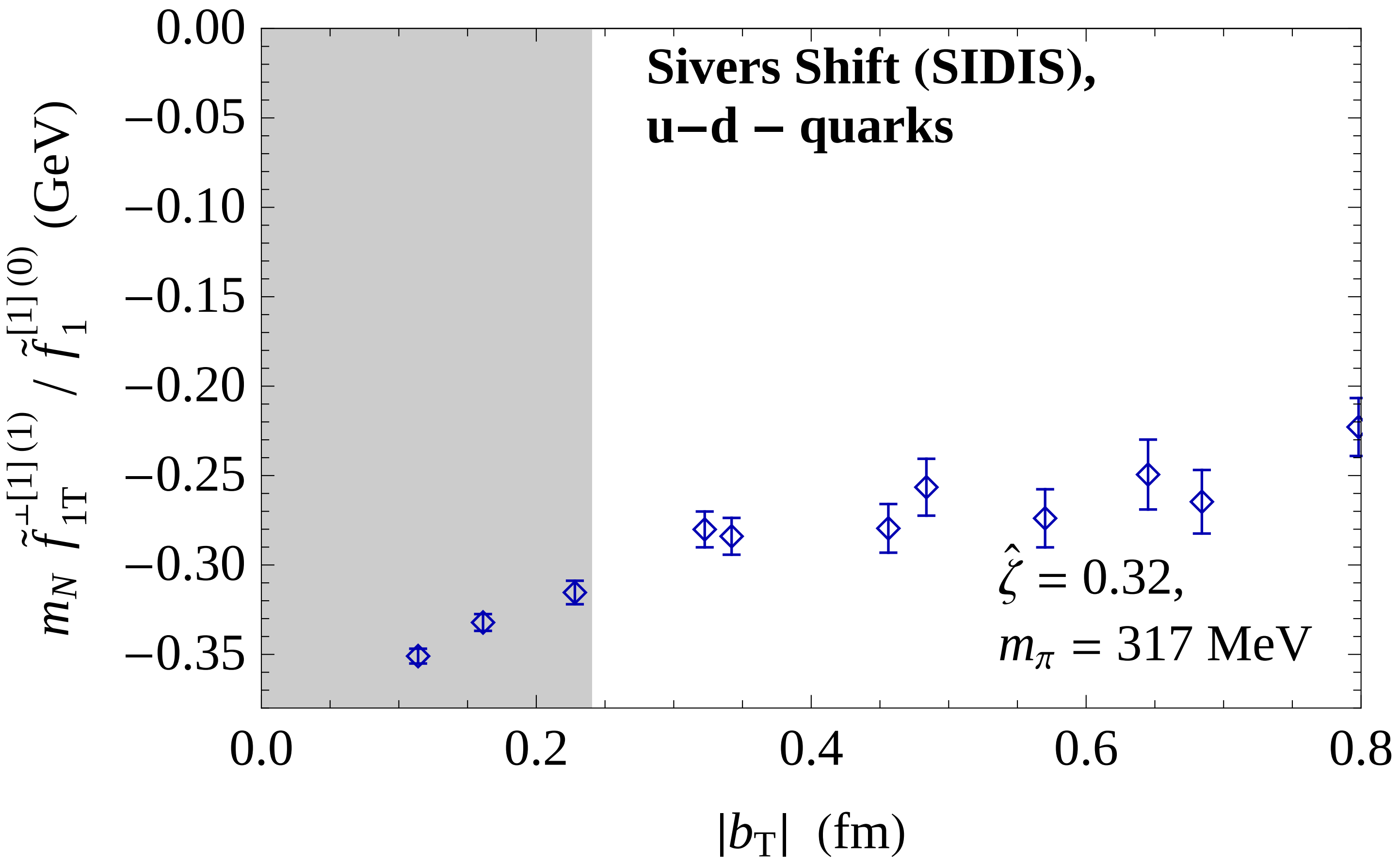




Results: Sivers shift

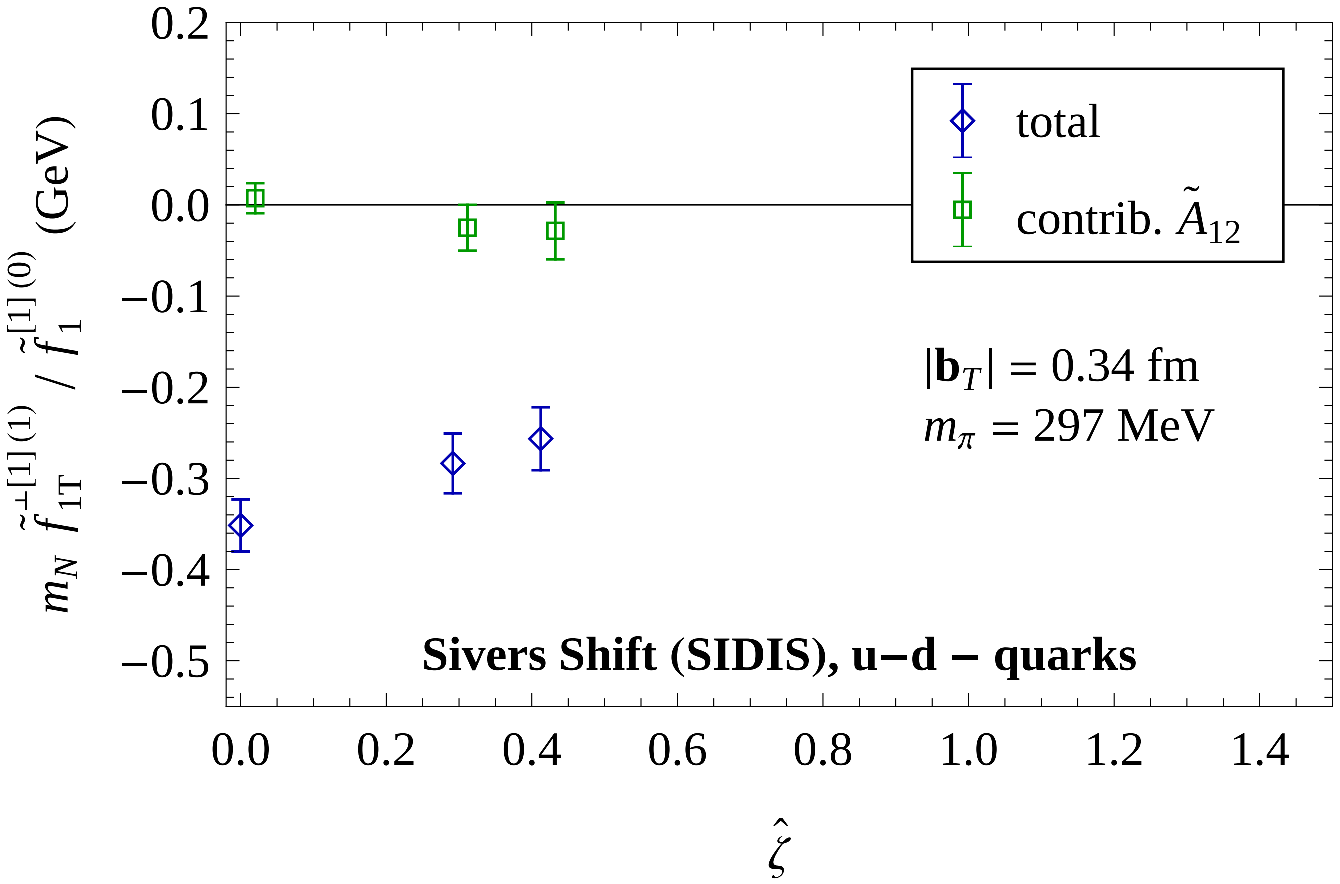
Dependence of SIDIS limit on $|b_T|$

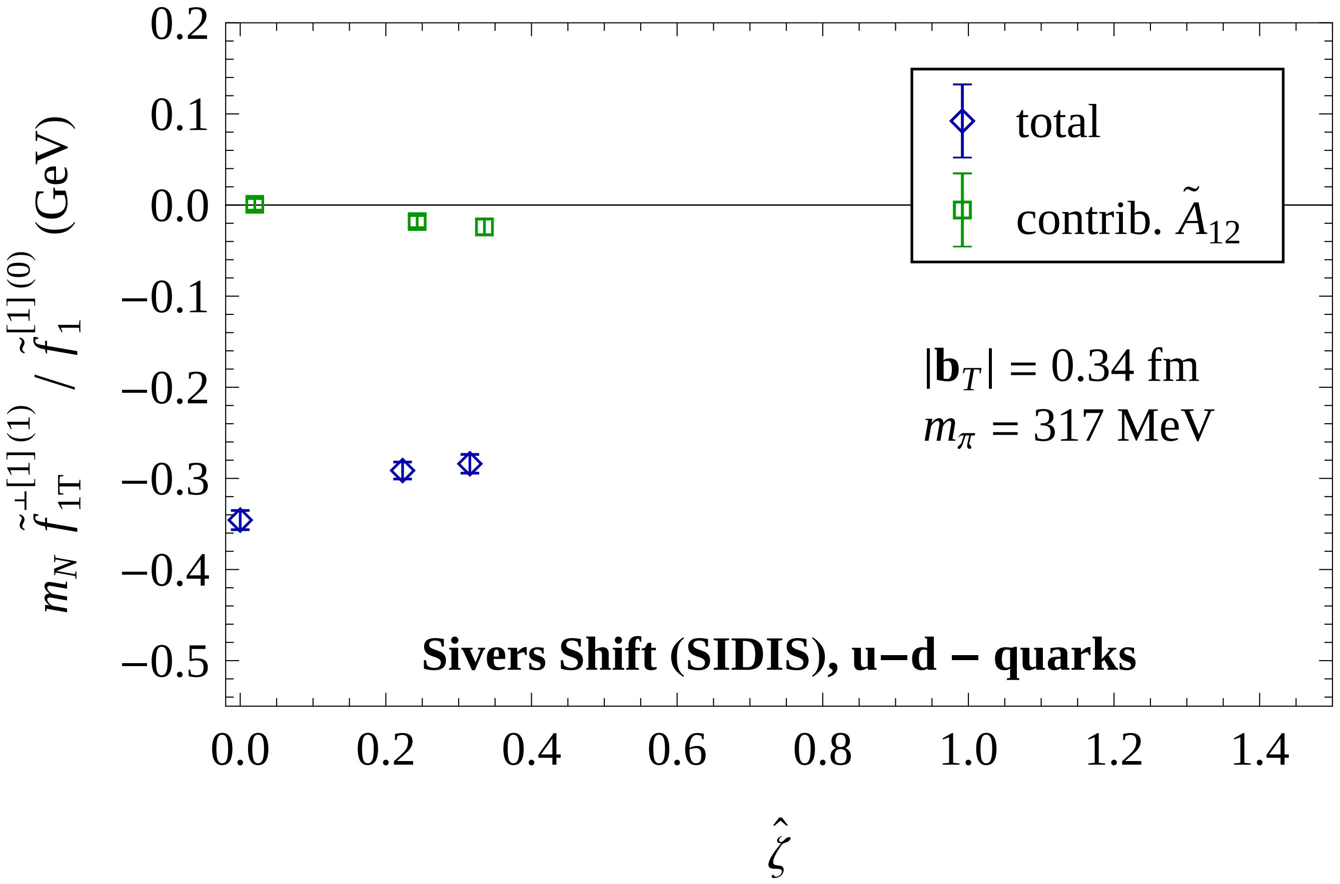




Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$

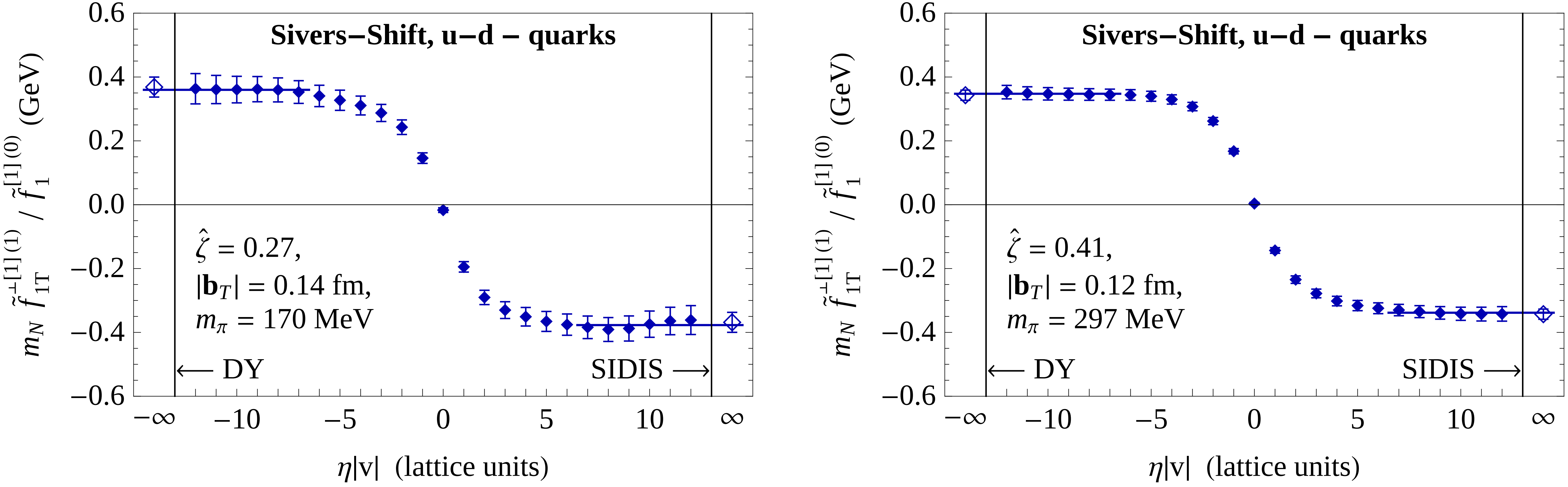




Dependence on the pion mass

Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$



Results: Sivers shift

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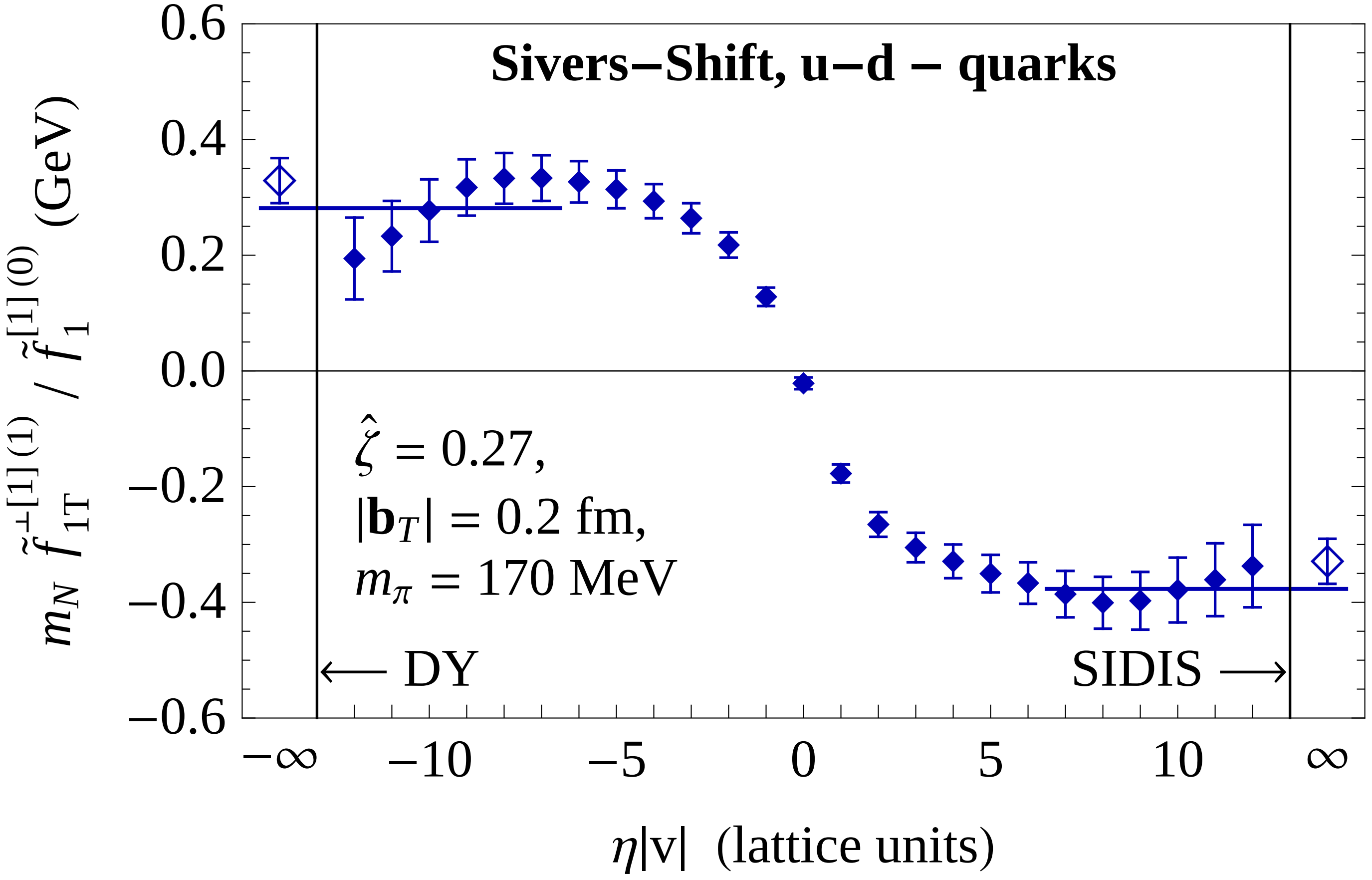

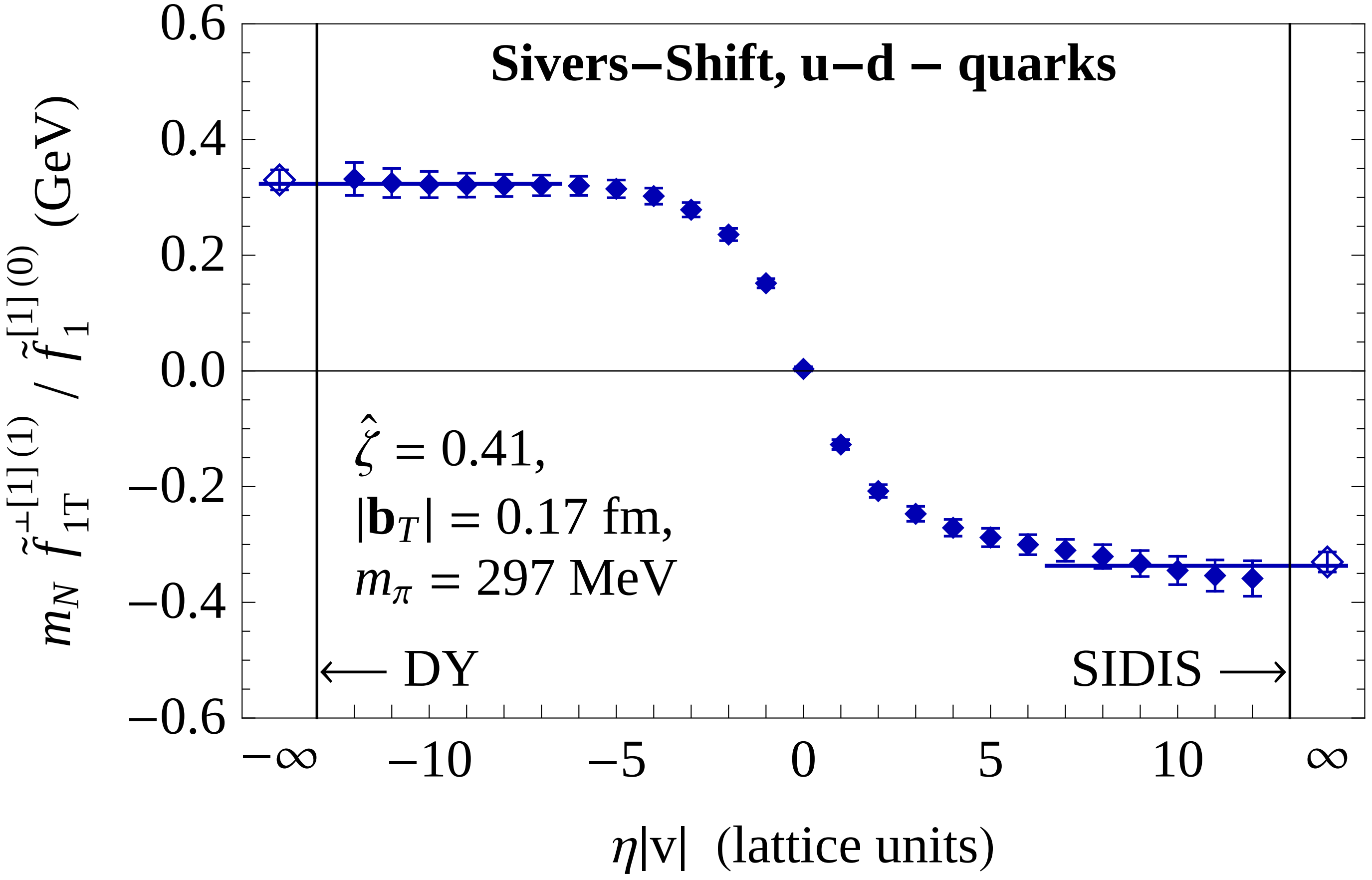

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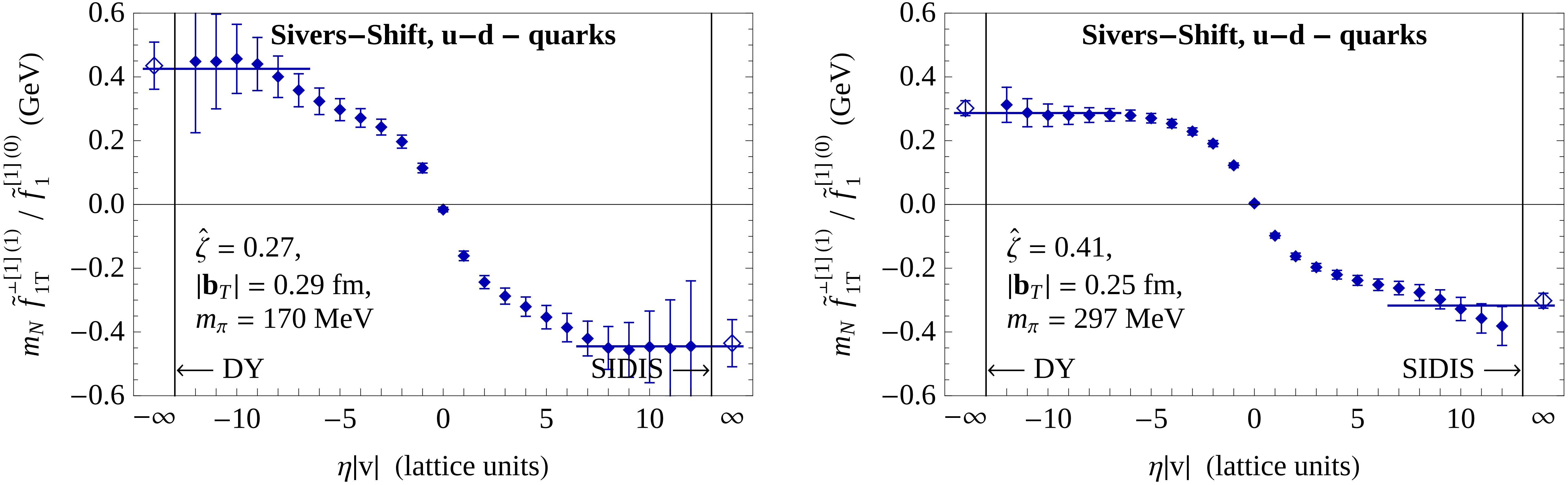

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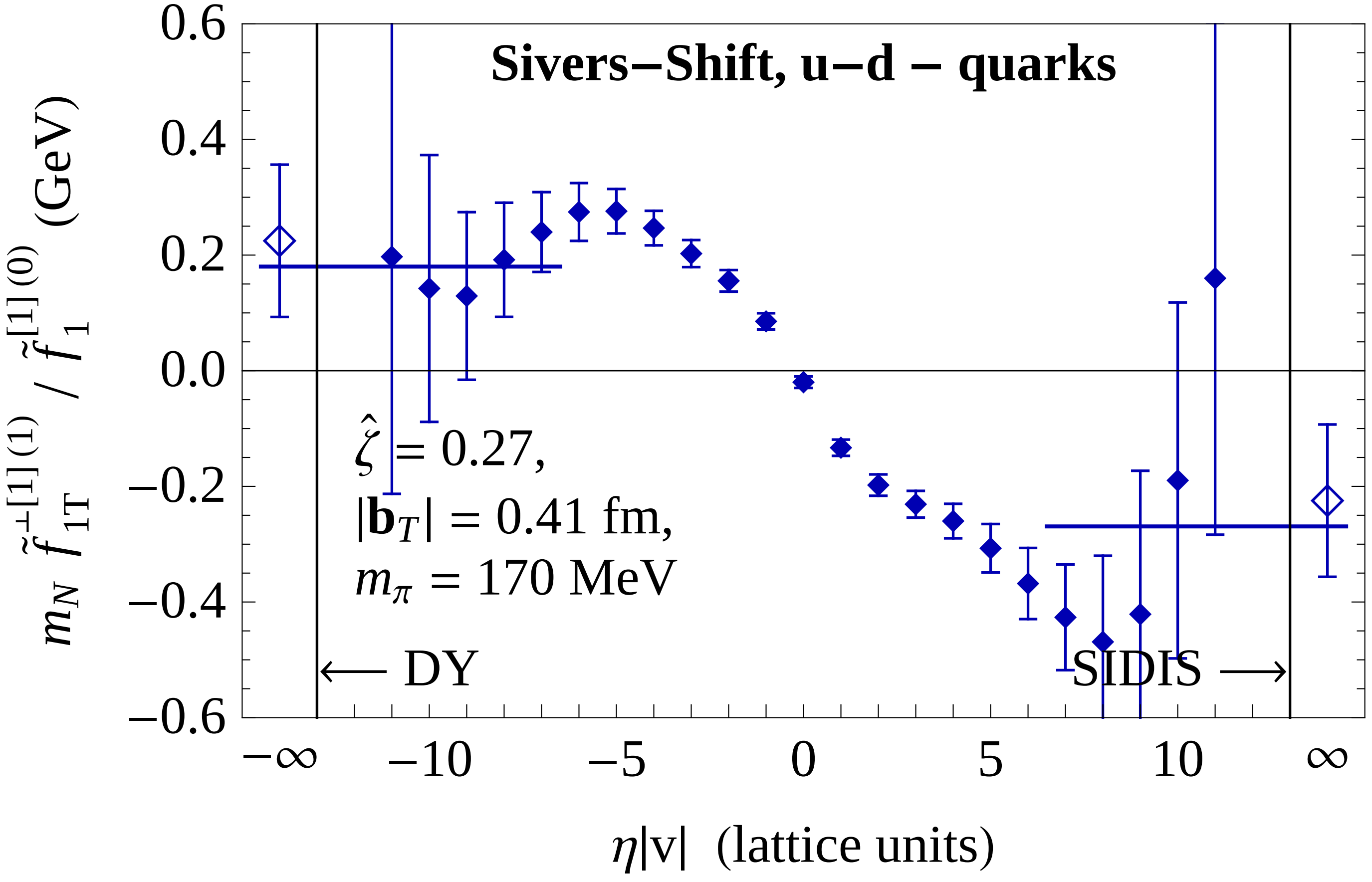

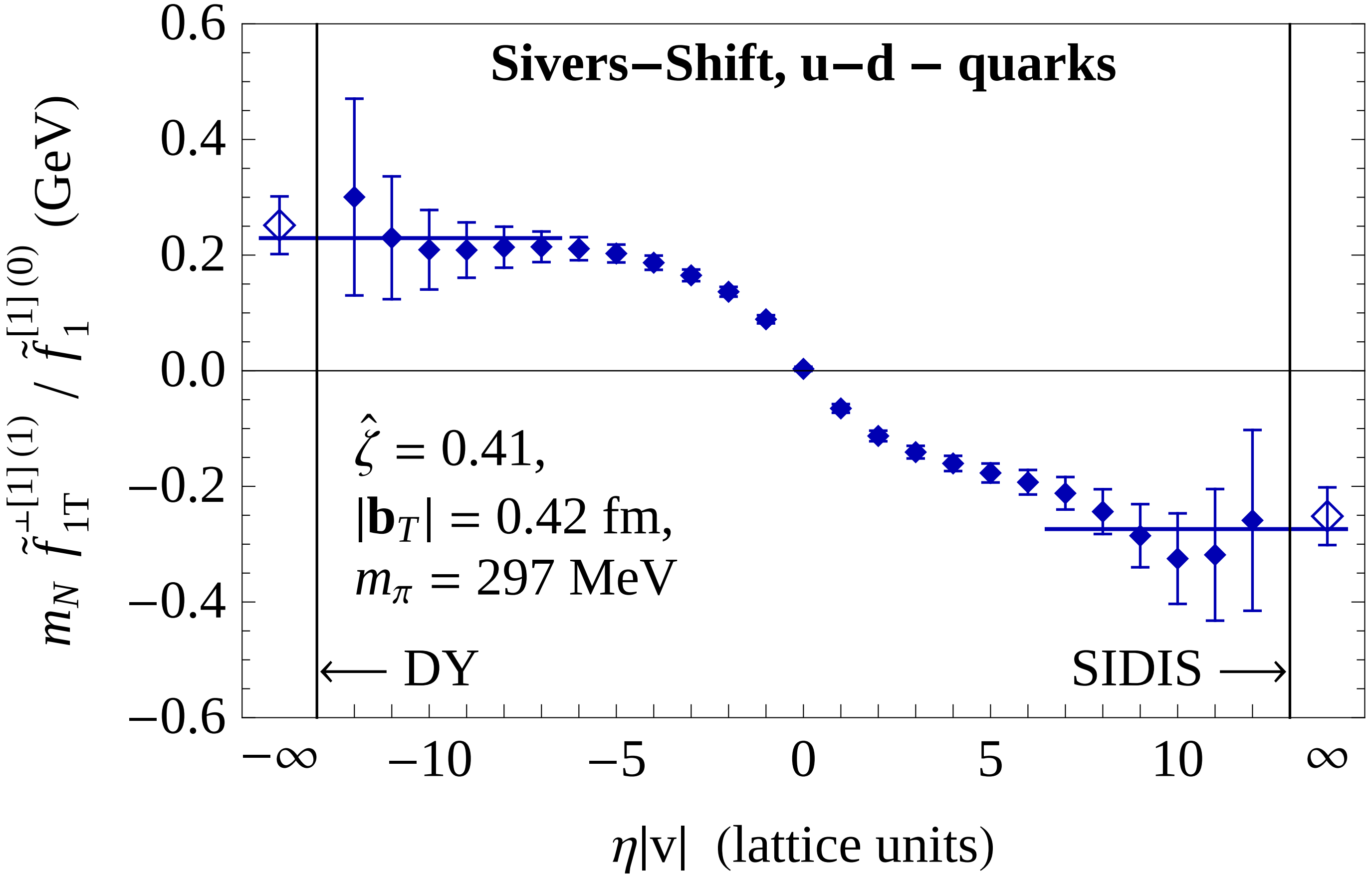

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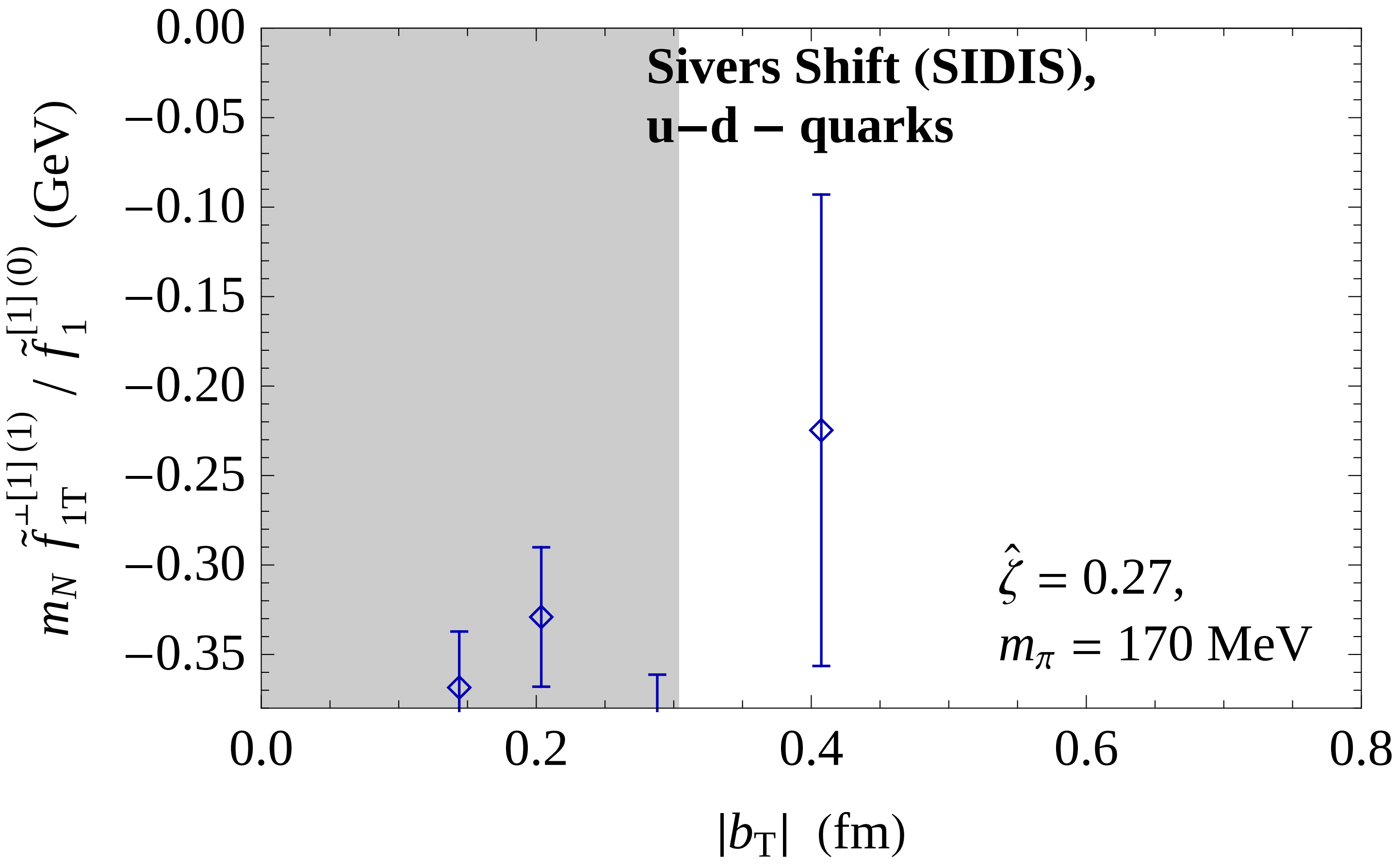
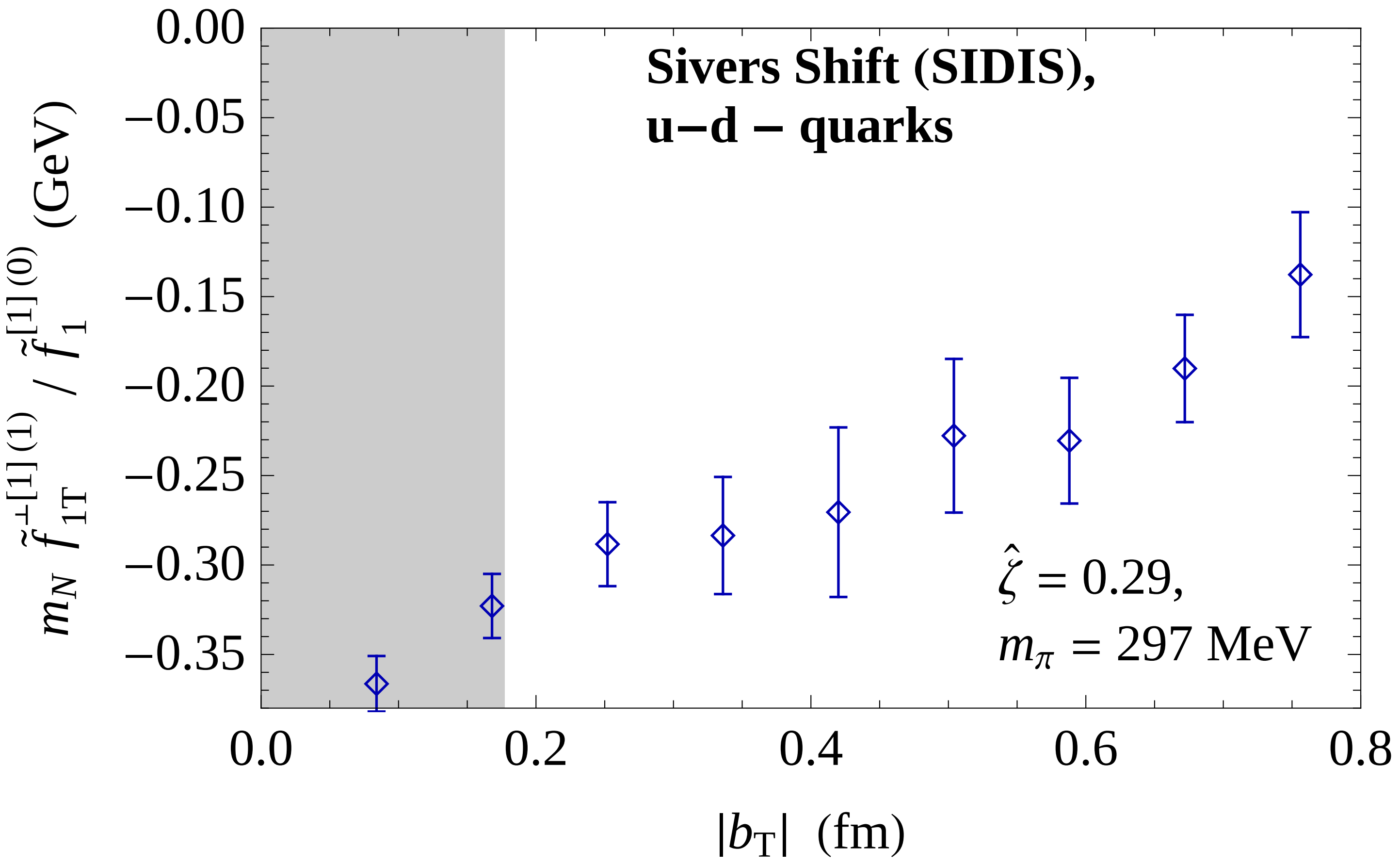
Dependence of SIDIS limit on $|b_T|$





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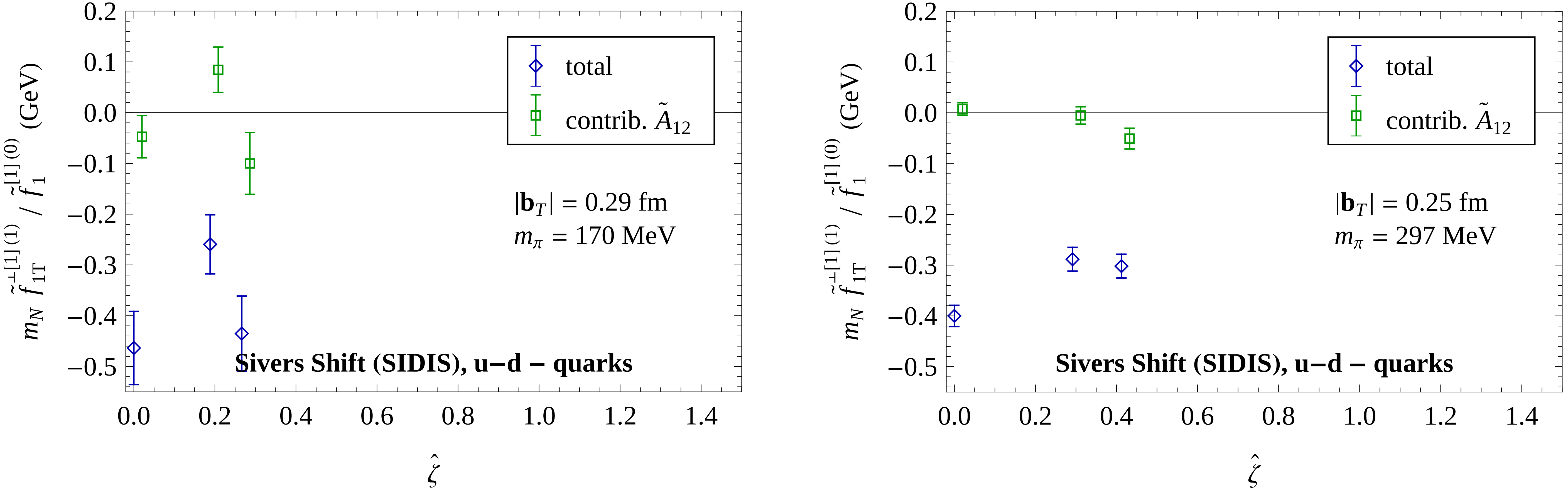
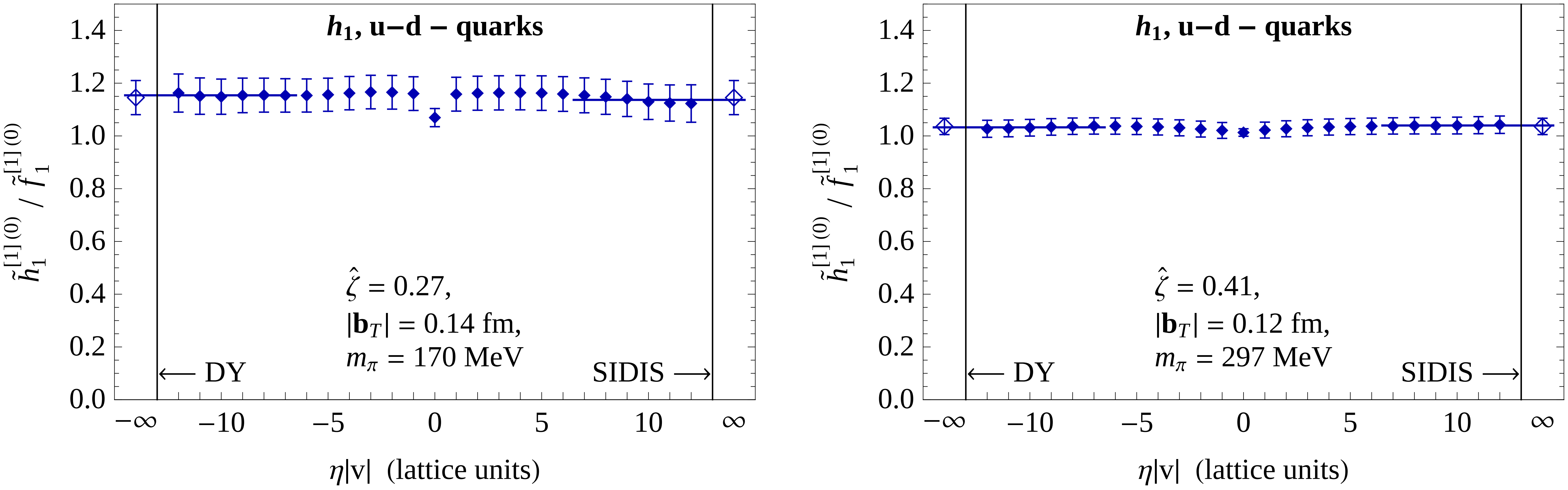


Results: Transversity

Dependence on staple extent; sequence of panels at different $|b_T|$



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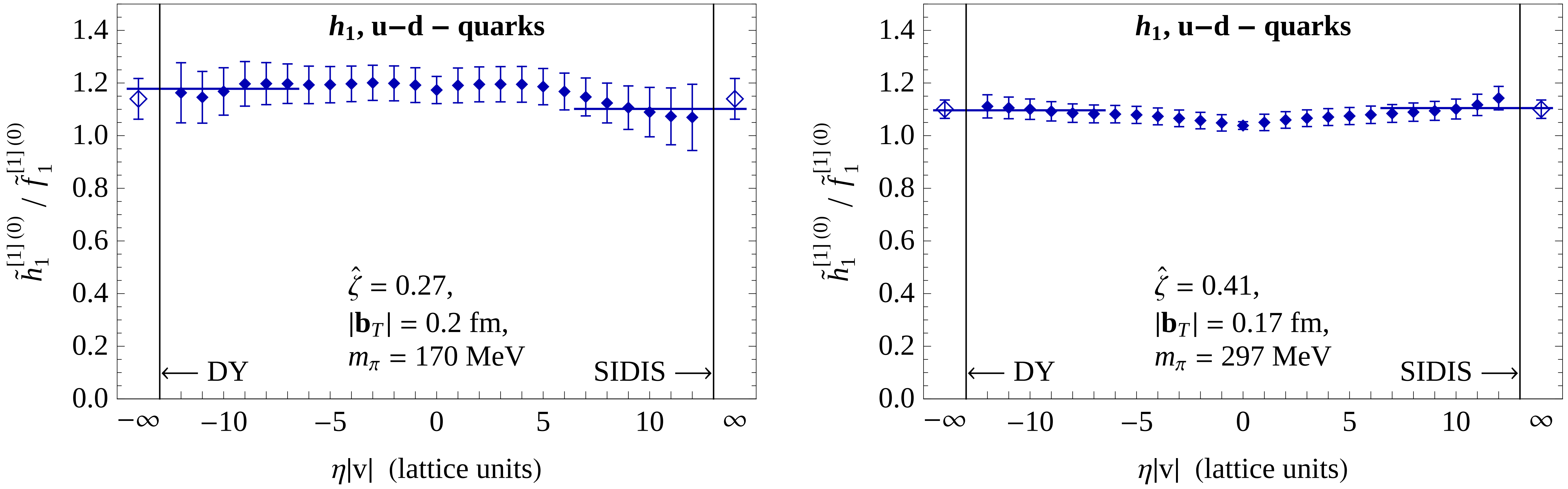

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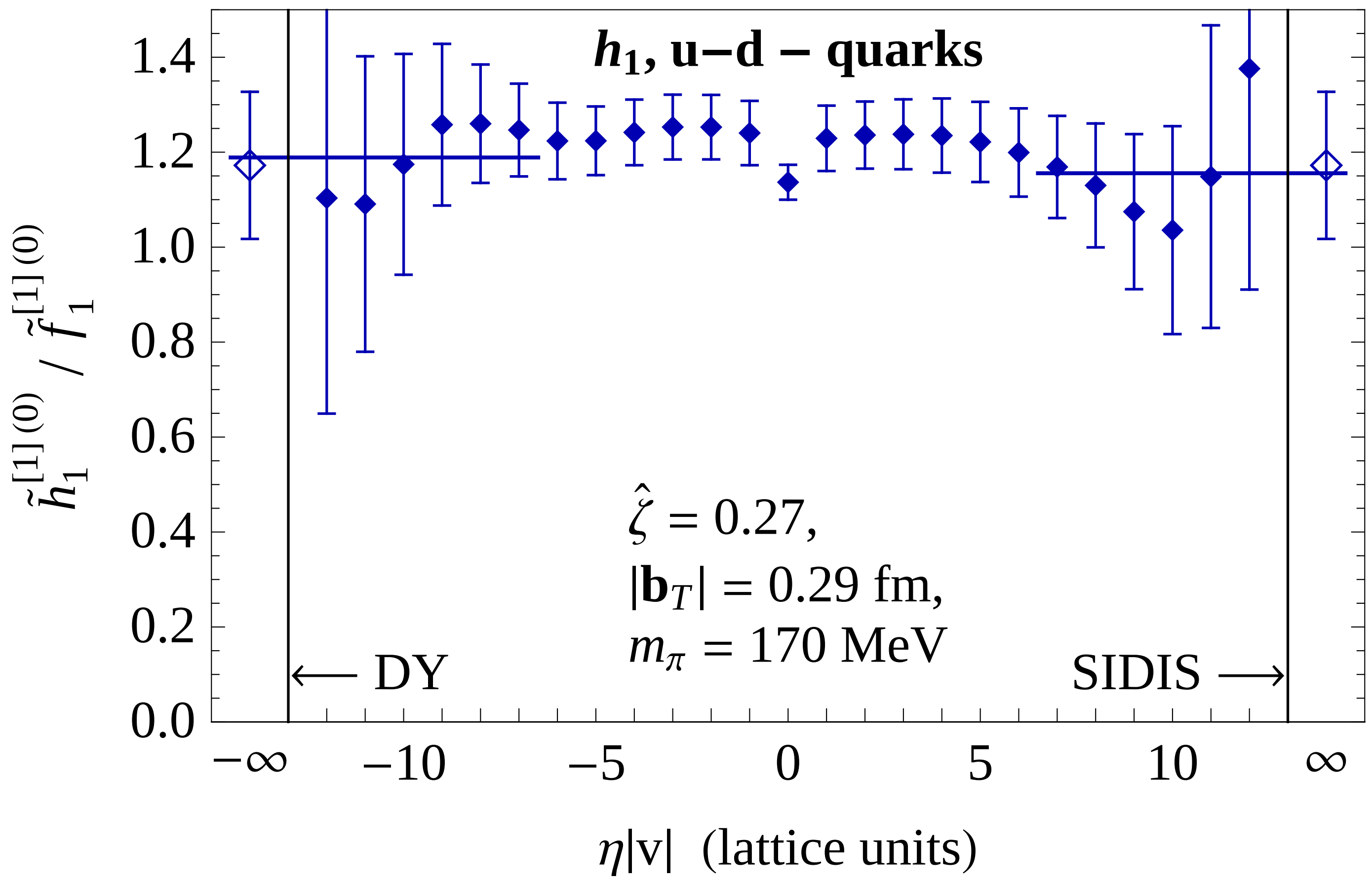

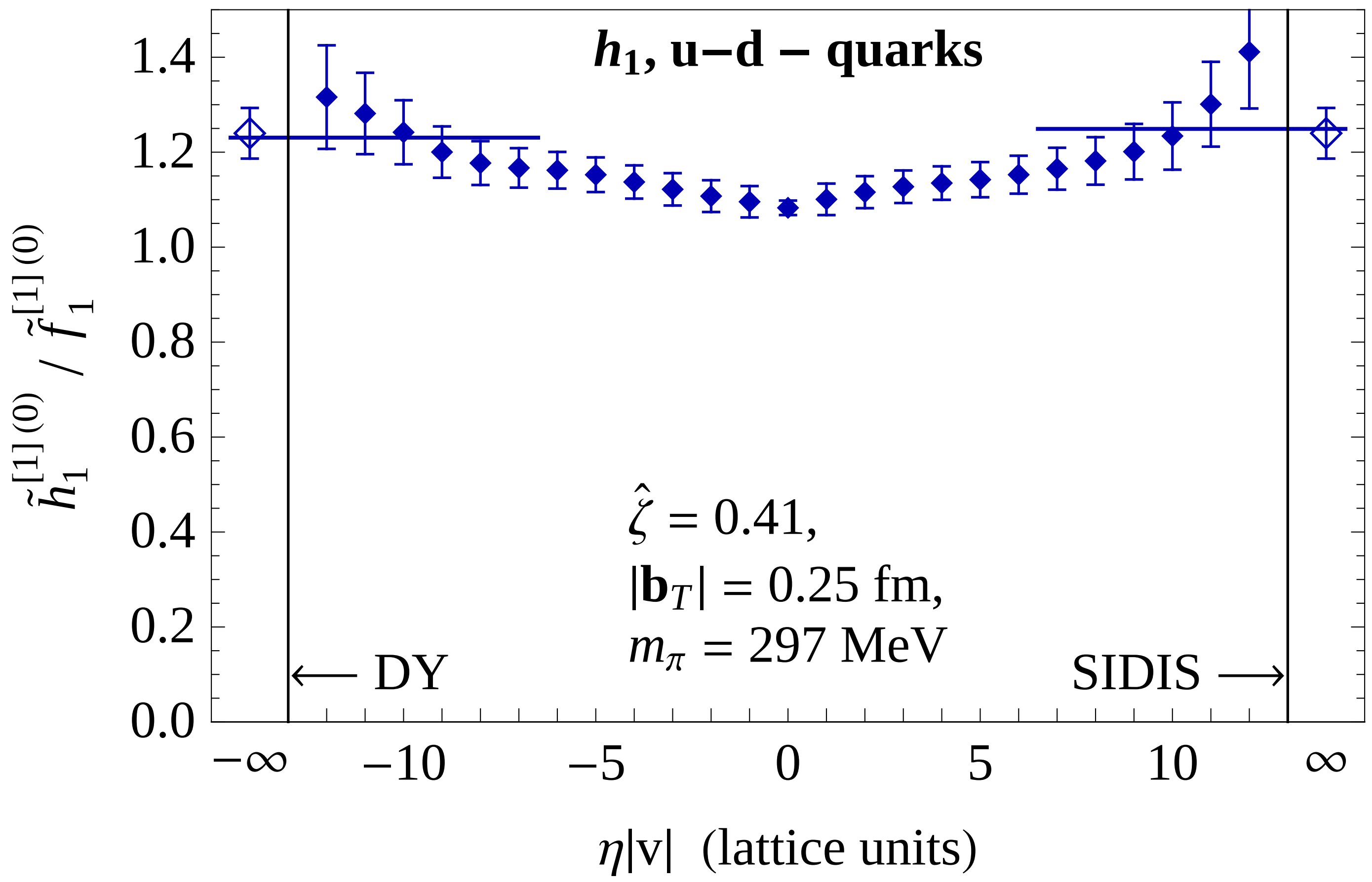

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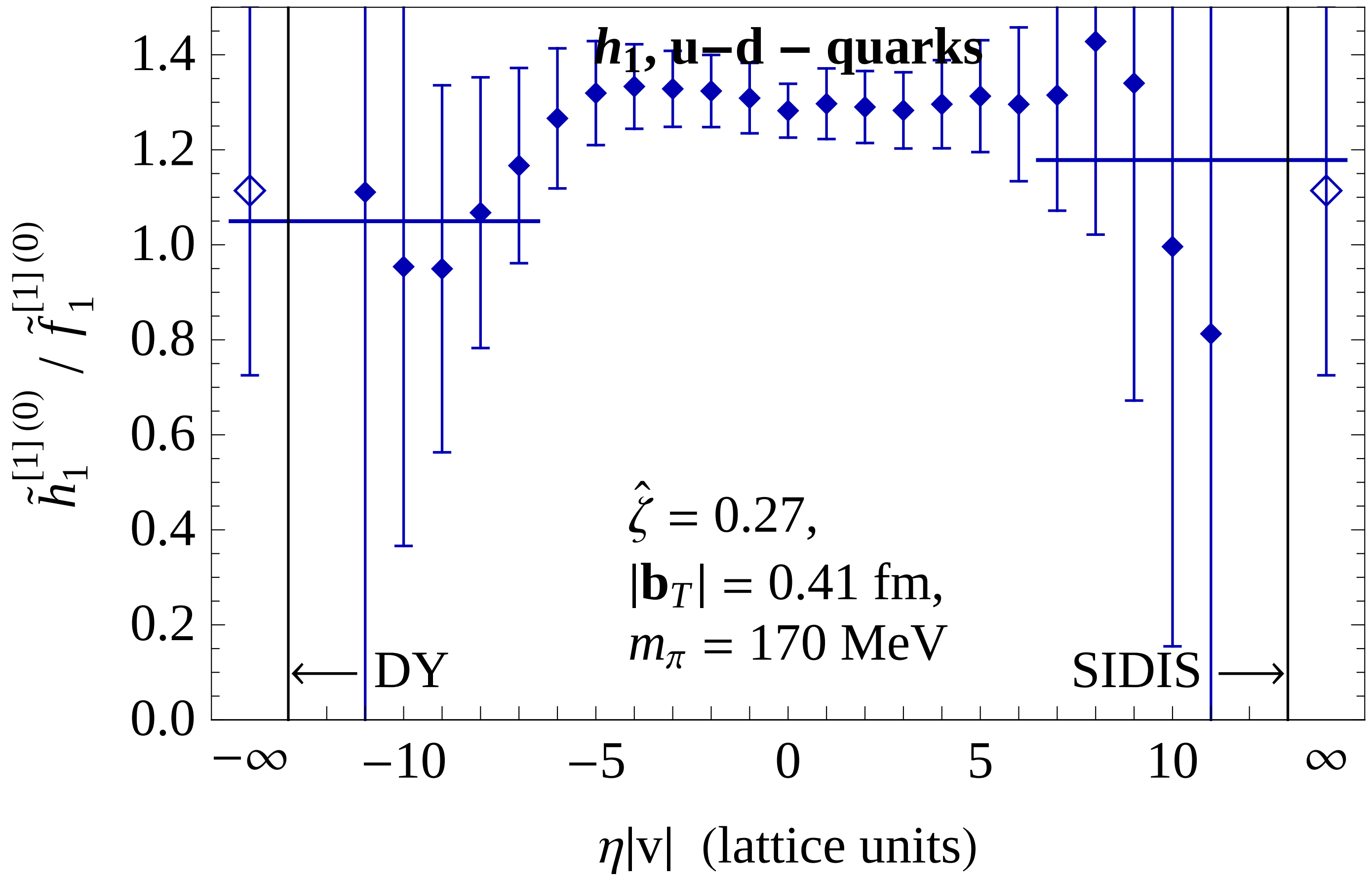
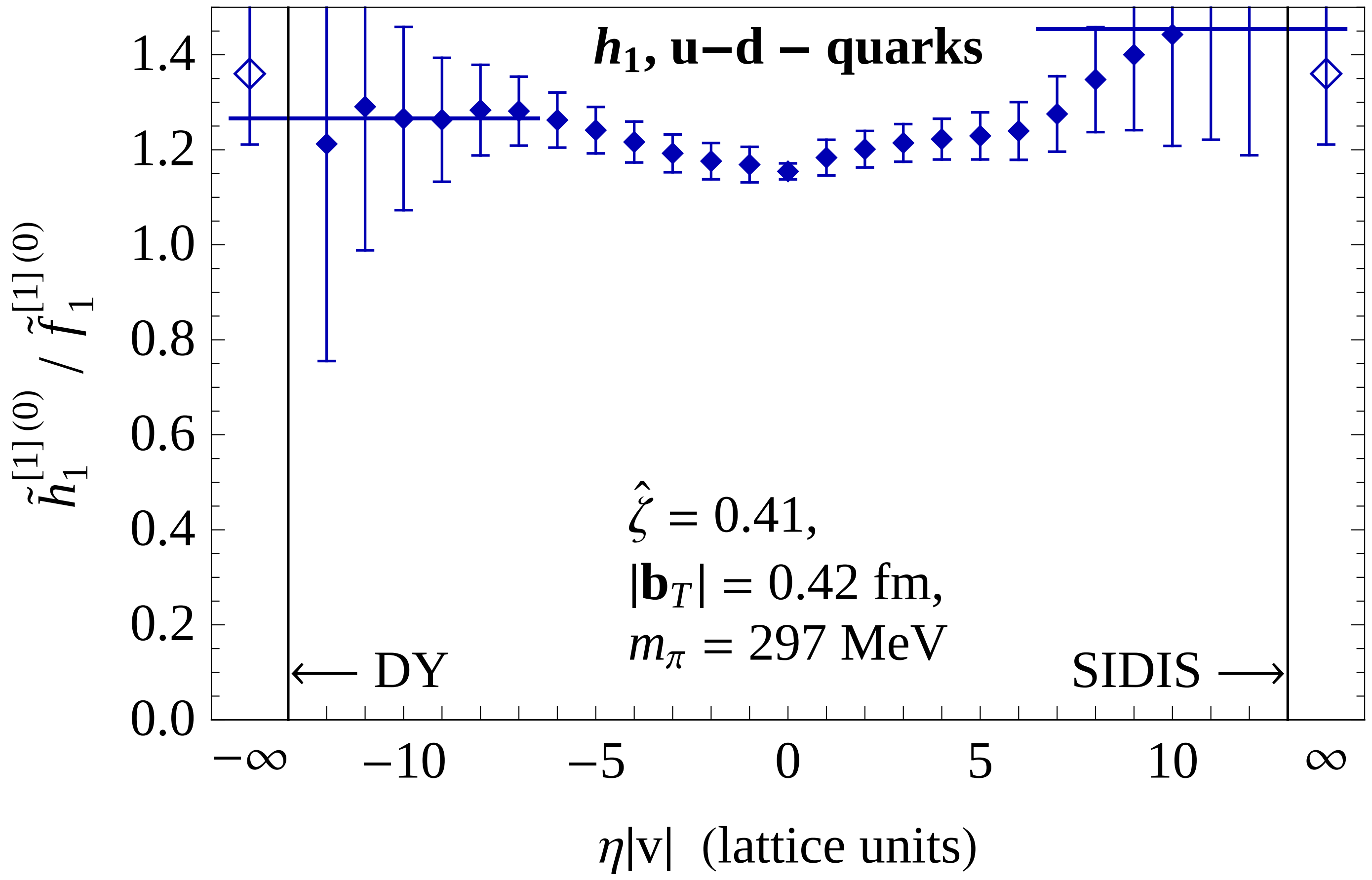
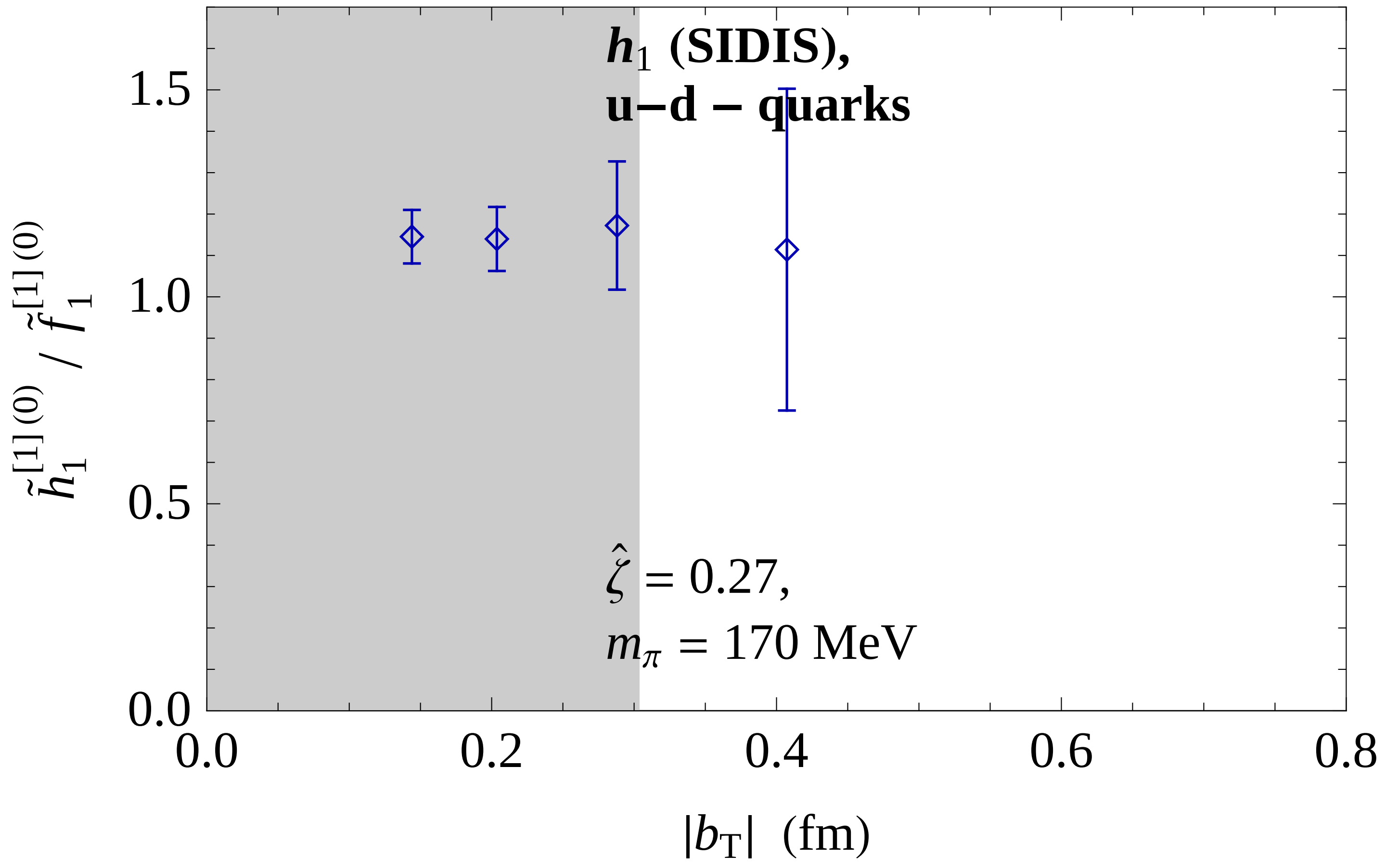
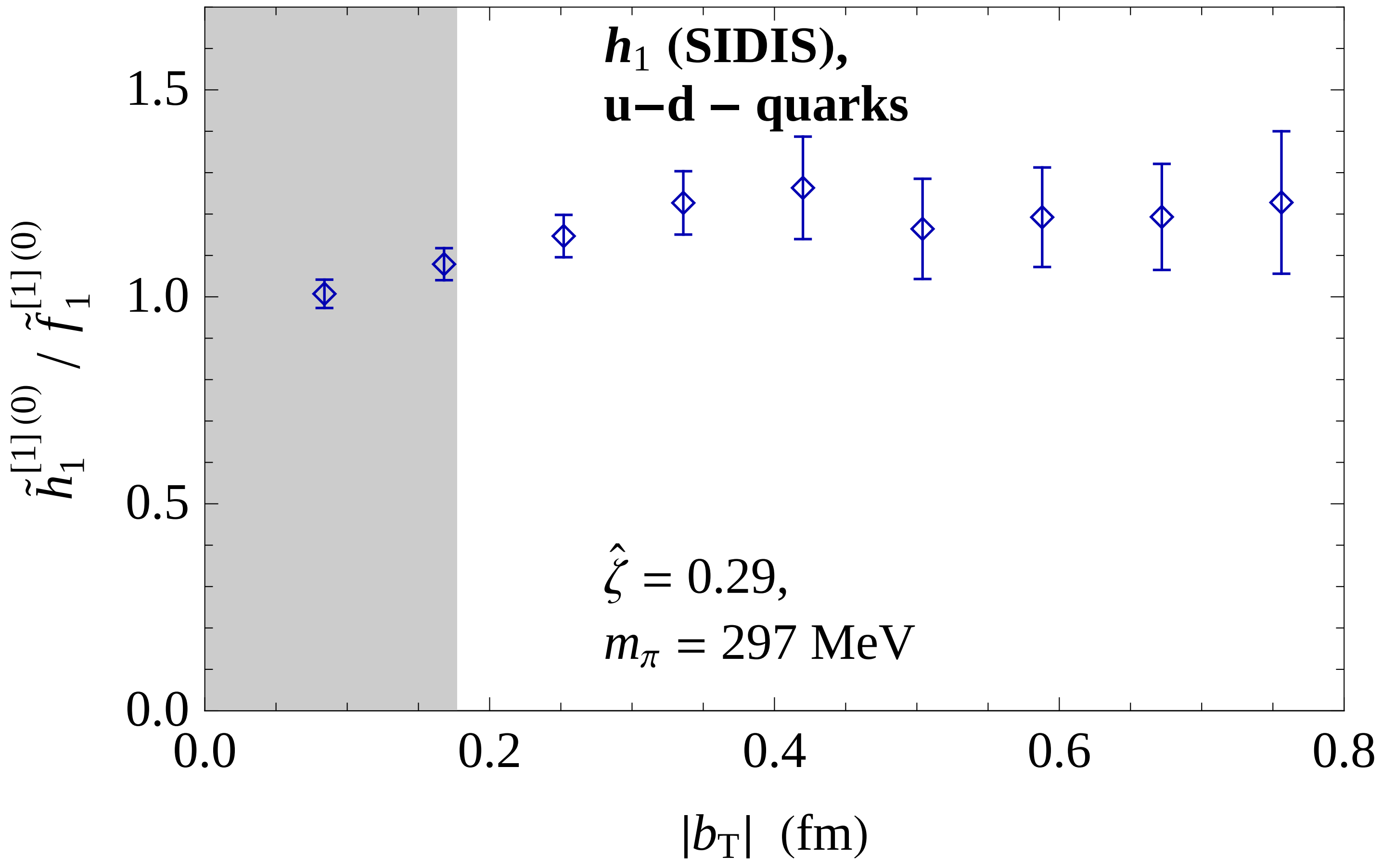




Results: Transversity

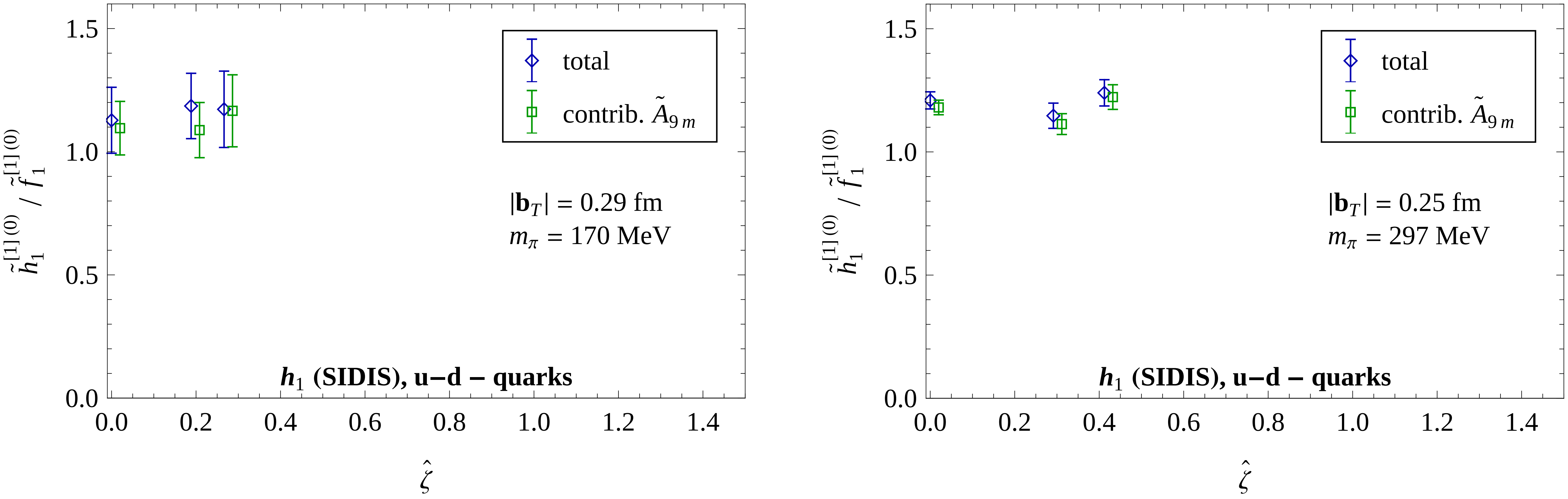
Dependence of SIDIS/DY limit on $|b_T|$





Results: Transversity

Dependence of SIDIS/DY limit on $\hat{\zeta}$

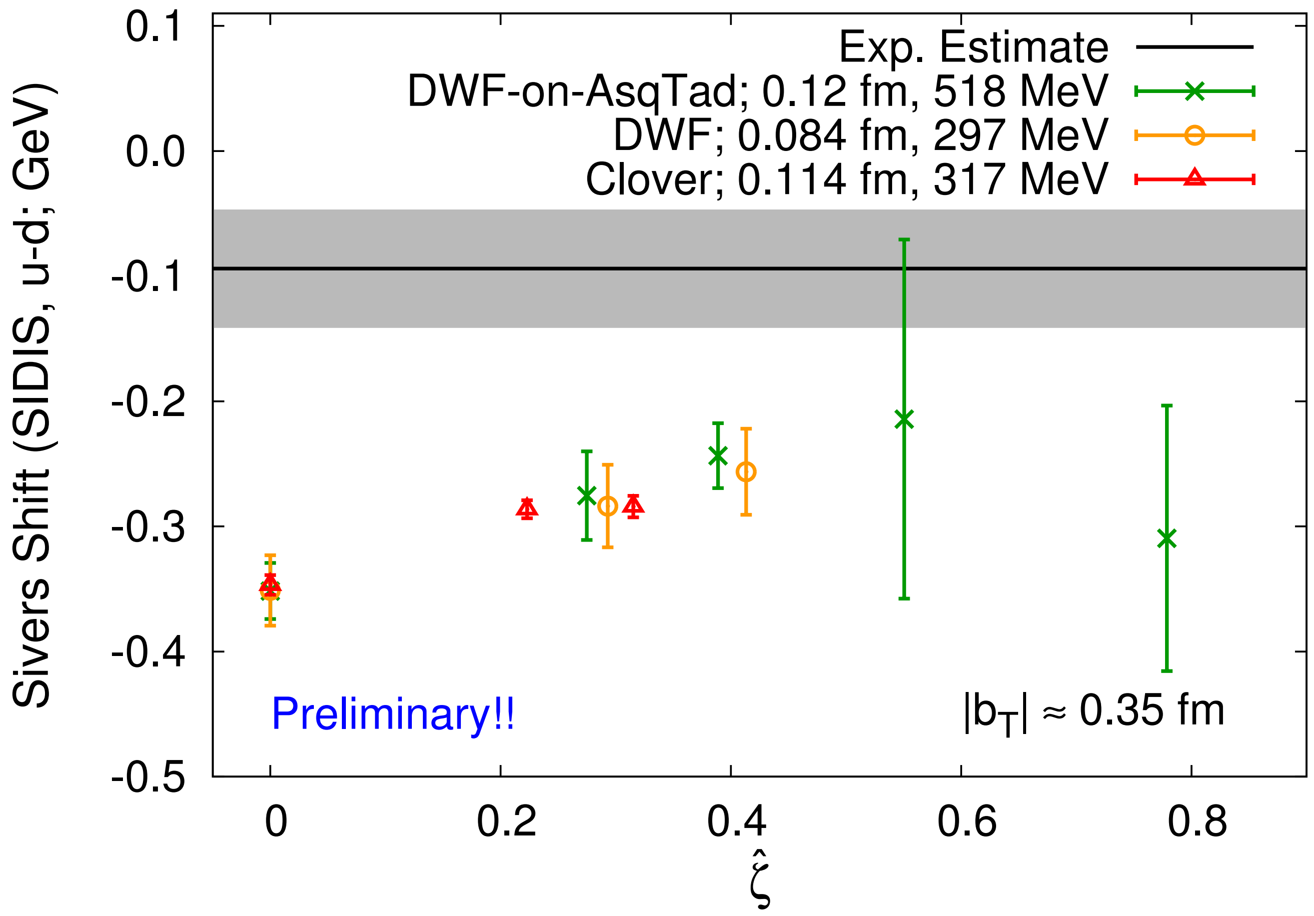


Progressing toward the physical pion mass

In preparation: Clover ensemble (K. Orginos and JLab collaborators) at 190 MeV pion mass

Results: Sivers shift summary

Dependence of SIDIS limit on $\hat{\zeta}$



Experimental value from global fit to HERMES, COMPASS and JLab data,
M. Echevarria, A. Idilbi, Z.-B. Kang and I. Vitev, Phys. Rev. D 89 (2014) 074013

Conclusions and Outlook

- Continued exploration of TMDs using bilocal quark operators with staple-shaped gauge link structures; exploration of challenges posed by $\hat{\zeta} \rightarrow \infty$ limit, discretization effects, physical limit.
- To avoid soft factors, multiplicative renormalization constants, considered appropriate ratios of Fourier-transformed TMDs ("shifts").
- These observables show no statistically significant variation under the considered changes of action, lattice spacing and pion mass, except at very short distances.
- Production on a clover ensemble (K. Orginos and collaborators) at 190 MeV pion mass in preparation.
- Generalization to mixed transverse momentum / transverse position observables (Wigner functions) will give direct access to quark orbital angular momentum; production complete, analysis in progress.

Quark orbital angular momentum in units of the number of valence quarks

$$\frac{L_z}{n} = \frac{2\epsilon_{ij}\frac{\partial}{\partial b_{T,i}}\frac{\partial}{\partial q_{T,j}}\,\langle P,S \mid \overline{\psi}(0)\gamma^{+}\mathcal{U}[0,b]\psi(b) \mid P',S\rangle\Big|_{b^+=b^-=0\,,\ q_T=0\,,\ b_T\to 0}}{\langle P,S \mid \overline{\psi}(0)\gamma^{+}\mathcal{U}[0,b]\psi(b) \mid P',S\rangle\Big|_{b^+=b^-=0\,,\ q_T=0\,,\ b_T\to 0}}$$

$P = p - q_T/2$, $P' = p + q_T/2$, p, S in z-direction, $p \to \infty$

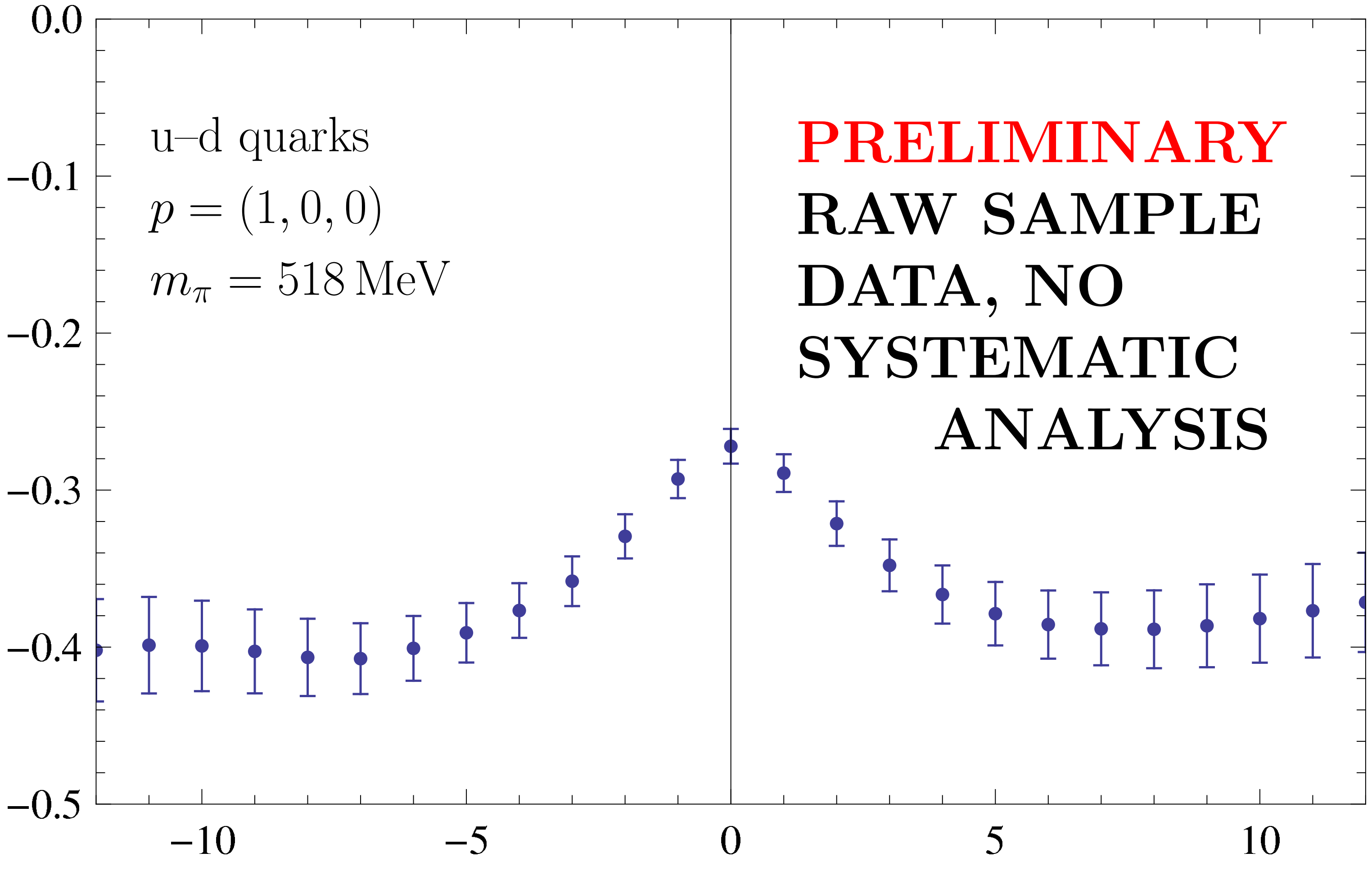
Y. Hatta, M. Burkardt:
Staple-shaped $\mathcal{U}[0,b] \longrightarrow$ Jaffe-Manohar OAM
Straight $\mathcal{U}[0,b] \longrightarrow$ Ji OAM

Connection to GTMDs –
A. Metz, M. Schlegel, C. Lorcé,
B. Pasquini …

M. Polyakov, S. Liuti:
Connection to twist three, direct lattice calculation
of Ji OAM at twist three

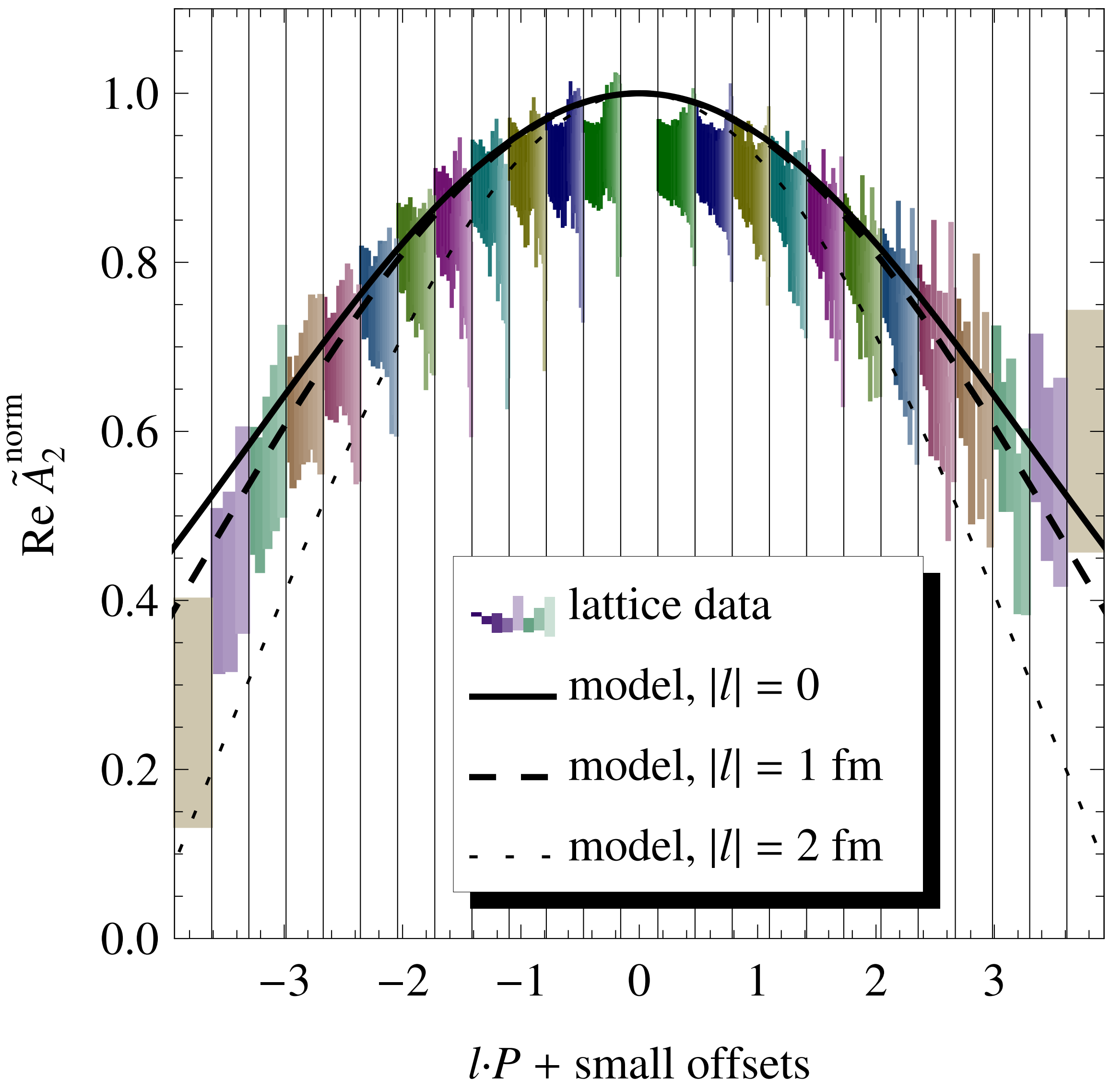
Quark orbital angular momentum in units of the number of valence quarks

$$\frac{L_z}{n} = \frac{2\epsilon_{ij}\frac{\partial}{\partial b_{T,i}}\frac{\partial}{\partial q_{T,j}} \langle P,S \mid \overline{\psi}(0)\gamma^{+}\mathcal{U}[0,b]\psi(b) \mid P',S\rangle\big|_{b^+=b^-=0\,,\ q_T=0\,,\ b_T\to 0}}{\langle P,S \mid \overline{\psi}(0)\gamma^{+}\mathcal{U}[0,b]\psi(b) \mid P',S\rangle\big|_{b^+=b^-=0\,,\ q_T=0\,,\ b_T\to 0}}$$



Accessing Bjorken-x dependence

(Fourier transform of) unpolarized distribution, up quarks, normalized to unity at $l \cdot P = 0$



$l \cdot P$: Variable Fourier conjugate to Bjorken x

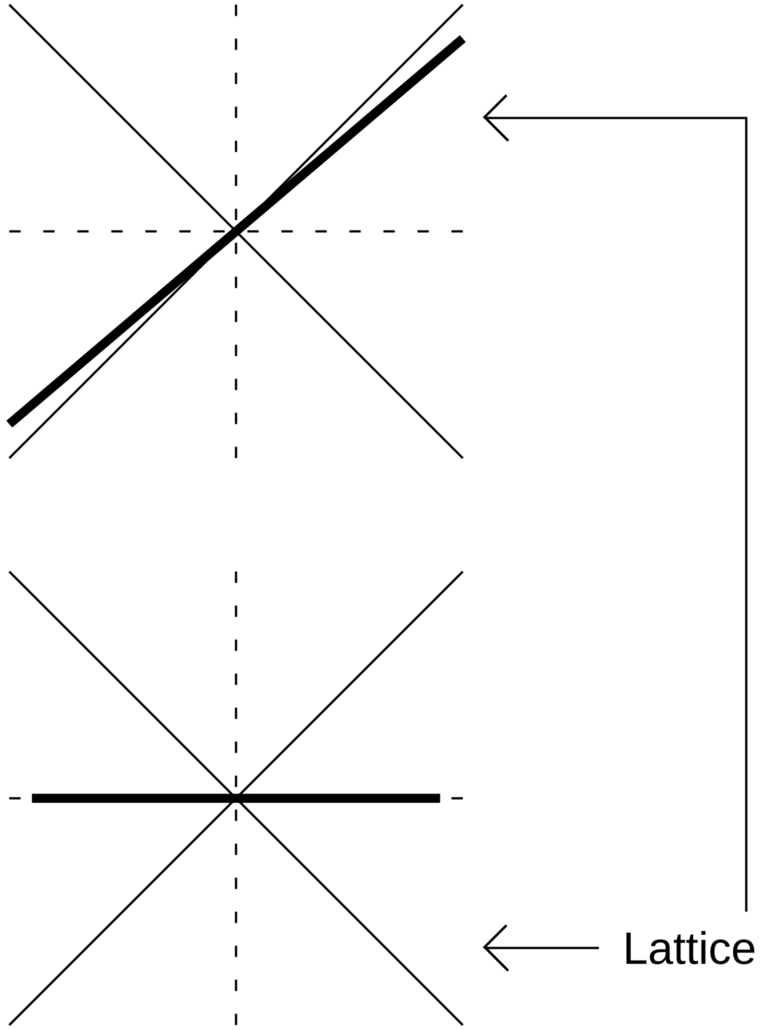
From: B. Musch, P. Hägler, J. Negele and A. Schäfer, Phys. Rev. **D 83** (2011) 094507.

Lattice: $m_\pi = 625\,\mathrm{MeV}$
Model curves: Spectator diquark model

Relation to Ji Large Momentum Effective Theory (LaMET)

Phenomenology

Lattice QCD



Collins TMD
factorization

TMD ratios

invariant
amplitudes

Ji LaMET
spatial soft factor

Lattice calculation